Number Density Interpretation of Dihadron Fragmentation Functions

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We present a new quantum field-theoretic definition of fully unintegrated dihadron fragmentation functions (DiFFs) as well as a generalized version for n-hadron fragmentation functions. We demonstrate that this definition allows certain sum rules to be satisfied, making it consistent with a number density interpretation. Moreover, we show how our corresponding so-called extended DiFFs that enter existing phenomenological studies are number densities and also derive their evolution equations. Within this new framework, DiFFs extracted from experimental measurements will have a clear physical meaning.

Introduction.—High-energy collisions of hadrons are central to understanding their femtoscale structure at the level of quarks and gluons (partons) within the theory of quantum chromodynamics (QCD). The critical ingredients that encode this information are parton distribution functions (PDFs) and fragmentation functions (FFs). A crucial property of PDFs and FFs is their interpretation as number densities in a parton model framework [1,2], which consequently allows one to derive certain sum rules [1–5]. For example, the unpolarized transverse momentum dependent (TMD) PDF $f_1^{i/N}(x, \vec{k}_T^2)$ gives the number density in the momentum fraction x and transverse momentum $\vec{k}_T$ of a parton $i = q$ or g in a nucleon N [1,2]. Similarly, the unpolarized TMD FF $D_1^{h/i}(z, \vec{P}_\perp^2)$ gives the number density in the momentum fraction z and transverse momentum $\vec{P}_\perp$ of a hadron h fragmenting from a parton i [1,2]. Since PDFs and FFs are number densities, one can also use them to calculate expectation values (see, e.g., Refs. [6–9]). The information contained in sum rules and expectation values are important pieces to understanding hadronic structure as well as constraining or cross-checking phenomenological extractions and model calculations of PDFs and FFs.

The most common type of FFs describe the situation where a single hadron h is detected in the final state, $i \to hX$ (X representing all undetected particles). Another intensely studied class of reactions analyzes the case of two hadrons h_1 and h_2 being detected from the same

parton-initiated jet, $i \to (h_1 h_2)X$, where dihadron FFs (DiFFs) become relevant [10–50]. The quantum field-theoretic definition of DiFFs at the fully unintegrated level (what we will call uDiFFs) was first written down almost 25 years ago in the pioneering paper of Bianconi, Boffi, Jakob, and Radici (BBJR) [20]. This work has been the basis for all subsequent dihadron-related research for observables sensitive to the relative transverse momentum of the two hadrons [21–25,27,31,32,35–39,41–48]. Unfortunately, the BBJR definition does not allow the uDiFFs, nor the so-called extended DiFFs (extDiFFs) that are the focus of existing phenomenological analyses, to retain a number density interpretation in a parton model framework.

The main purpose of this Letter is to disseminate a new definition of uDiFFs that corrects this issue. We justify its number density interpretation by explicitly proving certain sum rules. We also show our corresponding extDiFFs are number densities and derive their evolution equations. Given the existing electron-positron annihilation dihadron cross section data [51], dihadron transverse single-spin asymmetries in electron-positron annihilation [52], semi-inclusive deep-inelastic scattering (SIDIS) [53,54], and proton-proton collisions [55,56], and anticipated measurements of the proton-proton dihadron cross section and SIDIS dihadron multiplicities, one eventually will be able to perform rigorous fits of extDiFFs within QCD global analyses. These studies must be carried out within our new framework for the extracted extDiFFs to have a clear physical meaning—see Refs. [57,58].

New correlator definition of DiFFs.—We begin by briefly discussing two different reference frames that will be relevant for our analysis: the "parton frame" (p), where the fragmenting parton has no transverse momentum, and the "dihadron frame" (h), where the dihadron has no

transverse momentum. In both frames the parton has the same large minus-light-cone momentum component k^- [$V^\pm \equiv (V^0 \pm V^3)/\sqrt{2}$ for a generic vector V]. They are connected through the following Lorentz transformation (see, e.g., Ref. [2], Sec. 12.4.1): $V_p^- = V_h^- \equiv V^-$; $V_p^+ = (\vec{k}_T/k^-)^2 V^-/2 + V_h^+ - \vec{k}_T \cdot \vec{V}_T/k^-$; $\vec{V}_\perp = -(\vec{k}_T/k^-)V^- + \vec{V}_T$. We use $\perp(T)$ to denote transverse components in the parton (dihadron) frame. The parton frame is more natural for the formulation of fragmentation correlators (whether single hadron or dihadron) as number densities, whereas the dihadron frame is more practical for proofs of factorization needed for phenomenological applications.

The quantum field-theoretic correlator for the fragmentation of a parton i into two hadrons h_1 and h_2, after integrating over k^+, is defined as [20]

$$\Delta_{\alpha\beta}^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$
$$= \frac{1}{N_i} \sum_X \int \frac{d\xi^+ d^2\vec{\xi}_\perp}{(2\pi)^3} e^{ik\cdot\xi} \mathcal{O}_{\alpha\beta}^{h_1 h_2/i}(\xi) \Big|_{\xi^-=0}, \qquad (1)$$

where z_1 and z_2 are the fractions of the parton's longitudinal momentum carried by each hadron and $\vec{P}_{1\perp}$ and $\vec{P}_{2\perp}$ are the transverse momenta of the hadrons relative to the parton. For a quark, N_i is the number of quark colors $N_c = 3$, and

$$\mathcal{O}_{\alpha\beta}^{h_1 h_2/q}(\xi) = \langle 0|\mathcal{W}(\infty, \xi)\psi_{q,\alpha}(\xi^+, 0^-, \vec{\xi}_\perp)|P_1, P_2; X\rangle$$
$$\times \langle P_1, P_2; X|\bar{\psi}_{q,\beta}(0^+, 0^-, \vec{0}_\perp)\mathcal{W}(0, \infty)|0\rangle, \quad (2)$$

where ψ_q is the quark field, α and β are indices for the components of the field, and $\mathcal{W}$ is a Wilson line in the fundamental representation of SU(3) that ensures color gauge invariance [2,59]. A sum over color indices in Eq. (2) is implied. For a gluon, $N_i = N_c^2 - 1$, and

$$\mathcal{O}_{\alpha\beta}^{h_1 h_2/g}(\xi) = \langle 0|\mathcal{W}^{ba}(\infty, \xi)F_{+\alpha}^a(\xi^+, 0^-, \vec{\xi}_\perp)|P_1, P_2; X\rangle$$
$$\times \langle P_1, P_2; X|F_{+\beta}^c(0^+, 0^-, \vec{0}_\perp)\mathcal{W}^{cb}(0, \infty)|0\rangle,$$
$$\qquad (3)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$ is the field strength tensor involving the gluon field A and the Wilson lines are now in the adjoint representation of SU(3).

Throughout this Letter, we focus on the production of unpolarized hadrons. For the fragmentation of an unpolarized parton, we parametrize the correlator in Eq. (1) as

$$\frac{1}{64\pi^3 z_1 z_2} \text{Tr}\left[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\right]$$
$$= D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}), \qquad (4)$$

$$\frac{z}{32\pi^3 z_1 z_2 P_h^-}\delta_\perp^{ij}\Delta^{h_1 h_2/g, ij}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$
$$= D_1^{h_1 h_2/g}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}), \qquad (5)$$

where $z = z_1 + z_2$ is the total momentum fraction of the dihadron and $P_h = P_1 + P_2$. As we will show in the next section, the prefactor of $1/(64\pi^3 z_1 z_2)$ in Eq. (4) is crucial to justifying the number density interpretation of the quark uDiFFs [and similarly for the gluon case in Eq. (5)]. If one instead were to use a prefactor of $1/(4z)$, to be in full analogy with single-hadron fragmentation [1,59–61], the quark uDiFFs would not retain a number density interpretation. Indeed, the fact that the prefactors on the lhs of Eqs. (4) and (5) are needed was already recognized previously in the context of collinear DiFFs $D_1^{h_1 h_2/i}(z_1, z_2)$ [28,29].

Number density interpretation.—To justify that Eqs. (4) and (5) have the desired number density interpretation, we will derive sum rules involving our uDiFFs in a parton model framework. The proofs of the sum rules in this section are left for Supplemental Material [62]. We focus first on the number sum rule:

$$\int d\mathcal{PS}\, D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \langle\mathcal{N}(\mathcal{N}-1)\rangle,$$
$$\qquad (6)$$

where $\int d\mathcal{PS} = \sum_{h_1}\sum_{h_2}\int_0^1 dz_2 \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}$ and $\mathcal{N}$ is the total number of hadrons produced when the parton i fragments. Thus, $\langle\mathcal{N}(\mathcal{N}-1)\rangle$ is the expectation value for the total number of hadron pairs produced in the fragmentation of i. A sum over hadron spins must be included if either or both hadrons have nonzero spin. We remark that the labeling of the two hadrons as (h_1, h_2) or (h_2, h_1) is distinguishable and no factor of $1/2$ is needed in the rhs of Eq. (6). We note that the number sum rule Eq. (6) was first derived in Ref. [29]. A crucial step in our proof is being able to introduce the number operator

$$\hat{N}_{h_j} \equiv \int \frac{dP_j^- d^2\vec{P}_{j\perp}}{(2\pi)^3 2P_j^-}\hat{a}_{h_j}^\dagger \hat{a}_{h_j} = \int \frac{dz_j d^2\vec{P}_{j\perp}}{(2\pi)^3 2z_j}\hat{a}_{h_j}^\dagger \hat{a}_{h_j} \quad (7)$$

for each hadron ($j = 1$ or 2). This can be achieved only by having the specific prefactors on the lhs of Eqs. (4) and (5). Indeed, a derivation is not possible if a prefactor of $1/(4z) = 1/[4(z_1 + z_2)]$ is used on the lhs of Eq. (4).

The result in Eq. (6) gives a clear interpretation for the uDiFFs we defined in Eqs. (4) and (5): They are densities in the momentum fractions z_1 and z_2 and transverse momenta $\vec{P}_{1\perp}$ and $\vec{P}_{2\perp}$ for the number of hadron pairs $(h_1 h_2)$ fragmenting from a parton i. The uDiFF $D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$ encodes the dihadron fragmentation process for an unpolarized quark (γ^- projection of the correlator). The number density interpretation

also holds for the fragmentation of a longitudinally polarized quark ($\gamma^-\gamma^5$ projection) and a transversely polarized quark ($i\sigma^{i-}\gamma_5$ projection). The explicit parametrization of Eq. (1) in terms of quark and gluon uDiFFs for all parton polarizations, as functions of $(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp} \cdot \vec{P}_{2\perp})$, is given in Supplemental Material [62].

We can also derive a momentum sum rule involving uDiFFs and TMD FFs:

$$\sum_{h_1} \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp} z_1 D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp} \cdot \vec{P}_{2\perp})$$
$$= (1-z_2) D_1^{h_2/i}(z_2, \vec{P}_{2\perp}^2). \quad (8)$$

If either or both hadrons have nonzero spin, then a sum over the spin of h_1 must be included on the lhs of Eq. (8) [and Eq. (9) below]. Note that one can identify the ratio of the uDiFF to the TMD FF, $D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp} \cdot \vec{P}_{2\perp})/D_1^{h_2/i}(z_2, \vec{P}_{2\perp}^2)$, as a conditional number density in the momentum $(z_1, \vec{P}_{1\perp})$ for h_1 fragmenting from i given h_2 has fragmented from i with momentum $(z_2, \vec{P}_{2\perp})$. Further integrating Eq. (8) over $\vec{P}_{2\perp}$ yields

$$\sum_{h_1} \int_0^{1-z_2} dz_1 \, z_1 D_1^{h_1 h_2/i}(z_1, z_2) = (1-z_2) D_1^{h_2/i}(z_2). \quad (9)$$

The momentum sum rule Eq. (9) was first put forth in Refs. [11,14]. We also mention that the study of DiFFs has a close connection to double PDFs (DPDFs), where two partons emerge from a single nucleon. Indeed, an analogous sum rule to Eq. (9) exists for DPDFs, as was derived in Refs. [63,64]. The quantum field-theoretic derivation of the sum rule Eq. (8) at the unintegrated (transverse-momentum-dependent) operator level [from which Eq. (9) follows immediately] is a new aspect presented here for the first time.

One can readily generalize to n-hadron ($n \geq 1$) fragmentation in a way that retains a number density interpretation:

$$\frac{1}{4(16\pi^3)^{n-1} z_1 \ldots z_n} \text{Tr}\left[\Delta^{\{h_i\}_n/q}(\{z_i\}_n, \{\vec{P}_{i\perp}\}_n)\gamma^-\right]$$
$$= D_1^{\{h_i\}_n/q}(\{z_i\}_n, \{\vec{P}_{i\perp}^2\}_n, \{\vec{P}_{i\perp} \cdot \vec{P}_{j\perp}\}_n), \quad (10)$$

$$\frac{z}{2P_h^-(16\pi^3)^{n-1} z_1 \ldots z_n} \delta_\perp^{ij} \Delta^{\{h_i\}_n/g,ij}(\{z_i\}_n, \{\vec{P}_{i\perp}\}_n)$$
$$= D_1^{\{h_i\}_n/g}(\{z_i\}_n, \{\vec{P}_{i\perp}^2\}_n, \{\vec{P}_{i\perp} \cdot \vec{P}_{j\perp}\}_n), \quad (11)$$

where $z = z_1 + \cdots + z_n$, $P_h = P_1 + \cdots + P_n$, $\{h_i\}_n \equiv h_1 \ldots h_n$, $\{z_i\}_n \equiv z_1, \ldots, z_n$, $\{\vec{P}_{i\perp}\}_n \equiv \vec{P}_{1\perp}, \ldots, \vec{P}_{n\perp}$, $\{\vec{P}_{i\perp}^2\}_n \equiv \vec{P}_{1\perp}^2, \ldots, \vec{P}_{n\perp}^2$, $\{\vec{P}_{i\perp} \cdot \vec{P}_{j\perp}\}_n \equiv \vec{P}_{1\perp} \cdot \vec{P}_{2\perp}, \ldots, \vec{P}_{1\perp} \cdot \vec{P}_{n\perp}, \vec{P}_{2\perp} \cdot \vec{P}_{3\perp}, \ldots, \vec{P}_{2\perp} \cdot \vec{P}_{n\perp}$, etc. The correlators $\Delta^{\{h_i\}_n/i}(\{z_i\}_n, \{\vec{P}_{i\perp}\}_n)$ are

the natural extensions of Eqs. (2) and (3) to n hadrons, i.e., the final state is now $|P_1, \ldots, P_n; X\rangle$. The corresponding number sum rule reads

$$\int d\mathcal{PS}_n \, D_1^{\{h_i\}_n/i}(\{\cdots\}_n) = \left\langle \prod_{k=0}^{n-1} (\mathcal{N} - k) \right\rangle, \quad (12)$$

where $\int d\mathcal{PS}_n$ denotes the n-hadron version of $\int d\mathcal{PS}$ and we have abbreviated the arguments of the FF. Interestingly, the evolution of collinear n-hadron FFs was already studied some time ago [12,13], as well as more recently in Refs. [49,50], but no correlator definition was presented.

Connection to phenomenology.—In order to analyze measurements of dihadron observables, it becomes convenient to change to the dihadron frame [20,23]. In addition to P_h, we also introduce the relative momentum $R = (P_1 - P_2)/2$. The individual hadrons have masses M_1 and M_2, while the invariant mass (squared) of the dihadron is $M_h^2 = P_h^2$. Along with z, we form the variable $\zeta = (z_1 - z_2)/z$. The hadron momenta P_1 and P_2 can then be written as $P_1 = [(M_1^2 + \vec{R}_T^2)/((1+\zeta)P_h^-), ((1+\zeta)/2)P_h^-, \vec{R}_T]$ and $P_2 = [(M_2^2 + \vec{R}_T^2)/((1-\zeta)P_h^-), ((1-\zeta)/2)P_h^-, -\vec{R}_T]$. Note that one readily finds $\vec{R}_T^2 = ((1-\zeta^2)/4)M_h^2 - ((1-\zeta)/2)M_1^2 - ((1+\zeta)/2)M_2^2$. Because of this change of reference frames, one naturally thinks of uDiFFs as now depending on $(z, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T)$ rather than $(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp} \cdot \vec{P}_{2\perp})$.

Nevertheless, the form of the number sum rule in Eq. (6) allows us to generalize the idea of uDiFFs as number densities to any set of variables we choose. Consider making a change of variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(w, x, \vec{Y}, \vec{Z})$, where we understand w and x to be scalars and $\vec{Y}$ and $\vec{Z}$ to be two-dimensional vectors. Then Eq. (6) implies

$$D_1^{h_1 h_2/i}(w, x, \vec{Y}^2, \vec{Z}^2, \vec{Y} \cdot \vec{Z})$$
$$\equiv \mathcal{J} \cdot D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp} \cdot \vec{P}_{2\perp}) \quad (13)$$

is a number density in $(w, x, \vec{Y}, \vec{Z})$, where $\mathcal{J} = |\partial(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})/\partial(w, x, \vec{Y}, \vec{Z})|$ is the Jacobian for the change of variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(w, x, \vec{Y}, \vec{Z})$. Substituting Eq. (4) or (5) into the rhs of Eq. (13) then gives an operator definition of $D_1^{h_1 h_2/i}(w, x, \vec{Y}^2, \vec{Z}^2, \vec{Y} \cdot \vec{Z})$. In addition, integrating over one or more of the variables $(w, x, \vec{Y}, \vec{Z})$ will define a DiFF that is a number density in the remaining variables.

For example, if we change variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(z, \zeta, \vec{k}_T, \vec{R}_T)$, as is typically done when deriving factorization theorems used in phenomenology, then $\mathcal{J} = z^3/2$. Thus,

$$D_1^{h_1 h_2/q}(z, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T)$$
$$= \frac{z}{32\pi^3(1-\zeta^2)} \text{Tr}[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-] \quad (14)$$

is a number density in $(z, \zeta, \vec{k}_T, \vec{R}_T)$ [where we made use of $z_1 z_2 = z^2(1-\zeta^2)/4$], and similarly for the gluon case. Note that the arguments of the correlator on the rhs can be replaced with $z_{1(2)} = z(1 \pm \zeta)/2$ and $\vec{P}_{1(2)\perp} = -z(1 \pm \zeta)\vec{k}_T/2 \pm \vec{R}_T$.

We emphasize the distinction between our prefactor of $z/[32\pi^3(1-\zeta^2)]$ in Eq. (14) and the prefactor of $1/(4z)$ used by BBJR. The latter does not allow for the uDiFFs to retain a number density interpretation. The explicit parametrization of Eq. (1) in terms of quark and gluon uDiFFs for all parton polarizations, as functions of $(z, \zeta, \vec{k}_T, \vec{R}_T)$, is given in Supplemental Material [62].

The functions of interest in experimental measurements are the extDiFFs, which we define by changing variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(z, \zeta, \vec{k}_T, \vec{R}_T)$ (as above) and integrating over $\vec{k}_T$. In the quark sector, two twist-2 Dirac projections survive [16,20]:

$$\frac{z}{32\pi^3(1-\zeta^2)} \int d^2\vec{k}_T \text{Tr}\Big[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\Big]$$
$$= D_1^{h_1 h_2/q}(z, \zeta, \vec{R}_T^2), \quad (15)$$

$$\frac{z}{32\pi^3(1-\zeta^2)} \int d^2\vec{k}_T \text{Tr}\Big[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})i\sigma^{i-}\gamma_5\Big]$$
$$= -\frac{\epsilon_T^{ij} R_T^j}{M_h} H_1^{\sphericalangle h_1 h_2/q}(z, \zeta, \vec{R}_T^2), \quad (16)$$

where $\epsilon_T^{ij} = \epsilon^{-+ij}$ with $\epsilon_T^{12} = 1$. One should understand the lhs of Eqs. (15) and (16) as giving an operator definition of the extDiFFs where the integration over $\vec{k}_T$ has been explicitly carried out on the correlator in Eq. (1). In this case, we consider these objects within full QCD. We note that if one instead changes variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(z_1, z_2, \vec{k}_T, \vec{R}_T)$, integrating over $\vec{k}_T$ and $\vec{R}_T$ leads to the collinear DiFF $D_1^{h_1 h_2/q}(z_1, z_2)$, and the associated correlator matches that in Refs. [28,29]. We emphasize the existence of $H_1^{\sphericalangle h_1 h_2/q}(z, \zeta, \vec{R}_T^2)$, which is not present for fragmentation into a single hadron. This function has become important in the extraction of the transversity PDFs, which couple to it in dihadron observables [32,35,36,39,41,45,48,57,58]. The gluon extDiFFs are given in Supplemental Material [62].

Experimental measurements of dihadron observables are usually differential in (z, M_h) and integrated over ζ. The relevant DiFFs are then dependent on (z, M_h) [20,22–25,27]. We change variables from $(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ to $(z, \zeta, \vec{k}_T, M_h, \phi_{R_T})$, where ϕ_{R_T} is the azimuthal angle of $\vec{R}_T$. The Jacobian is $\mathcal{J} = z^3(1-\zeta^2)/8$. Using our

aforementioned prescription, we can define a DiFF that is a number density in (z, M_h):

$$D_1^{h_1 h_2/i}(z, M_h) \equiv \frac{\pi}{2} M_h \int_{-1}^{1} d\zeta (1-\zeta^2) D_1^{h_1 h_2/i}(z, \zeta, \vec{R}_T^2). \quad (17)$$

For completeness, we also write down our definition of $H_1^{\sphericalangle h_1 h_2/i}(z, M_h)$:

$$H_1^{\sphericalangle h_1 h_2/i}(z, M_h)$$
$$\equiv \frac{\pi}{2} M_h \int_{-1}^{1} d\zeta \frac{|\vec{R}_T|}{M_h}(1-\zeta^2) H_1^{\sphericalangle h_1 h_2/i}(z, \zeta, \vec{R}_T^2). \quad (18)$$

Given the number density interpretation of our DiFFs, one can compute expectation values for the ensemble of all $(h_1 h_2)$ pairs in the fragmentation of a parton i. As mentioned, two of the main variables that dihadron measurements are sensitive to are z and M_h. The expectation value of an arbitrary function $\mathcal{O}(z, M_h)$ of these variables can then be calculated as

$$\langle \mathcal{O}(z, M_h) \rangle^{h_1 h_2/i} = \int dz dM_h \mathcal{O}(z, M_h) D_1^{h_1 h_2/i}(z, M_h). \quad (19)$$

For example, one could compute the average value of z or of M_h for $\pi^+\pi^-$ pairs produced from the fragmentation of an unpolarized quark. The DiFF $D_1^{h_1 h_2/i}(z, M_h)$ can be extracted directly from experiment, e.g., using the cross section $d\sigma/dz dM_h$ for $e^+ e^- \to (h_1 h_2)X$ measured by Belle [51]. Our definition of uDiFFs allows us to establish a clear physical meaning for $D_1^{h_1 h_2/i}(z, M_h)$, which has been absent thus far in the literature.

Actually, calculating the leading-order cross section for $d\sigma/dz dM_h$ for $e^+ e^- \to (h_1 h_2)X$ serves as another verification of the number density interpretation of our new definition of $D_1^{h_1 h_2/i}(z, M_h)$. Starting from $P_1^0 P_2^0 d\sigma/d^3\vec{P}_1 d^3\vec{P}_2$, the result takes the form

$$\frac{d\sigma}{dz dM_h} = \hat{\sigma}_0^i D_1^{h_1 h_2/i}(z, M_h). \quad (20)$$

For $i = q$, $\hat{\sigma}_0^q = 4\pi\alpha_{\text{em}}^2 N_c e_q^2/(3s)$, which is the partonic cross section for $e^+ e^- \to \gamma \to q\bar{q}$, where α_{em} is the fine structure constant and $\sqrt{s}$ is the center-of-mass energy of the $e^+ e^-$ pair. A sum over quarks and antiquarks is then needed on the rhs of Eq. (20). For $i = g$, $\hat{\sigma}_0^g = [(\alpha_s^2 G_F^2)/(576\pi^3)]\{[m_e^2 s^2(N_c^2-1)]/(s-m_H^2)^2\}$, which is the partonic cross section for $e^+ e^- \to H \to gg$ (H being the Higgs boson) using an effective $H - g - g$ coupling [65–67], α_s is the strong coupling, G_F is the Fermi constant, and m_e (m_H) is the mass of the electron (Higgs). A factor of 2 is now needed on the rhs of Eq. (20), since both gluons have the ability to fragment into the dihadron.
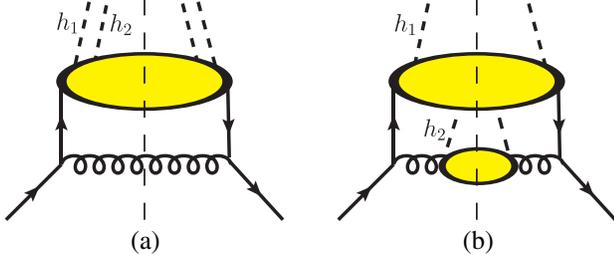
FIG. 1. Example diagrams of the (a) homogeneous and (b) inhomogeneous terms for the evolution of the extDiFF $D_1^{h_1 h_2/q}(z, \zeta, \vec{R}_T^2)$.

The structure of Eq. (20) is exactly what one expects if $D_1^{h_1 h_2/i}(z, M_h)$ is to be interpreted as a number density, i.e., the differential cross section equals the partonic cross section times the DiFF. We have also explicitly confirmed this feature for other sets of variables, including $d\sigma/dz_1 dz_2$ and $d\sigma/dz d\zeta d^2\vec{R}_T$ involving $D_1^{h_1 h_2/i}(z_1, z_2)$ and $D_1^{h_1 h_2/i}(z, \zeta, \vec{R}_T^2)$, respectively.

Evolution of extended DiFFs.—Since the extDiFFs in Eqs. (15) and (16) (and their gluon analogs) are the objects that enter most directly in existing phenomenological studies of dihadron observables, it is important to derive their evolution equations for our definition. Here, we analyze the $\mathcal{O}(\alpha_s)$ perturbative corrections to the dihadron fragmentation correlator, similar to what is done for the single-hadron case—see, e.g., Ref. [2], Sec. 12.10. The evolution of the DiFF correlator in Eq. (1) has two pieces: a "homogeneous term" involving only DiFFs [an example graph is given in Fig. 1(a)] and an "inhomogeneous term" involving single-hadron FFs [an example graph is given in Fig. 1(b)]. We have explicitly checked that the inhomogeneous term for the evolution of $D_1^{h_1 h_2/i}(z, \zeta, \vec{R}_T^2)$ is not ultraviolet divergent (see Supplemental Material [62]) and, therefore, does not contribute to the evolution of extDiFFs. The same conclusion was reached in Ref. [30]. However, this inhomogeneous term is needed to derive the full evolution for the collinear DiFFs $D_1^{h_1 h_2/i}(z_1, z_2)$ [12,13,15,26,28,29,49,50]. We also remark that, for extDiFFs, inhomogeneous diagrams will contribute at $\mathcal{O}(\alpha_s^2)$ and higher orders of evolution.

For collinear PDFs and FFs [e.g., $f_1^{i/N}(x)$ and $D_1^{h/i}(z)$], evolution is a perturbative process for the $1 \to 2$ splitting of a parton and is independent of the target (in the case of PDFs) or final state (in the case of FFs)—see, e.g., Ref. [2], Secs. 9.3.1 and 12.9. This observation, along with the structure of the correlator in Eq. (1), the fact that the extDiFFs are obtained by integrating over $\vec{k}_T$, and the conclusion that only the homogeneous term contributes to their evolution, makes clear that the splitting functions for extDiFFs will be the same as those for a parton fragmenting into a single hadron. The final result reads

$$\frac{\partial \mathcal{D}^{h_1 h_2/i}(z, \zeta, \vec{R}_T^2; \mu)}{\partial \ln \mu^2}$$
$$= \sum_{i'} \int_z^1 \frac{dw}{w} \mathcal{D}^{h_1 h_2/i'}\left(\frac{z}{w}, \zeta, \vec{R}_T^2; \mu\right) P_{i \to i'}(w), \quad (21)$$

where $\mathcal{D} = D_1$ or $H_1^{\sphericalangle}$ and $P_{i \to i'}(w)$ are the unpolarized timelike splitting kernels [68] when $\mathcal{D} = D_1$ or the transversely polarized splitting kernels [69] when $\mathcal{D} = H_1^{\sphericalangle}$. We note from Eqs. (17) and (18) it is clear that $D_1^{h_1 h_2/i}(z, M_h)$ and $H_1^{\sphericalangle h_1 h_2/i}(z, M_h)$ obey the same evolution equations as Eq. (21), since the ζ dependence there is not altered in the evolution.

Evolution equations for extDiFFs were previously derived in Ref. [30] (and commented on in Ref. [31]). No correlator was explicitly given, though, preventing us from unambiguously comparing to those results. We will instead emphasize that the prefactor used to parametrize the dihadron correlator in terms of DiFFs does affect the evolution kernel. For example, if on the lhs of Eq. (4) a prefactor of $1/(4z)$ is used [instead of $1/(64\pi^3 z_1 z_2)$], then the integrand of Eq. (21) would not have the $1/w$ factor. We also mention that the evolution of $D_1^{h_1 h_2/i}(z_1, z_2)$ was derived in Refs. [15,26,28,29] at leading order and recently in Refs. [49,50] at next-to-leading order. We briefly discuss in Supplemental Material [62] how, starting from extDiFFs, one can reproduce the leading-order result.

Conclusions.—We have introduced a new quantum field-theoretic definition for fully unintegrated dihadron fragmentation functions, as well as a generalized version for n-hadron fragmentation, that retains a number density interpretation. We have justified this by proving certain number and momentum sum rules. Moreover, we have developed a simple prescription for how to define operators for uDiFFs that are number densities in any variables of interest. In particular, we established a clear physical meaning for the function $D_1^{h_1 h_2/i}(z, M_h)$ as a number density in (z, M_h), which was not possible with prior definitions in the literature. The definitions in Eqs. (4) and (5) will also be beneficial as the starting point for possible factorization theorems beyond leading order for processes involving uDiFFs and extDiFFs. In addition, we derived the $\mathcal{O}(\alpha_s)$ evolution equations for our extDiFFs.

With DiFFs now rigorously established as number densities through this Letter, one can achieve a deeper understanding of hadronic structure through phenomenological extractions of extDiFFs. Indeed, there are electron-positron annihilation dihadron cross section data [51] available sensitive to $D_1^{h_1 h_2/i}(z, M_h)$, as well as several measurements of dihadron transverse single-spin asymmetries sensitive to $H_1^{\sphericalangle h_1 h_2/i}(z, M_h)$ in electron-positron annihilation [52], semi-inclusive deep-inelastic scattering [53,54], and proton-proton collisions [55,56]. A simultaneous global analysis

in our framework of all the aforementioned data can be found in Refs. [57,58].

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