Quantum Many-Body Scars in Dual-Unitary Circuits

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Dual-unitary circuits are a class of quantum systems for which exact calculations of various quantities are possible, even for circuits that are nonintegrable. The array of known exact results paints a compelling picture of dual-unitary circuits as rapidly thermalizing systems. However, in this Letter, we present a method to construct dual-unitary circuits for which some simple initial states fail to thermalize, despite the circuits being "maximally chaotic," ergodic, and mixing. This is achieved by embedding quantum many-body scars in a circuit of arbitrary size and local Hilbert space dimension. We support our analytic results with numerical simulations showing the stark contrast in the rate of entanglement growth from an initial scar state compared to nonscar initial states. Our results are well suited to an experimental test, due to the compatibility of the circuit layout with the native structure of current digital quantum simulators.

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Introduction.—Understanding how thermalization arises in closed quantum systems is a fundamental problem in many-body physics. Currently, our best understanding is based on the eigenstate thermalization hypothesis, which roughly states that thermalization occurs because the individual eigenstates of the unitary propagator appear thermal with respect to local observables [1–5]. In this framework, failure to thermalize is due to the presence of nonthermal eigenstates. *Strong ergodicity breaking* occurs in systems where a significant fraction of all the eigenstates are nonthermal, while *weak ergodicity breaking* arises if the fraction of nonthermal eigenstates is exponentially small in system size [6–9]. Strong ergodicity breaking is generally observed in systems with an extensive number of local conserved quantities, such as integrable [10] or many-body localized (MBL) systems [11,12], while weak ergodicity breaking is usually observed in systems with nonthermal eigenstates known as *quantum many-body scars* (QMBS) [7–9,13–17].

Because of the complex nature of interacting many-body dynamics, exact results are notoriously hard to come by. Significant progress has been made in recent years, by studying a special class of models, called *dual-unitary (DU) circuits*. These are a class of quantum circuits constructed as a brickwork pattern of two-qudit gates, which are unitary in both temporal and spatial directions. This special property enables the exact calculation of some system properties that would ordinarily be prohibitively hard to calculate [18–35].

For example, exact calculation of the spectral form factor has shown that interacting DU circuits are "maximally chaotic," in the sense that the spectral form factor agrees with the predictions of random matrix theory at all timescales [19,26]. Exact results also indicate that DU circuits are fast scramblers of quantum information, since two-time correlation functions and out-of-time correlators spread at their maximal possible velocities [20,21]. In a similar spirit, from a certain class of solvable initial states it has been shown that entanglement growth occurs at the maximal rate [23,25,32] and that any finite subsystem thermalizes to its maximally mixed (i.e., infinite temperature) reduced density matrix in a short finite time [25]. Moreover, the exact results on two-time correlation functions allow a rigorous classification of DU circuits in terms of their ergodic and mixing properties [20,29].

The above exact properties seem to indicate that generic DU circuits are rapidly thermalizing systems [36]. This is further supported by the observation that it is impossible to induce MBL through disorder in DU circuits [19,26], leaving integrability as the only known mechanism of strong ergodicity breaking in DU circuits. However, to the best of our knowledge, so far there has been no demonstration of weak ergodicity breaking in DU circuits nor any arguments against it as in the case of MBL.

In this Letter, we show that weak ergodicity breaking is indeed possible in DU circuits. Using a projector-embedding approach, initially proposed for continuous-time dynamics [6], we provide an explicit construction to insert QMBS in DU circuits of arbitrary system size and arbitrary local Hilbert space dimension. We demonstrate our construction with examples that embed a single, a few, or exponentially many QMBS in this class of circuits. Our construction shows that provably maximally chaotic, ergodic, and mixing systems can support QMBS. Despite the rapid scrambling properties of such systems, if the system is initialized in the QMBS subspace, then all quantum information remains localized in the subspace. Our analytical results are supported by numerical calculations for the entanglement growth, showing a striking difference in the dynamics between scar and nonscar initial states. Our work paves the way for future theoretical as well as experimental investigations of weak ergodicity breaking in quantum circuits, where dual-unitarity can be leveraged [37,38]. Because of the native structure of current digital quantum simulators, some of the proposed models can be adapted directly to current devices.

Dual-unitary circuits.—The basic building blocks of the circuits considered in this work are unitary operators $\hat{U}$, acting on two qudits of arbitrary local Hilbert space dimension d. The *dual* $\tilde{U}$ of a unitary is defined by a reordering of the subsystem indices $\langle k| \otimes \langle l|\tilde{U}|i\rangle \otimes |j\rangle = \langle j| \otimes \langle l|\hat{U}|i\rangle \otimes |k\rangle$ [20] and can be physically interpreted as an exchange of the spatial and temporal dimensions of $\hat{U}$. A gate $\hat{U}$ is DU if and only if both $\hat{U}$ and its dual $\tilde{U}$ are unitary. Any two-qudit gate of the form

$$\hat{U}^{\mathrm{DU},1} = (\hat{u}^+ \otimes \hat{u}^-)\hat{S}\,\hat{V}(\hat{v}^- \otimes \hat{v}^+) \tag{1}$$

is dual-unitary [35,39]. (See Supplemental Material [40], Sec. I, for further details.) Here, $\hat{S}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle$ is the SWAP operator, $\hat{u}^{\pm}$ and $\hat{v}^{\pm}$ are arbitrary single-qudit unitaries, and

$$\hat{V} = \exp\left\{ i \sum_{j=0}^{d-1} \hat{h}^{(j)} \otimes |j\rangle\langle j| \right\} \tag{2}$$

is an entangling gate with arbitrary single-qudit Hermitian operators $\hat{h}^{(j)}$. We rewrite the nonentangling unitaries as

$$\hat{u}^+ \otimes \hat{u}^- = \exp\{i(\hat{f}^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}^-)\}, \tag{3}$$

$$\hat{v}^- \otimes \hat{v}^+ = \exp\{i(\hat{g}^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}^+)\}, \tag{4}$$

in terms of the single-qudit Hermitian operators $\hat{f}^{\pm}$ and $\hat{g}^{\pm}$. With this parametrization, a dual-unitary gate is specified by $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$. Note that, for any $\hat{U}^{\mathrm{DU},1}$, the gate

$$\hat{U}^{\mathrm{DU},2} = \hat{S}\hat{U}^{\mathrm{DU},1}\hat{S} \tag{5}$$

is also DU. Generally, this expression is not expressible in the form of Eq. (1), thus giving a distinct parametrization.

A DU circuit is a quantum circuit in a "brickwork" geometry, in which all of the two-qudit gates are DU. We consider an even number N of qudits, where each qudit has a local Hilbert space $\mathbb{C}^d$. The qudit sites are labeled by $n = 0, 1, 2, \ldots, N-1$, and we impose periodic boundary conditions $n \equiv n + N$. The basic building blocks of the dynamics are local DU gates $\hat{U}_{n,n+1}$. A single time step is implemented by a Floquet unitary operator $\hat{\mathbb{U}} = \hat{\mathbb{U}}_o\hat{\mathbb{U}}_e$ which is a layer of DU gates across even-odd bonds $\hat{\mathbb{U}}_e = \otimes_{j=0}^{N/2-1} \hat{U}_{2j,2j+1}$ and a layer of DU gates across odd-even bonds $\hat{\mathbb{U}}_o = \otimes_{j=1}^{N/2} \hat{U}_{2j-1,2j}$. We set all gates in the even layer to be identical to each other and in the form of Eq. (1), $\hat{U}_{0,1} = \hat{U}_{2,3} = \cdots = \hat{U}^{\mathrm{DU},1}$, and all gates in the odd layer to be identical to each other and in the form of Eq. (5), i.e., $\hat{U}_{1,2} = \hat{U}_{3,4} = \cdots = \hat{U}^{\mathrm{DU},2} = \hat{S}\hat{U}^{\mathrm{DU},1}\hat{S}$. However, this two-site translation invariance is not essential for our results, which are also valid if the gates vary from site to site. Evolution for $t \in \mathbb{Z}$ time steps is generated by powers of the Floquet operator $\hat{\mathbb{U}}^t$.

Embedding QMBS in DU circuits.—In order to construct DU circuits with QMBS, we employ a projector embedding method, similar to the construction initially proposed by Shiraishi and Mori for continuous-time dynamics in Ref. [6]. The essential idea is to use projectors in the generators of the unitary gates in Eqs. (2)–(4) so that a chosen set of target states evolves by an elementary brickwork circuit of two-qudit swap gates, while states outside the target set evolve by more complicated dynamics.

Let $\hat{P}_{n,n+1}$ denote two-qudit projectors acting on neighboring sites n and $n+1$ and define the extended projectors acting on the total system of N qudits as $\hat{\mathbb{P}}_{n,n+1} \equiv \hat{\mathbb{I}}_{0,n-1} \otimes \hat{P}_{n,n+1} \otimes \hat{\mathbb{I}}_{n+2,N-1}$, where $\hat{\mathbb{I}}_{i,j}$ is the identity acting on all qudits in the range $i, i+1, \ldots, j-1, j$. The common kernel of all projectors is the set of states that are simultaneously annihilated by all projectors:

$$\mathcal{K} = \{|\psi\rangle : \hat{\mathbb{P}}_{n,n+1}|\psi\rangle = 0, \ \forall \ n\}. \tag{6}$$

Our target set of states $\mathcal{T}$, which we wish to embed in our circuit as QMBS, is the subset of $\mathcal{K}$ that is invariant under the action of even and odd layers of SWAP gates:

$$\mathcal{T} = \{|\psi\rangle : |\psi\rangle \in \mathcal{K}, \hat{\mathbb{S}}_e|\psi\rangle \in \mathcal{K}, \hat{\mathbb{S}}_o|\psi\rangle \in \mathcal{K}\}, \tag{7}$$

where $\hat{\mathbb{S}}_e = \otimes_{j=0}^{N/2-1} \hat{S}_{2j,2j+1}$ is the even layer of SWAP gates and $\hat{\mathbb{S}}_o = \otimes_{j=1}^{N/2} \hat{S}_{2j-1,2j}$ is the odd layer.

We now outline a construction to embed $\mathcal{T}$ as a set of scars in our DU circuit. To do this, we impose three conditions on the generators $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$ that define the DU gates in Eqs. (1)–(5) and on the projectors $\{\hat{P}_{n,n+1}\}$ that define the target subspace:

$$\hat{P}_{n,n+1}(\hat{f}_n^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}_{n+1}^-)\hat{P}_{n,n+1} = \hat{f}_n^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}_{n+1}^-, \tag{8}$$

$$\hat{P}_{n,n+1}(\hat{g}_n^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}_{n+1}^+)\hat{P}_{n,n+1} = \hat{g}_n^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}_{n+1}^+, \tag{9}$$

$$\hat{P}_{n,n+1}\left(\sum_{j=0}^{d-1}\hat{h}_n^{(j)}\otimes|j\rangle\langle j|_{n+1}\right)\hat{P}_{n,n+1}=\sum_{j=0}^{d-1}\hat{h}_n^{(j)}\otimes|j\rangle\langle j|_{n+1}. \tag{10}$$

Clearly, two-qudit gates satisfying conditions (8)–(10) are still of the DU form in Eq. (1). Conditions (8)–(10) ensure that the unitaries $\hat{u}_n^+\otimes\hat{u}_{n+1}^-$, $\hat{v}_n^+\otimes\hat{v}_{n+1}^-$, and $\hat{V}_{n,n+1}$ act trivially on all states in the kernel $\mathcal{K}$ of the projectors $\hat{P}_{n,n+1}$. However, the dynamics are not necessarily closed in the subspace $\mathcal{K}$. This is because the parametrization of Eq. (1) includes the SWAP operators $\hat{S}_{n,n+1}$ which can potentially take states out of $\mathcal{K}$. So, the QMBS subspace is the subspace $\mathcal{T}\subset\mathcal{K}$ that is invariant under even $\hat{\mathbb{S}}_e$ and odd $\hat{\mathbb{S}}_o$ layers of SWAP operators. For any initial state $|\psi\rangle\in\mathcal{T}$, the circuit will then act as $\hat{\mathbb{U}}|\psi\rangle=\hat{\mathbb{S}}|\psi\rangle$, where $\hat{\mathbb{S}}=\hat{\mathbb{S}}_o\hat{\mathbb{S}}_e$, corresponding to integrable dynamics in the target subspace $\mathcal{T}$, even if the overall circuit $\hat{\mathbb{U}}$ is nonintegrable and results in complicated dynamics for initial states that are not in $\mathcal{T}$. The action of $\hat{\mathbb{U}}$ in $\mathcal{T}$ can, thus, be seen as a permutation of the qudits, and many dynamical properties can, therefore, be studied exactly.

This prescription embeds QMBS in a DU circuit, which allows us to use all known exact results for these models. One of the key results for DU circuits is the rigorous classification in terms of their ergodic and mixing properties, based on the spectrum of a map $\mathcal{M}$ which determines the dynamical correlation functions [20,29]. We can show that our construction leads to QMBS in provably ergodic and mixing many-body systems, since all dynamical correlation functions decay to their infinite temperature values at long times. Further details are provided in Supplemental Material [40], Sec. II.

Example A: Single QMBS.—We first illustrate our construction with the simplest possible example, for which the set of projectors is

$$\hat{P}_{n,n+1}=\hat{\mathbb{I}}_{n,n+1}-|0\rangle\langle 0|_n\otimes|0\rangle\langle 0|_{n+1}, \tag{11}$$

for all $n=0,1,\ldots,N-1$. The common kernel $\mathcal{K}$ of all projectors consists of a single state $|0\rangle^{\otimes N}$, which is invariant under the action of any SWAP gates so that the target space is the single state $\mathcal{T}=\{|0\rangle^{\otimes N}\}$.

We choose $\{\hat{f}^{\pm},\hat{g}^{\pm},\hat{h}^{(j)}\}$ to be random Hermitian matrices, apart from some rows and columns $\langle i|\hat{f}^{\pm}|0\rangle=\langle i|\hat{g}^{\pm}|0\rangle=\langle i|\hat{h}^{(0)}|0\rangle=0$, $i\in\{0,1,\ldots,d-1\}$, that are set to zero to ensure that conditions (8)–(10) are satisfied (see Supplemental Material [40], Sec. III, for further details). This choice gives a generic DU gate which we expect to give a rapidly thermalizing circuit. Moreover, we can prove that for this example our circuit is ergodic and mixing [20,29] (see Supplemental Material [40], Sec. II). However, by our construction the target state $|0\rangle^{\otimes N}$ will be a nonthermal eigenstate of the circuit $\hat{\mathbb{U}}|0\rangle^{\otimes N}=|0\rangle^{\otimes N}$.

Although this construction embeds a QMBS for any system size N, we demonstrate our results with numerical simulations on finite size systems. In Fig. 1(a), we plot the half-system bipartite entanglement entropy $S(|\varphi_\alpha\rangle)=-\mathrm{Tr}[\hat{\rho}_\alpha\log(\hat{\rho}_\alpha)]$ of the eigenstates $|\varphi_\alpha\rangle$ of the Floquet unitary $\hat{\mathbb{U}}$. Here, $\hat{\rho}_\alpha=\mathrm{Tr}_{0,(N/2)-1}|\varphi_\alpha\rangle\langle\varphi_\alpha|$ is the reduced density matrix of the eigenstate obtained by tracing out the first $N/2$ qudits of the system. We see the single QMBS $|0\rangle^{\otimes N}$ with zero entanglement, separated from the rest of

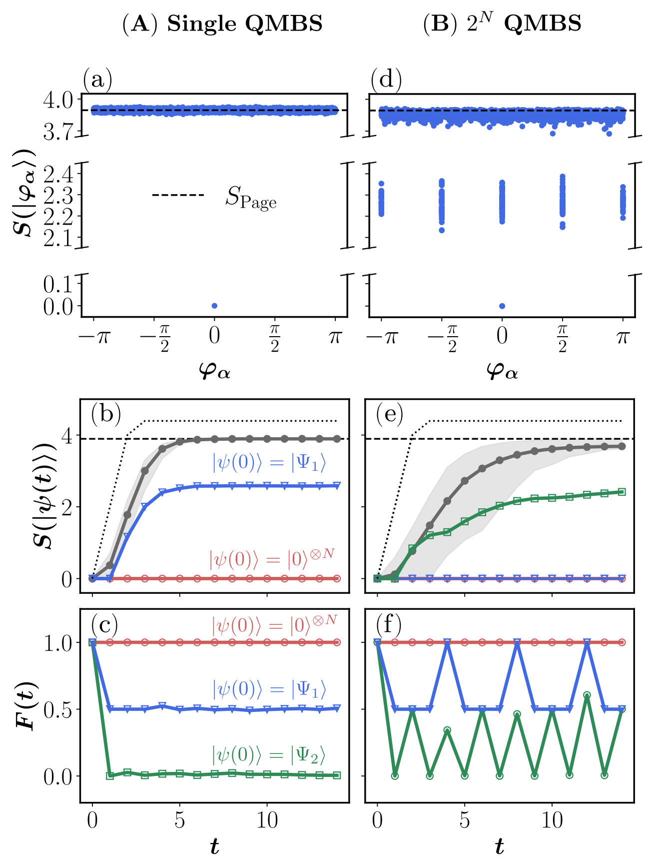

FIG. 1. Embedding a single QMBS (A) and an exponential number of QMBS (B) in a DU circuit. The top row (a),(d) shows the bipartite entanglement of eigenstates $|\varphi_\alpha\rangle$ of the Floquet unitary $\hat{\mathbb{U}}$ in each case. The second row (b),(e) shows the growth of bipartite entanglement, starting from different separable states. The gray line is the average entanglement growth for 100 random product states of the form $|\psi(0)\rangle=|i_0,i_1,\ldots,i_{N-1}\rangle$, $i_n\in\{0,1,\ldots,d-1\}$, while the surrounding gray region shows the 5–95 percentile range. The dashed black line in (a), (b), (d), and (e) is the Page entropy, $S_{\mathrm{Page}}=[N\ln(d)-1]/2$. The dotted line in (b) and (d) shows the maximum possible rate of entanglement growth, saturating at the maximum value $S_{\max}=N\ln(d)/2$. "Solvable" initial states have this maximal entanglement growth in the thermodynamic limit [23,25,43]. The bottom row (c),(f) shows the overlap with the initial state $F(t)=|\langle\psi(0)|\psi(t)\rangle|$. The blue line in (b), (c), (e), and (f) shows the results for $|\Psi_1\rangle=|0\rangle^{N-1}\otimes[(|0\rangle+|d-1\rangle)/\sqrt{2}]$, and the green line for $|\Psi_2\rangle=|0,0,d-1,d-1\rangle^{\otimes N/4-1}\otimes|0,0,d-1\rangle\otimes(|d-1\rangle+|1\rangle)/\sqrt{2}$. [Parameters: $N=8$ and $d=3$.]

the spectrum. All the other eigenstates are concentrated around the Page entropy $S_{\rm Page}$, which is the expected value of entanglement entropy for a random pure state [42]. Further evidence of the QMBS can be seen in the dynamics. In Fig. 1(b), we show that, for the initial QMBS state $|\psi(0)\rangle = |0\rangle^{\otimes N}$, the bipartite entanglement $S[|\psi(t)\rangle]$ does not grow in time and in Fig. 1(c) that the fidelity $F(t) = |\langle\psi(0)|\hat{\mathbb{U}}^t|\psi(0)\rangle|$ remains constant at $F = 1$. By contrast, a typical random initial product state of the form $|\psi(0)\rangle = |\vec{i}\rangle \equiv |i_0, i_1, \ldots, i_{N-1}\rangle$, with i_n chosen uniformly at random from the set $\{0, 1, \ldots, d-1\}$, will give rapid growth in entanglement that saturates at the Page value $S_{\rm Page}$ and will give a fidelity that decays rapidly to zero.

We further devise an initial state with intermediate slow dynamics. We consider the product state $|\psi(0)\rangle = |\Psi_1\rangle \equiv |0\rangle^{N-1} \otimes [(|0\rangle + |d-1\rangle)/\sqrt{2}]$. This initial state is partly overlapping with the QMBS $|0\rangle^{\otimes N}$ and is partly overlapping with the thermalizing state $|0\rangle^{\otimes(N-1)} \otimes |d-1\rangle$. Correspondingly, the entanglement entropy saturates to a value intermediate between zero and the thermal value, while the fidelity approaches $F \approx 1/2$ after $t \sim 1$ time step.

This construction can be extended to embed $k < d$ QMBS of the form $\{|i\rangle^{\otimes N}\}_{i=0}^{k-1}$ into a DU circuit with local Hilbert space dimension d. The main adjustment is, instead of Eq. (11), to use the set of projectors $\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} - \sum_{i=0,}^{k-1} |i,i\rangle\langle i,i|_{n,n+1}$. We provide numerical demonstrations for the $k = 2$ case in Supplemental Material [40], Sec. V, as well as a discussion on how the circuit can be modified to break the eigenphase degeneracy between the QMBS [40], Sec. IV. We also show that the QMBS in the Floquet unitary $\hat{\mathbb{U}}$ have counterpart *dual* QMBS in the dual Floquet unitary $\tilde{\mathbb{U}}$ [40], Sec. VI.

Example B: Exponentially many QMBS.—A more complex example is obtained by using the projectors:

$$\begin{aligned}\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} &- |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1} \\ &- |0\rangle\langle d-1|_n \otimes |d-1\rangle\langle 0|_{n+1} \\ &- |d-1\rangle\langle 0|_n \otimes |0\rangle\langle d-1|_{n+1} \\ &- |d-1\rangle\langle d-1|_n \otimes |d-1\rangle\langle d-1|_{n+1}, \qquad (12)\end{aligned}$$

for all n. We choose the DU gate generators $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$ to be random, apart from a few rows and columns that are set to zero to ensure that conditions (8)–(10) are satisfied $[\langle i|\hat{f}^{\pm}|j\rangle = \langle i|\hat{g}^{\pm}|j\rangle = \langle i|\hat{h}^{(j)}|j\rangle = 0,\ j \in \{0, d-1\},\ i \in \{0, 1, \ldots, d-1\}]$.

The common kernel $\mathcal{K}$ of the projectors in Eq. (12) is the 2^N-dimensional space spanned by the set of all N-qudit product states of the form $|\vec{i}\rangle = |i_0, i_1, \ldots, i_{N-1}\rangle$ with $i_n \in \{0, d-1\}$. We note that this space, though exponentially large in the system size N, is still an exponentially small fraction $(2/d)^N$ of the full Hilbert space when $d > 2$. Therefore, the fraction of nonthermal eigenstates constitutes a measure-zero set in the thermodynamic limit and, thus, leads to weak ergodicity breaking, consistent with the definition in Refs. [7,14]. It is also invariant under the action of $\hat{\mathbb{S}}_{e/o}$, so that our target space is $\mathcal{T} = \mathcal{K}$. However, this does not mean that the product states $|\vec{i}\rangle$, $i_n \in \{0, d-1\}$, are QMBS since, in general, $\hat{\mathbb{U}}|\vec{i}\rangle = \hat{\mathbb{S}}|\vec{i}\rangle \neq |\vec{i}\rangle$. Instead, the QMBS are found by diagonalizing the SWAP circuit $\hat{\mathbb{S}}$ in the subspace spanned by $\{|\vec{i}\rangle\}_{i_n \in \{0,d-1\}}$.

In Fig. 1(d), we plot the entanglement entropy of the eigenstates of $\hat{\mathbb{U}}$. The bulk of the eigenstates have an entropy close to the Page value $S_{\rm Page}$. However, the 2^N states in $\mathcal{T}$ have a lower entanglement. There are, in fact, four QMBS that have zero entanglement. These are $|0\rangle^{\otimes N}$, $|d-1\rangle^{\otimes N}$, $|0, d-1\rangle^{\otimes N/2}$, and $|d-1, 0\rangle^{\otimes N/2}$, which are eigenstates of the SWAP circuit $\hat{\mathbb{S}}$. The remaining $2^N - 4$ QMBS have nonzero entanglement but are still clearly separated from the bulk of the spectrum. In Supplemental Material [40], Sec. VII, we prove that the entanglement entropy of these QMBS can be bounded by $S \leq \log(N/2)$, a subvolume law scaling [44].

The presence of exponentially many QMBS can also be observed in the dynamics, as shown in the entanglement entropy growth and fidelity dynamics in Figs. 1(e) and 1(f). When the system is initialized in a typical random product state (gray in the plot), it rapidly thermalizes to a highly entangled state. Conversely, for an initial product state in the QMBS subspace $\mathcal{T}$, the entanglement growth is completely suppressed, since the SWAP circuit $\hat{\mathbb{S}}$ can map only product states to product states. For instance, the initial states $|\psi(0)\rangle = |0\rangle^{\otimes N}$ and $|\psi(0)\rangle = |\Psi_1\rangle \equiv |0\rangle^{N-1} \otimes (|0\rangle + |d-1\rangle)/\sqrt{2}$ [red and blue lines, respectively, in Fig. 1(e)] show zero entanglement growth, as they are both in the QMBS subspace $\mathcal{T}$.

However, the frozen entanglement from initial product states in $\mathcal{T}$ does not capture all the information about the dynamics in the QMBS subspace nor the variety of dynamical behaviors that are possible. For instance, evolution from $|\psi(0)\rangle = |\Psi_1\rangle = |0\rangle^{N-1} \otimes (|0\rangle + |d-1\rangle)/\sqrt{2}$ is characterized by an oscillating fidelity $F = |\langle\psi(0)|\psi(t)\rangle|$. This is due to the fact that $|\Psi_1\rangle$, although it is in the QMBS subspace, is a superposition of multiple QMBS. The $|0\rangle^{\otimes N}$ component of $|\Psi_1\rangle$ is a QMBS and does not decay, while the $|0\rangle^{\otimes N-1} \otimes |d-1\rangle$ component is a superposition of QMBS and leads to fidelity oscillations with a period $T = N/2$. Alternatively, by initializing in $|\Psi_2\rangle = |0, 0, d-1, d-1\rangle^{\otimes N/4-1} \otimes |0, 0, d-1\rangle \otimes (|d-1\rangle + |1\rangle)/\sqrt{2}$ (green in Fig. 1), the system undergoes more rapid oscillatory dynamics. This initial state is an equal superposition of a state $|0, 0, d-1, d-1\rangle^{\otimes N/4}$ which is in the QMBS subspace and one that is not $|0, 0, d-1, d-1\rangle^{\otimes N/4-1} \otimes |0, 0, d-1, 1\rangle$. While the former leads to revivals in fidelity with period $T = 2$, the latter contribution rapidly decays to zero.

In Supplemental Material [40], Sec. VIII, we report further numerical results, including the dynamics of local observables, illustrating the presence of QMBS.

Discussion.—Dual-unitary circuits are a paradigmatic model for investigations into many-body phenomena due to the abundance of available exact results. In this Letter, we have shown that these models can host QMBS, which lead to weak ergodicity breaking in a provably maximally chaotic system. We provide a systematic way to embed QMBS into DU circuits and highlight the contrast with the rest of the spectrum via numerical simulations.

The presented results motivate several fundamental questions with respect to QMBS, DU circuits, and chaotic quantum many-body systems, in general. Because of the fact that for $d > 2$ the used parametrization is not complete for DU gates, we expect that there are more DU circuit instances which can host QMBS. However, it is not certain whether the proposed embedding approach will work for DU circuits constructed with gates which lie outside the used form in Eq. (1) for $d > 2$. Furthermore, even with our parametrization, our embedding approach is probably not exhaustive. Further theoretical investigations are required to obtain a more complete picture of QMBS in DU circuits and their properties.

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