Extended Black Hole Thermodynamics from Extended Iyer-Wald Formalism

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In recent years, there has been significant interest in the field of extended black hole thermodynamics, where the cosmological constant and/or other coupling parameters are treated as thermodynamic variables. Drawing inspiration from the Iyer-Wald formalism, which reveals the intrinsic and universal structure of conventional black hole thermodynamics, we illustrate that a proper extension of this formalism also unveils the underlying theoretical structure of extended black hole thermodynamics. As a remarkable consequence, for any gravitational theory described by a diffeomorphism invariant action, it is always possible to construct a consistent extended thermodynamics using this extended formalism.

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Introduction.—Black hole thermodynamics plays a central role in understanding the characteristics of quantum gravity. In recent years, the so-called extended black hole thermodynamics has been proposed and substantially developed [1–3]. For instance, by treating the cosmological constant Λ as a thermodynamic variable, the first law of thermodynamics of the anti–de Sitter (AdS)-Schwarzschild black hole can be expressed as

$$\tilde{\delta} M = T\tilde{\delta} S + V\tilde{\delta} P, \tag{1}$$

where $P \equiv -(\Lambda/8\pi)$ and $V = (4\pi/3)r_h^3$ are commonly referred to as the pressure and thermodynamic volume in the literature. However, literally interpreting Λ as pressure could be misleading in some sense [4]. A better strategy is to put Λ on equal footing with other couplings in the action. Note that we use $\tilde{\delta}$ to represent variations in the extended thermodynamics, distinguishing it from the conventional variation δ that satisfies the thermodynamic law $\delta M = T\delta S$.

At first glance, this extension may seem perplexing, as Λ was traditionally considered a fixed parameter in the theory. However, as suggested in Refs. [5,6], the cosmological constant could become a dynamical variable in gauged supergravity and string theories. And a recent work [7] illustrated that the variation of the cosmological constant could be induced by tuning the brane tension in a braneworld model. Therefore, it is generally possible to circumvent the problem by taking Λ as an external parameter controlled by a more comprehensive theory.

The extended black hole thermodynamics has offered a multitude of intriguing physical implications and applications. It serves as a fertile ground for investigating black hole phase transitions, triggering the emergence of a research direction known as black hole chemistry [8–10]. Moreover, within the framework of AdS/CFT, the holographic dual of the extended black hole thermodynamics on the conformal field theory (CFT) side has been widely explored. The AdS/CFT correspondence dictates that $c \propto (l^{D-2}/G_N)$, so the cosmological constant is not directly related to the CFT pressure but rather to the central charge; see [11,12] for details. Besides, the extended thermodynamics can also find applications in the study of holographic complexity, weak cosmic censorship conjecture, and other research fields [13–15].

In fact, it is not only the cosmological constant that can be considered as a thermodynamic variable; other coupling parameters in higher curvature theories of gravity, such as those in Lovelock gravity, can also be regarded as thermodynamic variables [16–20]. Given the successes and universality of the extended thermodynamics, a natural question arises as to whether there exists a fundamental theoretical framework that underlies it. We will show that the answer is affirmative.

In this Letter, we propose a robust formalism that guarantees the presence of the extended thermodynamic law for any diffeomorphism invariant theory of gravity. More significantly, it also provides a novel and systematic approach for computing the conjugate quantities associated with the couplings. In the past, the thermodynamic volume was primarily derived through the thermodynamic relation

$V^{\rm th} \equiv (\partial M/\partial P)_{S,J,\ldots}$, or extracted from the Komar integrals that possess an important geometric interpretation and are closely linked to the Smarr relations [5]. In contrast, our formalism enables an independent calculation of such quantities, which has unexpectedly resolved an ambiguity between the thermodynamic volume and the geometric volume present in the literature. All the detailed calculations are provided in the Supplemental Material [21].

Iyer-Wald formalism for conventional black hole thermodynamics.—As is well known, for any diffeomorphism invariant gravitational theory, the conventional black hole thermodynamic law can be easily obtained by employing the Iyer-Wald formalism [22–24].

Consider the Lagrangian denoted by $\mathbf{L} = L\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is the volume form of the $D = d + 1$ dimensional space-time. Its variation with respect to the dynamic fields $\phi \equiv \{g_{\mu\nu}, \psi\}$ can be expressed as

$$\delta\mathbf{L} = \mathbf{E}^{\phi}\delta\phi + d\mathbf{\Theta}[\delta\phi], \tag{2}$$

where $\mathbf{E}^{\phi}$ represents the equations of motion of the fields, and $d\mathbf{\Theta}[\delta\phi]$ is a total derivative term. The differential forms on space-time are written in boldface, and below we implicitly use the on shell condition $\mathbf{E}^{\phi} = 0$.

For an arbitrary, fixed vector ξ, the familiar Noether current $\mathbf{J}_{\xi}$ is defined as

$$\mathbf{J}_{\xi} \equiv \mathbf{\Theta}[\mathcal{L}_{\xi}\phi] - \xi \cdot \mathbf{L}. \tag{3}$$

Under the on shell condition, $d\mathbf{J}_{\xi} = 0$, so the Noether charge $\mathbf{Q}_{\xi}$ can be constructed from $\mathbf{J}_{\xi} = d\mathbf{Q}_{\xi}$. Then, requiring ξ to be Killing, two pivotal formulas can be proven:

$$d(\delta\mathbf{Q}_{\xi} - \xi \cdot \mathbf{\Theta}[\delta\phi]) = 0, \tag{4}$$

$$d\mathbf{Q}_{\xi} = -\xi \cdot \mathbf{L}. \tag{5}$$

The proof of Eq. (4) has been left to the Supplemental Material [21]; Eq. (5) is a direct consequence of the Killing property $\mathcal{L}_{\xi}\phi = 0$.

For a stationary black hole with a bifurcate Killing horizon, we integrate the two formulas over a hypersurface $\mathcal{V}_r$ extending from the bifurcation surface denoted by S_h to another codimension-2 surface S_r. Owing to Gauss's theorem, Eqs. (4) and (5) become

$$\int_{S_r} (\delta\mathbf{Q}_{\xi_H} - \xi_H \cdot \mathbf{\Theta}[\delta\phi]) - \int_{S_h} \delta\mathbf{Q}_{\xi_H} = 0, \tag{6}$$

$$\int_{S_r} \mathbf{Q}_{\xi_H} - \int_{S_h} \mathbf{Q}_{\xi_H} = -\int_{\mathcal{V}_r} \xi_H \cdot \mathbf{L}, \tag{7}$$

where we have applied the horizon Killing vector ξ_H that vanishes on S_h. Owing to the unique characteristics of the horizon Killing vector and the bifurcation surface, one can show that $\int_{S_h} \delta\mathbf{Q}_{\xi_H} = T\delta S$ and similarly $\int_{S_h} \mathbf{Q}_{\xi_H} = TS$ for the gravity sector, which actually is the origin of the definition of Wald entropy [22,23]. Thus it is essential to use ξ_H in order to establish a connection with the thermodynamic properties of a black hole. Thereupon, the first law of black hole thermodynamics follows from Eq. (6), and the Smarr relation follows from Eq. (7).

We emphasize that Eqs. (6) and (7) are valid for any surface S_r of radius r that encompasses the black hole, which is not necessarily the spatial infinity. In realistic calculations, all the terms dependent on r will cancel out precisely in the final result, as a distinctive feature of the covariant formalism. Thus we do not have to worry about the potential divergent behaviors caused by $r \to \infty$, particularly in the context of AdS black holes.

An extension of the Iyer-Wald formalism.—Let us consider a general diffeomorphism invariant gravitational theory with the Lagrangian

$$L = \frac{1}{16\pi}(R - 2\Lambda) + \sum_m \alpha_m F_m[g_{\mu\nu}, \nabla_\rho, R_{\mu\nu\rho\sigma}], \tag{8}$$

where F_m represents the higher curvature term. In the spirit of effective field theory, it is permissible to add all possible diffeomorphism invariant curvature terms to the Einstein-Hilbert action [16,25]. Without confusion, we also use the subscript m to count the number of the curvatures and half of the number of ∇_ρ's in F_m, so that the coupling parameter α_m has the dimension $[L]^{2(m-1)}$. We neglect the matter field sectors; thus $\phi = g_{\mu\nu}$ is the only dynamical field.

The operator $\tilde{\delta}$ allows for variations in the cosmological constant Λ and other couplings α_m, in contrast to the conventional variation δ. Technically, we define $\tilde{\delta}g_{\mu\nu} = \delta g_{\mu\nu} + (\partial g_{\mu\nu}/\partial\Lambda)\tilde{\delta}\Lambda + \sum_m (\partial g_{\mu\nu}/\partial\alpha_m)\tilde{\delta}\alpha_m$, which naturally arises in analyzing $\tilde{\delta}\mathbf{L}(g_{\mu\nu}, \Lambda, \alpha_m)$. First, $\tilde{\delta}\mathbf{L}$ includes a term $(\partial\mathbf{L}/\partial g_{\mu\nu})\delta g_{\mu\nu}$ induced by varying $g_{\mu\nu}$ while fixing Λ and α_m's. Second, it includes the terms $(\partial\mathbf{L}/\partial g_{\mu\nu})(\partial g_{\mu\nu}/\partial\Lambda)\tilde{\delta}\Lambda + (\partial\mathbf{L}/\partial\Lambda)\tilde{\delta}\Lambda$ induced by varying Λ, either from the explicit dependence of $\mathbf{L}$ on Λ, or from an implicit dependence of $\mathbf{L}$ on Λ through $g_{\mu\nu}$. And $\tilde{\delta}\mathbf{L}$ also includes similar terms caused by varying $\tilde{\delta}\alpha_m$. Adding them together, one easily identifies the presence of $(\partial\mathbf{L}/\partial g_{\mu\nu})\tilde{\delta}g_{\mu\nu}$. Accordingly, the complete variation is

$$\begin{aligned}\tilde{\delta}\mathbf{L} &= \frac{\partial\mathbf{L}}{\partial g_{\mu\nu}}\tilde{\delta}g_{\mu\nu} + \frac{\partial\mathbf{L}}{\partial\Lambda}\tilde{\delta}\Lambda + \sum_m \frac{\partial\mathbf{L}}{\partial\alpha_m}\tilde{\delta}\alpha_m \\ &= \mathbf{E}^{\phi}\tilde{\delta}\phi + d\mathbf{\Theta}[\tilde{\delta}\phi] - \frac{\tilde{\delta}\Lambda}{8\pi}\boldsymbol{\epsilon} + \sum_m F_m\boldsymbol{\epsilon}\tilde{\delta}\alpha_m. \end{aligned}\tag{9}$$

Utilizing this expression of $\tilde{\delta}\mathbf{L}$, we reexamine the derivation of Eq. (4) as given in the Supplemental Material [21]. It leads to

$$d(\tilde{\delta}\mathbf{Q}_\xi - \xi\cdot\mathbf{\Theta}[\tilde{\delta}\phi]) = \left(\frac{\tilde{\delta}\Lambda}{8\pi} - \sum_m F_m \tilde{\delta}\alpha_m\right)\xi\cdot\boldsymbol{\epsilon}. \quad (10)$$

By applying the horizon Killing vector ξ_H and integrating it over $\mathcal{V}_r$, we obtain

$$\int_{S_r}\left(\tilde{\delta}\mathbf{Q}_{\xi_H} - \xi_H\cdot\mathbf{\Theta}[\tilde{\delta}\phi]\right) - \int_{S_h}\tilde{\delta}\mathbf{Q}_{\xi_H} = \frac{\tilde{\delta}\Lambda}{8\pi}\int_{\mathcal{V}_r}\xi_H\cdot\boldsymbol{\epsilon} - \sum_m \tilde{\delta}\alpha_m\int_{\mathcal{V}_r}F_m\xi_H\cdot\boldsymbol{\epsilon}. \quad (11)$$

This formula yields the extended first law of black hole thermodynamics, similar to how Eq. (6) produces the conventional first law. As mentioned, in realistic calculations all the terms dependent on r (including potentially divergent terms) precisely cancel out, producing the desired result.

Since we are mainly concerned about the AdS black holes, we also introduce an alternative method to regularize the divergences, which involves subtracting the contribution of the pure AdS as the regulator. For a pure AdS background, Eq. (11) still holds by omitting the horizon term. By subtracting it from Eq. (11), the divergent terms can be eliminated. Then taking the limit $r\to\infty$ removes the other r-dependent terms. This procedure defines the regularized integral $\int^{(\mathrm{reg})} \equiv \int^{(BH)} - \int^{(\mathrm{AdS})}$ and leads to the regularized version of Eq. (11) as

$$\int_{S_\infty}^{(\mathrm{reg})}\left(\tilde{\delta}\mathbf{Q}_{\xi_H} - \xi_H\cdot\mathbf{\Theta}[\tilde{\delta}\phi]\right) - T\tilde{\delta}S = V\tilde{\delta}P + \sum_m V_m\tilde{\delta}\alpha_m, \quad (12)$$

where $P\equiv -(\Lambda/8\pi)$, the horizon term $\int_{S_h}\tilde{\delta}\mathbf{Q}_{\xi_H}$ has been formally identified as $T\tilde{\delta}S$, and the geometric volume V and its generalizations V_m are defined respectively as

$$V \equiv -\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_H\cdot\boldsymbol{\epsilon}, \quad (13)$$

$$V_m \equiv -\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}F_m\xi_H\cdot\boldsymbol{\epsilon}. \quad (14)$$

Amazingly, Eq. (12) has almost been the form of the extended first law. We only need to evaluate the first term of Eq. (12). Notice that the background subtraction can also be applied in the analysis of conventional thermodynamics, which can be viewed as a special case of the extended formalism with $\tilde{\delta}P = 0$ and $\tilde{\delta}\alpha_m = 0$.

From our derivation, it is clear that the term $V\tilde{\delta}P$ in Eq. (12) comes from the explicit dependence of the Lagrangian on Λ. Meanwhile, during the evaluation of $\int_{S_\infty}^{(\mathrm{reg})}\left(\tilde{\delta}\mathbf{Q}_{\xi_H} - \xi_H\cdot\mathbf{\Theta}[\tilde{\delta}\phi]\right)$, additional terms proportional to $\tilde{\delta}P$, denoted by $\Delta V\tilde{\delta}P$, may also arise due to the dependence of the dynamical fields ϕ on Λ. Combining them together, we will get the thermodynamic volume $V^{\mathrm{th}} = V + \Delta V$, which is consistent with the long-lasting experience that the thermodynamic volume V^{th} is not necessarily equal to the geometric volume V. From this perspective, $V\tilde{\delta}P$ is just a normal term, while the additional terms are the truly interesting ones, as they explain the distinction between V^{th} and V. In a similar manner, the conjugate quantities $V_m^{\mathrm{th}} = V_m + \Delta V_m$ of α_m's can also be deduced.

As before, the Smarr relation can be derived from Eq. (7). This formula remains unchanged in the discussion of the extended black hole thermodynamics, since it is evaluated on a given metric and has nothing to do with the variations. However, a scaling argument combined with the extended first law is sufficient to deduce the Smarr relation. Therefore, we can make use of Eq. (7) for cross-checking.

AdS-Schwarzschild black hole.—Now we start to study some concrete examples, which are limited to four dimensions for simplicity. In Einstein gravity with a negative cosmological constant Λ, the Lagrangian is $L = (1/16\pi)(R - 2\Lambda)$. From the variation $\delta(\sqrt{-g}L) = \sqrt{-g}E^{\mu\nu}\delta g_{\mu\nu} + \sqrt{-g}\nabla_\mu\Theta^\mu$, one can read off

$$\Theta^\mu[\delta g_{\mu\nu}] \equiv \frac{1}{16\pi}(g^{\mu\alpha}\nabla^\nu\delta g_{\alpha\nu} - g^{\alpha\beta}\nabla^\mu\delta g_{\alpha\beta}), \quad (15)$$

and construct the Noether charge as

$$Q_\xi^{\mu\nu} \equiv -\frac{1}{16\pi}(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu). \quad (16)$$

Their differential forms are given by $\mathbf{\Theta}\equiv(1/3!)\Theta^\mu\epsilon_{\mu\nu\alpha\beta}dx^\nu\wedge dx^\alpha\wedge dx^\beta$, and $\mathbf{Q}_\xi = (1/2!2!)Q_\xi^{\mu\nu}\epsilon_{\mu\nu\alpha\beta}dx^\alpha\wedge dx^\beta$.

For the AdS-Schwarzschild black hole, the horizon Killing vector is $\xi_H = (\partial/\partial t)$, and the horizon radius is denoted by r_h. Substituting Eqs. (15) and (16) into the unregularized formula (11), we get

$$\tilde{\delta}M + \cancel{\frac{r^3}{6}\tilde{\delta}\Lambda} - T\tilde{\delta}S = \cancel{\frac{4\pi r^3}{3}\frac{\tilde{\delta}\Lambda}{8\pi}} - \frac{4\pi r_h^3}{3}\frac{\tilde{\delta}\Lambda}{8\pi}. \quad (17)$$

Alternatively, the regularized formula (12) also yields

$$\tilde{\delta}M = T\tilde{\delta}S + V\tilde{\delta}P, \quad (18)$$

with $V = (4\pi/3)r_h^3$. Both approaches produce the same final result, but the latter approach is more straightforward. So we will mainly employ the regularized formulas. Note that there are no additional terms proportional to $\tilde{\delta}P$ in this case; thus the geometrical volume V and the thermodynamic volume V^{th} coincide with each other.

The extended first law, together with a scaling argument, can lead to the Smarr relation. Here the scaling behavior is given by $\eta M(S,P)=M(\eta^2 S,\eta^{-2}P)$. Taking the derivative with respect to η, one gets $M=2(\partial M/\partial S)S-2(\partial M/\partial P)P$. By virtue of Eq. (18), it becomes

$$M=2TS-2PV. \tag{19}$$

Note this Smarr relation can also be derived from Eq. (7).

AdS-Kerr black hole.—The AdS-Kerr black hole is a nontrivial example that deserves detailed inspection. The metric is listed in the Supplemental Material [21], parametrized by the mass parameter m, rotational parameter a, and AdS radius l. In fact, there were some controversies even around the conventional first law of thermodynamics for the AdS-Kerr case; see [26,27] for details. Nevertheless, both the conventional first law and its extension can be naturally derived in the present formalism.

Here, the horizon Killing vector is $\xi_H=(\partial/\partial t)+\Omega_H(\partial/\partial\phi)$. The AdS-Kerr space-time has a nonvanishing angular velocity at infinity: $\Omega_\infty=-(a/l^2)$. Thus, we can make the decomposition $\xi_H=\xi_t+\Omega\xi_\phi$, where $\xi_t\equiv(\partial/\partial t)+\Omega_\infty(\partial/\partial\phi)$, $\xi_\phi\equiv(\partial/\partial\phi)$, and $\Omega\equiv\Omega_H-\Omega_\infty$. By the linearity property of the formalism, the two parts of ξ_H can be analyzed separately. First, we have $\int_{S_\infty}^{(\mathrm{reg})}(\delta\mathbf{Q}_{\xi_t}-\xi_t\cdot\mathbf{\Theta}[\delta\phi])=\delta M$, where $M=\{m/[1-(a^2/l^2)]^2\}$. Second, by noticing ξ_ϕ is tangential to the integrating surface, there is $\int_{S_\infty}^{(\mathrm{reg})}(\delta\mathbf{Q}_{\xi_\phi}-\xi_\phi\cdot\mathbf{\Theta}[\delta\phi])=\delta\int_{S_\infty}^{(\mathrm{reg})}\mathbf{Q}_{\xi_\phi}=-\delta J$, where $J=\{ma/[1-(a^2/l^2)]^2\}$. Therefore we get the conventional first law

$$\delta M-\Omega\delta J-T\delta S=0. \tag{20}$$

Notably, the angular velocity that appears in the black hole thermodynamics is Ω rather than Ω_H.

Next we derive the extended black hole thermodynamics from the formula (12). When evaluating $\int_{S_\infty}^{(\mathrm{reg})}(\tilde{\delta}\mathbf{Q}_{\xi_\phi}-\xi_\phi\cdot\mathbf{\Theta}[\tilde{\delta}\phi])$, because ξ_ϕ is tangential to the integrating surface, its second term vanishes. This leads to a total variation $\tilde{\delta}\int_{S_\infty}^{(\mathrm{reg})}\mathbf{Q}_{\xi_\phi}$, which is identified as $-\tilde{\delta}J$ by definition. Then we have to explicitly calculate $\int_{S_\infty}^{(\mathrm{reg})}(\tilde{\delta}\mathbf{Q}_{\xi_t}-\xi_t\cdot\mathbf{\Theta}[\tilde{\delta}\phi])$ associated with ξ_t, which gives

$$\frac{l^4}{(l^2-a^2)^2}\tilde{\delta}m+\frac{4al^4m}{(l^2-a^2)^3}\tilde{\delta}a-\frac{a^2lm(a^2+3l^2)}{(l^2-a^2)^3}\tilde{\delta}l. \tag{21}$$

Unlike the conventional case analyzed above, now the expression is nonintegrable. By the knowledge of calculus, if extracting a total variation $\tilde{\delta}M$ from the nonintegrable expression, there *inevitably* remains an additional term. Indeed, Eq. (21) turns out to be $\tilde{\delta}M-(4\pi/3)Ma^2\tilde{\delta}P$, where $P\equiv-(\Lambda/8\pi)=(3/8\pi)(1/l^2)$. Accordingly, we find the extended first law

$$\tilde{\delta}M=T\tilde{\delta}S+\Omega\tilde{\delta}J+\left(V+\frac{4\pi}{3}Ma^2\right)\tilde{\delta}P. \tag{22}$$

Thus, in the AdS-Kerr case, the thermodynamic volume $V^{\mathrm{th}}=V+(4\pi/3)Ma^2$ is not equal to the geometric volume V. This result (22) coincides with that given in [5], which was determined from thermodynamic relations. In stark contrast, in our formalism, the thermodynamic volume V^{th} and the first law are deduced even *without* resorting to the specific expressions of T, S, Ω, J, and V.

Taking the derivative of the scaling behavior $\eta M(S,J,P)=M(\eta^2 S,\eta^2 J,\eta^{-2}P)$ with respect to η, one gets $M=2(\partial M/\partial S)S+2(\partial M/\partial J)J-2(\partial M/\partial P)P$. Using the extended first law (22), one can read off the Smarr relation

$$M=2TS+2\Omega J-2\left(V+\frac{4\pi}{3}Ma^2\right)P. \tag{23}$$

Now let us rederive this Smarr relation from Eq. (7) for cross-checking. Applying $\xi_H=\xi_t+\Omega\xi_\phi$, the regularized version of Eq. (7) can be put into the form

$$2\int_{S_\infty}^{(\mathrm{reg})}\mathbf{Q}_{\xi_t}+2\Omega\int_{S_\infty}^{(\mathrm{reg})}\mathbf{Q}_{\xi_\phi}-2\int_{S_h}\mathbf{Q}_{\xi_H}=-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_H\cdot\mathbf{L}. \tag{24}$$

We have multiplied a factor of 2, because now the first term is just the standard Komar mass formula, which is regularized for the case of the AdS black hole. By explicit calculation, we find

$$M_K\equiv 2\int_{S_\infty}^{(\mathrm{reg})}\mathbf{Q}_{\xi_t}=M+\frac{8\pi}{3}Ma^2P, \tag{25}$$

and $-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_H\cdot\mathbf{L}=-2PV$. Substituting them into Eq. (24), it reduces to $[M+(8\pi/3)Ma^2P]-2\Omega J-2TS=-2PV$, which surely recovers the Smarr relation (23).

Now we can identify the presence of additional terms ΔV in the first law (22) and ΔM in the Smarr relation (23). Specifically, $\Delta M\equiv M-M_K$ is defined as the difference between the canonical energy and the Komar energy, while $\Delta V\equiv V^{\mathrm{th}}-V$ is defined as the difference between the thermal volume and the geometric volume. They play an essential role in the extended thermodynamics, i.e., unless ΔM and ΔV have been appropriately taken into account, a well-defined Smarr relation and extended first law cannot be achieved.

We notice that there have been some efforts to address the extended first law from some general formalisms [8,28–30]. But, as far as we know, there was no similar analysis for the AdS-Kerr case that accurately reproduces Eq. (22). As we stressed earlier, the additional terms are truly nontrivial and interesting. Therefore, to distinguish our approach from others, below we make further discussions about how these terms could emerge within our formalism.

Firstly, below Eq. (21), we have exemplified how the additional term ΔV arises from a reexamination about the integrability condition. Acting $\tilde{\delta}$ on a physical quantity such as M requires an extra derivative term $(\partial M/\partial\Lambda)\tilde{\delta}\Lambda$, compared to δM. One must take this into account in constructing the total variation $\tilde{\delta}M$. Even if the conventional expression $\int_{S_\infty}^{(\mathrm{reg})}(\delta\mathbf{Q}_{\xi_t}-\xi_t\cdot\mathbf{\Theta}[\delta\phi])$ can be identified as δM under the integrability condition, the extended expression $\int_{S_\infty}^{(\mathrm{reg})}(\tilde{\delta}\mathbf{Q}_{\xi_t}-\xi_t\cdot\mathbf{\Theta}[\tilde{\delta}\phi])$ may become nonintegrable; thus it cannot be naively identified as $\tilde{\delta}M$. This point is very crucial, yet it can be easily ignored.

Secondly, the additional terms ΔV and ΔM reflect the nontrivial asymptotic behaviors of the AdS-Kerr case. These additional terms are actually the terms that survive in the limit $r\to\infty$, when evaluating the corresponding formulas such as Eqs. (12) and (24), in addition to the regular terms. Moreover, our analysis reveals a close correlation between ΔM and ΔV, given by $\Delta M=-2P\Delta V$. Soon we will see that it can be viewed as a special case of Eq. (30) with $n=0$. Interestingly, although the situation $\Delta M\neq 0$ may appear problematic for someone attempting to calculate the canonical energy M from the Komar mass formula, the difference is essential in the context of the extended black hole thermodynamics.

Asymptotically-AdS black hole in higher curvature gravity.—The extended formalism is equally applicable to higher curvature theories of gravity, where the coupling parameters α_m are treated as thermodynamic variables. We introduce a new subscript $n=\{0,m\}$ and denote $P=-(\Lambda/8\pi)=\alpha_0$, $F_0=1$ as well as $V_0=V$, so that our results could be put into a unified form. We make a general analysis, and then study a concrete example.

The variation of the Lagrangian (8) results in modifications to the Einstein field equation, which could be rather complicated to solve. However, it is common practice to treat the couplings of the higher curvature terms as small quantities and solve for black hole solutions perturbatively and iteratively [25].

For simplicity, we restrict our analysis to the static and spherically symmetric black hole solutions, where the horizon Killing vector is simply $\xi_t=(\partial/\partial t)$. The expressions of Θ^μ and $Q^{\mu\nu}$ for the higher curvature theory can be obtained using standard techniques [23]. Subsequently, we evaluate the integral $\int_{S_\infty}^{(\mathrm{reg})}(\tilde{\delta}\mathbf{Q}_{\xi_t}-\xi_t\cdot\mathbf{\Theta}[\tilde{\delta}\phi])$ in Eq. (12). In general, we expect it reduces to the form $\tilde{\delta}M-\sum_n\Delta V_n\tilde{\delta}\alpha_n$, where $\Delta V_n\tilde{\delta}\alpha_n$ represents the additional term proportional to $\tilde{\delta}\alpha_n$. Accordingly, Eq. (12) simplifies to a general form of the extended first law

$$\tilde{\delta}M=T\tilde{\delta}S+\sum_n(V_n+\Delta V_n)\tilde{\delta}\alpha_n. \tag{26}$$

From the extended first law and the scaling behavior $\eta M(S,\ldots,\alpha_n)=M(\eta^2S,\ldots,\eta^{2(n-1)}a_n)$, we deduce the Smarr relation

$$M=2TS+2\sum_n(n-1)(V_n+\Delta V_n)\alpha_n. \tag{27}$$

Let us rederive the Smarr relation from Eq. (23) with $\Omega=0$. It reduces to $M_K-2TS=-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_t\cdot\boldsymbol{L}$. Because M_K is not necessarily equal to the canonical energy M, we represent it as $M_K=M-\Delta M$. Next, we handle with $-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_t\cdot\boldsymbol{L}=-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}d^3x\sqrt{-g}[(R/16\pi)+\sum_n\alpha_nF_n]$, which involves the sum of the on shell integrals of the Einstein–Hilbert term and the higher curvature terms. Interestingly, there exists a formula that establishes a relationship among such integrals [31]. For an asymptotically AdS black hole, the formula can be generalized as

$$\int\frac{R}{16\pi}=\sum_n(n-2)\int\alpha_nF_n, \tag{28}$$

where $\int$ is an abbreviation of $\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}d^3x\sqrt{-g}$ [32]. Thus we have $-2\int_{\mathcal{V}_\infty}^{(\mathrm{reg})}\xi_t\cdot\mathbf{L}=-2\sum_n(n-1)\int\alpha_nF_n=2\sum_n(n-1)\alpha_nV_n$. This leads to

$$M=2TS+\Delta M+2\sum_n(n-1)\alpha_nV_n. \tag{29}$$

Comparing with Eq. (27), we find there must be

$$\Delta M=2\sum_n(n-1)\alpha_n\Delta V_n. \tag{30}$$

Once again, there exists a close correlation between the additional terms ΔV_n and ΔM.

As an example, consider a model described by the Lagrangian

$$L=\frac{1}{16\pi}(R-2\Lambda)+\alpha_2R_{\mu\nu}R^{\mu\nu}+\alpha_3R^{\rho\sigma}_{\mu\nu}R^{\alpha\beta}_{\rho\sigma}R^{\mu\nu}_{\alpha\beta}. \tag{31}$$

We have solved the black hole solution perturbatively around the AdS-Schwarzschild metric, up to second order in the couplings, as given in the Supplemental Material [21]. Evaluating Eq. (12), we get the extended first law $\tilde{\delta}M=T\tilde{\delta}S+V\tilde{\delta}P+(V_2+\Delta V_2)\tilde{\delta}\alpha_2+(V_3+\Delta V_3)\tilde{\delta}\alpha_3$, where

$$\Delta V_2 = \frac{32}{3}\pi M\Lambda + \alpha_3 \frac{8192}{27}\pi^2 M\Lambda^3, \tag{32}$$

$$\Delta V_3 = \frac{32}{3}\pi M\Lambda^2 + \alpha_3 \frac{10240}{27}\pi^2 M\Lambda^4. \tag{33}$$

On the other hand, we obtain $\Delta M \equiv M - M_K = (64/3)\pi M\Lambda(\alpha_2 + 2\alpha_3\Lambda) + (8192/27)\pi^2\Lambda^3 M(2\alpha_2\alpha_3 + 5\alpha_3^2\Lambda)$. One can easily verify the equality $\Delta M = 2\alpha_2\Delta V_2 + 4\alpha_3\Delta V_3$, as alluded to in Eq. (30).

Notice that, when the equation of motion has been modified by the higher curvature terms, the pure AdS solution may possess an effective cosmological constant Λ_e that deviates from Λ. In view of some motivation, one may prefer to utilize Λ_e instead of Λ as the thermodynamic variable [16]. It is straightforward to change the variables by substituting the relation $\Lambda = \Lambda(\Lambda_e, \alpha_2, \alpha_3)$ into the extended first law at hand.

Concluding remarks.—In this Letter, we have fulfilled an elegant derivation of extended thermodynamics from the extended Iyer-Wald formalism, thereby establishing a robust foundation for the extended thermodynamics.

In extended black hole thermodynamics, an interesting observation is that the thermodynamic volume $V^{\rm th}$ may not necessarily be equal to the geometric volume V. This observation is naturally explained within our formalism. We have shown that $V\tilde{\delta}P$ comes from the explicit dependence of the Lagrangian on Λ, while the evaluation of $\int_{S_\infty}^{(\rm reg)} \left(\tilde{\delta}\mathbf{Q}_{\xi_t} - \xi_t \cdot \mathbf{\Theta}[\tilde{\delta}\phi]\right)$ yields additional terms $\Delta V\tilde{\delta}P$ by a careful reexamination of the integrability condition. This contributes a novel way to determine ΔV and $V^{\rm th}$, in contrast to those approaches relying on manipulating the thermodynamic relations. The argument applies equally well to the conjugate quantities $V_m^{\rm th} = V_m + \Delta V_m$ for other couplings α_m in the theory.

In addition, we uncover a connection between the additional terms ΔV_n and ΔM, as indicated in Eq. (30). As explained, these terms capture the nontrivial asymptotic behaviors of the bulk theory. Through dual thermodynamics, these terms and the relation among them may also be important on the CFT side, which is worthy of deep study in the future.

We have only analyzed a limited number of examples in this Letter. However, the formalism is flexible and can be readily applied to more complex scenarios. This includes gravitational theories coupled with diverse matter fields, as well as gravitational theories in higher dimensions, such as Lovelock gravity [17–20]. In the Supplemental Material [21], we present an analysis for Gauss-Bonnet gravity as a specific case of Lovelock gravity in $D = 5$ dimensions [33]. We have demonstrated that our extended formalism successfully derives the extended thermodynamics for this theory, which yields accurate expressions for all the thermodynamic quantities. This example is particularly valuable, as it highlights the effectiveness of our formalism in obtaining extended thermodynamics and its applicability at any order in the couplings of higher curvature theories.

Given that our formalism offers an efficient approach for calculating physical quantities in extended thermodynamics, it can potentially aid in analyzing the reverse isoperimetric inequality in different scenarios. The inequality and its refined version suggest a connection between the thermodynamic volume and black hole entropy [5,34]. It would be interesting to explore the inequality and address the intricacies beyond the domain of Einstein gravity.

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