

Conformal Invariance and Multifractality at Anderson Transitions in Arbitrary Dimensions

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Multifractals arise in various systems across nature whose scaling behavior is characterized by a continuous spectrum of multifractal exponents Δ_q . In the context of Anderson transitions, the multifractality of critical wave functions is described by operators O_q with scaling dimensions Δ_q in a field-theory description of the transitions. The operators O_q satisfy the so-called Abelian fusion expressed as a simple operator product expansion. Assuming conformal invariance and Abelian fusion, we use the conformal bootstrap framework to derive a constraint that implies that the multifractal spectrum Δ_q (and its generalized form) must be quadratic in its arguments in any dimension $d \geq 2$.

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Multifractal (MF) measures with intricate scaling arise in such diverse subjects as dynamical chaos [1,2], weather and climate [3], turbulence [4–8], fractal growth [9–12], critical clusters in statistical mechanics [13–15], disordered magnets and other random critical points [16,17], Anderson transitions (ATs) [18–25], mathematical finance [26,27], random energy landscapes [28,29], Gaussian multiplicative chaos [30], and rigorous approaches to conformal field theory (CFT) [31,32].

A MF measure $\mu(\mathbf{r})$ is characterized by the scaling of its moments with the system size L : $\int d^d r \mu^q(\mathbf{r}) \sim L^{-\tau_q}$, with a continuum of exponents τ_q that depend *nonlinearly* on q . MF moments $\mu^q(\mathbf{r})$ can be represented by local operators $O_q(\mathbf{r})$ in a scale-invariant field theory, with scaling dimensions, also called the *MF spectrum*, $\Delta_q \equiv \tau_q - d(q-1) + q\Delta_1$ [33].

Similar to critical phenomena, one may expect the scale invariance to be enhanced to conformal invariance (though this is not guaranteed [34,35]), in which case, MF properties can be described by a CFT. Our main result is that in this situation, and under the assumption of *Abelian fusion* [see Eq. (4)] that is valid for ATs, in *any dimensionality* $d \geq 2$, the MF spectrum Δ_q *must be parabolic*; see Eq. (6) below. Our result is general and should apply to all MF measures that obey conformal invariance and Abelian fusion.

Our Letter is motivated by and of particular significance to the study of MF wave functions at ATs [18,36], where the parabolicity of Δ_q was predicted in a $d = 2$ CFT [37]. This prediction was tested analytically and numerically, and was found to be violated at two-dimensional (2D) ATs in various symmetry classes [38–44]. This has led to the understanding that conformal invariance might be lost at these critical points. Similarly, numerical studies of multifractality in $d = 3, 4, 5$ have found strong deviations from parabolicity [45–48], but there has not been any prediction

in $d > 2$ from a CFT perspective. Our Letter provides such a prediction.

Multifractals and field theory.—We first recall properties of MF spectra that follow from general principles. The function τ_q is nondecreasing and convex, which implies the existence of $q_* > 0$ such that $\Delta_{q_*} = 0$ [49]. Further constraints follow from studying *MF correlators* in a field theory via the relation [33,50]

$$\overline{\mu^{q_1}(\mathbf{r}_1) \dots \mu^{q_n}(\mathbf{r}_n)} \propto \langle O_{q_1}(\mathbf{r}_1) \dots O_{q_n}(\mathbf{r}_n) \rangle. \quad (1)$$

The overbar denotes spatial or disorder average, while the angular brackets denote a field-theory expectation value.

Of crucial importance is the additive, or *Abelian*, nature of the operator product expansion (OPE) of two MF operators O_{q_1} and O_{q_2} [33,50,51]:

$$O_{q_1}(\mathbf{r})O_{q_2}(0) \propto |\mathbf{r}|^{\Delta_{q_1+q_2}-\Delta_{q_1}-\Delta_{q_2}} O_{q_1+q_2}(0) + \dots, \quad (2)$$

where the ellipsis denotes subleading operators. As a consequence, a MF correlator $\langle \prod_i O_{q_i}(\mathbf{r}_i) \rangle$ scales as $L^{-\Delta_{q_1+q_2+\dots}}$ in the infrared. In the $L \rightarrow \infty$ limit, only *charge-neutral correlators* with $\Delta_{q_1+q_2+\dots} = 0$ can be studied by field-theory methods [41].

When conformal invariance is present, it fixes two-point functions: $\langle O_{q_1}(\mathbf{r})O_{q_2}(0) \rangle = \delta_{\Delta_{q_1}, \Delta_{q_2}} |\mathbf{r}|^{-2\Delta_{q_1}}$. This form is consistent with the OPE (2) if $\Delta_{q_1+q_2} = 0$ and $\Delta_{q_1} = \Delta_{q_2}$. Given the convexity of the MF spectrum, $\Delta_q \neq \Delta_{-q}$, and the only consistent choice is $q_2 = q_* - q$. Then we get the symmetry relation [49]

$$\Delta_q = \Delta_{q_*-q}. \quad (3)$$

More generally, only MF correlators with $\sum_i q_i = q_*$ are consistent with conformal invariance [41]. The relation (3)

is on more rigorous footing for ATs, where it follows from the Weyl symmetry (5) of the critical theory and does not rely on conformal invariance.

Multifractality at ATs.—ATs between metals and insulators, as well as between topologically distinct localized phases, are a major focal point in the study of disordered systems [18]. Critical properties at ATs are notoriously difficult to study because of the strongly coupled nature of the critical points.

A remarkable property of ATs is the multifractality of critical wave functions, or the local density of states $\nu(\mathbf{r})$ whose moments scale as $\overline{\nu^q(\mathbf{r})} \sim L^{-\Delta_q}$. There are more general combinations P_γ of critical wave functions [41–43,52] labeled by vectors $\gamma = (q_1, \dots, q_n)$ of complex numbers q_i , with scaling dimensions Δ_γ . MF properties at ATs are under better control than in general multifractals, since they can be rigorously established within the field theories of ATs, the nonlinear sigma models on cosets \mathcal{G}/\mathcal{K} of certain Lie supergroups [18,53–56]. In these models, P_γ are represented by gradientless composite operators O_γ [19–21,52]. A key fact is that O_γ can be constructed as highest-weight vectors under the action of the Lie superalgebra of \mathcal{G} with weights γ [52]. Then, the \mathcal{G} symmetry of the target space (assumed not broken at the critical point) implies Abelian fusion

$$O_{\gamma_1} \times O_{\gamma_2} \sim O_{\gamma_1+\gamma_2} + \dots, \quad (4)$$

where the ellipsis denotes now derivatives of $O_{\gamma_1+\gamma_2}$ and not general subleading operators as in Eq. (2).

The \mathcal{G} symmetry also leads to the *Weyl symmetry* of the MF spectra $\Delta_\gamma = \Delta_{w\gamma}$, $w \in W$ [52]. The Weyl group W acts in the space of weights γ and is generated by

$$q_i \rightarrow -c_i - q_i, \quad q_i \rightarrow q_j + (c_j - c_i)/2. \quad (5)$$

The coefficients c_i of the half-sum of the positive roots $\rho_b = \sum_{j=1}^n c_j e_j$ in a standard basis e_j are known for all families of symmetric superspaces [41–44,49,52,57]. The Weyl symmetry implies the existence of the operator $O_{-\rho_b}$ with vanishing scaling dimension $\Delta_{-\rho_b} = 0$. The corresponding neutrality condition for generalized MF correlators $\langle \prod_i O_{\gamma_i} \rangle$ is $\sum_i \gamma_i = -\rho_b$. The simple MF operators O_q and the spectrum Δ_q corresponds to $\gamma = (q, 0, 0, \dots, 0)$. In this case, $c_1 = -q_*$, and the Weyl symmetry reduces to Eq. (3).

The Weyl symmetry is fully supported by numerical and analytical results for various symmetry classes and dimensions $d \geq 2$ [38,39,41–45,57–64].

Multifractality and CFT in $d = 2$.—2D CFTs possess the infinite-dimensional Virasoro symmetry. In this setting, the ellipsis in Eq. (4) represents Virasoro descendants and leads to a single Virasoro block in a four-point function of MF operators, and a Vafa-Lewellen [65,66] constraint on the MF spectra. The unique solution of this constraint

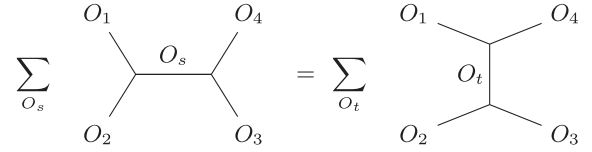


FIG. 1. A schematic representation of the s - t crossing equation.

subject to the symmetries (5) is the parabolic spectrum [37,41,67]

$$\Delta_\gamma = -b \sum_i q_i (q_i + c_i), \quad \Delta_q = bq(q_* - q), \quad (6)$$

where the parameter b cannot be determined from symmetry considerations alone. The second equation above is the simplification of the first for the simple MF spectrum Δ_q . In $d = 2$, the MF operators appear as vertex operators in a *Coulomb gas theory* (a Gaussian free field with a background charge) [37].

The central result of this Letter is that, once we assume conformal invariance and Abelian fusion, Eqs. (6) hold for MF spectra at ATs in any dimension $d \geq 2$.

The *conformal bootstrap* program [68] has brought the study of higher-dimensional CFTs into the limelight with extensive work on both analytical and numerical fronts. The bootstrap philosophy attempts to solve the *crossing symmetry* conditions coming from associativity of the OPE, with inputs from global symmetry and expected fusion rules for the operators. Crossing symmetry relates possible ways (or channels) of reducing a four-point function $\langle \prod_{i=1}^4 O_i(\mathbf{r}_i) \rangle$ to two-point functions by replacing pairs of operators with their OPEs (see Fig. 1). The s -channel fusion ($1 \rightarrow 2, 3 \rightarrow 4$) and the t -channel fusion ($1 \rightarrow 4, 2 \rightarrow 3$) result in two expansions of the four-point function and give the crossing equation $\sum_{O_s} \lambda_{12}^{O_s} \lambda_{34}^{O_s} W_{O_s} = \sum_{O_t} \lambda_{14}^{O_t} \lambda_{23}^{O_t} W_{O_t}$. The factors W_O are fully determined by conformal symmetry, while the CFT data $\{\Delta_i, \lambda_{ij}^k\}$ consisting of scaling dimensions and OPE coefficients are to be found. Solutions $\{\Delta_i, \lambda_{ij}^k\}$ fully define consistent CFTs. The s and t channels are obtained from each other by interchanges of indices of the operators (and their points of insertion): $s \leftrightarrow t \equiv 1 \leftrightarrow 3$. Accordingly, starting with a function $f^{(s)} \equiv f(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4)$ of four ordered arguments, we obtain, by permuting $1 \leftrightarrow 3$, another function $f^{(t)} \equiv f(\mathbf{r}_3, \mathbf{r}_2; \mathbf{r}_1, \mathbf{r}_4)$. Using this notation, we can write the four-point function as a product of a conformally covariant kinematic factor $\mathbb{K}_4^{(c)}$ and a G function

$$\left\langle \prod_{i=1}^4 O_i(\mathbf{r}_i) \right\rangle = \mathbb{K}_4^{(s)} G^{(s)} = \mathbb{K}_4^{(t)} G^{(t)}. \quad (7)$$

The G functions depend on the cross ratios

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2}, \quad v = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2}, \quad \text{where } r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|, \quad (8)$$

and operator labels. The cross ratios get transformed upon crossing so that $G^{(s)} = G_{12,34}(u, v)$, $G^{(t)} = G_{32,14}(v, u)$. In terms of the G functions, the s - t crossing equation is

$$G^{(s)} u^{-\frac{\Delta_1+\Delta_2}{2}} = G^{(t)} v^{-\frac{\Delta_2+\Delta_3}{2}}. \quad (9)$$

Much of the bootstrap formalism is geared toward solving Eq. (9) self-consistently for unitary CFTs. Since any putative CFT for MF correlators contains infinitely many relevant operators and, thus, is nonunitary, we resort to novel, unorthodox methods that focus on the G function and various physical inputs (similar to the ‘‘inverse bootstrap’’ method in [69]). We start by studying the Coulomb gas theories in the language of modern conformal bootstrap and use them as signposts to generalize the notion of Abelian fusion to higher dimensions. Then we show that the generalized Abelian fusion and crossing symmetry together yield a constraint on the spectrum of scaling dimensions in any d that is analogous to the Vafa-Lewellen constraints [65,66] known in CFT in $d = 2$. Finally, additional physical assumptions specific to MF observables allow us to solve the constraint, leading to a quadratic dependence of the MF spectrum Δ_γ on q_i in any dimension d .

Coulomb gas theories with global conformal blocks.—In $d = 2$, the Coulomb gas theories arise out of breaking the $U(1)$ symmetry of the free boson ϕ by including a background charge Q in the action [70,71]. A Coulomb gas CFT can be defined [71,72] in any dimension $d \in \mathbb{N}$ by considering an action with a possibly nonlocal kinetic term $\propto \phi(-\square)^{(d/2)}\phi$. Such CFTs also arise as limits of *generalized free fields*, where the scaling dimension of the field ϕ is tuned to $\Delta_\phi = 0$. In this limit, $\langle \phi\phi \rangle$ is logarithmic in any dimension which allows us to study vertex operators $V_\alpha \sim e^{d\alpha\phi}$. Following the conventions in [71,72], the scaling dimension of V_α is $\Delta_\alpha = d\alpha(Q - \alpha)$, and the multipoint functions satisfying the charge neutrality $\sum_i \alpha_i = Q$ are $\langle \prod V_{\alpha_i}(\mathbf{r}_i) \rangle = \prod_{i < j} r_{ij}^{-2d\alpha_i\alpha_j}$.

Next, we derive the OPE of vertex operators in terms of primaries of the global conformal group by studying the conformal block expansion. Consider a four-point function of vertex operators which can be written in the form (7) with the G function

$$G_{\text{CG}}^{(s)} = u^{\frac{1}{2}\Delta_{\alpha_1+\alpha_2}} v^{\frac{1}{2}(\Delta_{\alpha_2+\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_3})}. \quad (10)$$

This function is explicitly crossing symmetric [satisfies Eq. (9)] and has a convergent conformal block expansion [68] in the s channel in any dimension d ,

$$G_{\text{CG}}^{(s)} = \sum_O \lambda_{12}^O \lambda_{34}^O g_{\Delta_O, l_O}(u, v). \quad (11)$$

The conformal blocks g_{Δ_O, l_O} are often written as functions of (z, \bar{z}) related to the cross ratios (u, v) by

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (12)$$

In the s -channel limit, $r_{12} \approx r_{34} \ll r_{13} \approx r_{24} \approx r_{23} \approx r_{14}$, and thus, $u \rightarrow 0, v \rightarrow 1$; see Eq. (8). Then, $z, \bar{z} \rightarrow 0$, and the G function (10) has the form

$$G^{(s)} = (z\bar{z})^{\frac{\Delta^{(s)}}{2}} f(z, \bar{z}), \quad (13)$$

where $f(z, \bar{z})$ is a Taylor series symmetric in (z, \bar{z}) , and $\Delta^{(s)} = \Delta_{\alpha_1+\alpha_2}$. In the Supplemental Material [73] and Ref. [74], we use the leading behavior of the conformal blocks in the s channel [75,76] to show that any G function of the form (13) admits the conformal block expansion

$$G^{(s)} = \sum_{n, l \geq 0} \mu^{(n, l)} g_{\Delta^{(s)}+2n+l, l}(z, \bar{z}) \quad (14)$$

in arbitrary dimensions $d \geq 2$. Conversely, any G function that can be expanded as in Eq. (14) can also be written in the form of Eq. (13).

Let us denote global primaries as $[\tau, l]$ specifying their twist $\tau \equiv \Delta - l$ and spin l . Then we say that the expansion (14) contains just one twist family [69] consisting of the leading primary $[\Delta^{(s)}, 0]$ and subleading operators $[\Delta^{(s)} + 2n, l]$ which are constructed from its derivatives. The superscript of the product of the OPE coefficients $\mu^{(n, l)} \equiv \lambda_{12}^{(n, l)} \lambda_{34}^{(n, l)}$ identifies the operator $[\Delta^{(s)} + 2n, l]$ in the twist family.

Expanding the Coulomb gas G function (10) in global conformal blocks gives the OPE of $V_{\alpha_1} \times V_{\alpha_2}$ as [73]

$$[\Delta_{\alpha_1}, 0] \times [\Delta_{\alpha_2}, 0] \sim \sum_{n, l \geq 0} \lambda_{12}^{(n, l)} [\Delta_{\alpha_1+\alpha_2} + 2n, l], \quad (15)$$

where n, l are non-negative integers, and the $(n, l) = (0, 1)$ term is absent in the OPE. For two identical operators ($\alpha_1 = \alpha_2$), their OPE is completely specified by the conformal block expansion since we can extract the squared OPE coefficients (see Ref. [73] for explicit expressions in the $d = 2$ and $d = 4$ cases).

Generalized Abelian fusion.—In the strict sense, Abelian fusion (4) cannot hold in CFTs in $d > 2$, since an OPE written with finitely many *global* conformal primaries cannot satisfy crossing [77–80]. Thus, we need to generalize the notion of Abelian fusion to $d > 2$. Global conformal block expansions of Coulomb gas correlators exhibit certain features that we adopt as the *definition* of Abelian fusion in $d > 2$: (1) All primary MF operators can be grouped into twist families, and (2) the OPE of any two leading MF primaries contains only one twist family:

$$[\Delta_1, 0] \times [\Delta_2, 0] \sim \sum_{n, l \geq 0} \lambda_{12}^{(n, l)} [\Delta + 2n, l]. \quad (16)$$

The generalized Abelian fusion (16) and the related conformal block expansion (14) constrain a general scalar four-point G function to have the form (13). Since $z\bar{z} = u$

and $z + \bar{z} = u + 1 - v$ form a basis in the ring of symmetric functions of z and \bar{z} , Eq. (13) can also be written as

$$G^{(s)} = u^{\frac{\Delta^{(s)}}{2}} \sum_{n \geq 0} f_n^{(s)}(v) u^n. \quad (17)$$

The functions f_n are arbitrary so far, and quantities with the superscript (s) depend on the external dimensions in a channel-covariant manner. At this point, one can make a further simplification by assuming that the functions $f_r(v)$ can be represented as (possibly infinite) sums of power laws in v , i.e.,

$$G^{(s)} = u^{\frac{\Delta^{(s)}}{2}} \sum_{n,m \geq 0} C_{nm} v^{\sigma_m^{(s)}} u^n, \quad (18)$$

where σ_m are unrelated real numbers, and the coefficients C_{nm} do not depend on the cross ratios (u, v) or the external dimensions. The Coulomb gas theories are of this form with a single term; see Eq. (10). The generalized free field correlators [79] with $\Delta_\phi \geq 0$ are similarly composed of sums of power laws in u and v , although they do not satisfy Abelian fusion (integer gaps in powers of u). The above ansatz appeared for the case of a correlator of identical operators in Ref. [69], which discussed the idea of building crossing-symmetric G functions. Similarly, the authors of Ref. [80] use a version where the coefficients C_{nm} are functions of $\log u, \log v$ (the logarithms come from anomalous dimensions of the subleading operators in the twist family). As our definition of Abelian fusion, Eq. (16), exactly fixes the dimensions of all subleading operators, the logarithms are unnecessary in our treatment.

Constraints on the G function from crossing.—Substituting the Abelian G function (18) into Eq. (9), we obtain an equation that enforces a structure on the G function understandable in terms of crossing-symmetric building blocks [69], such that any truncation up to (N, M) of the double sum in Eq. (18) is also crossing symmetric; see the Supplemental Material [73] for details. The result is the G function

$$G^{(s)}(u, v) = u^{\frac{\Delta^{(s)}}{2}} v^{\frac{\Delta^{(t)}}{2} - \frac{\Delta_2 + \Delta_3}{2}} \left(\sum_{k \geq 0} S_k (uv)^k + \sum_{j \geq 1, k \geq 0} D_{jk} (uv)^k (u^j + v^j) \right), \quad (19)$$

where we use S_k and D_{jk} to represent the coefficients of the crossing-symmetric terms and pairs, respectively. Thus, we adopt Eq. (19) as the generic form of the Abelian G function (18) that also satisfies s - t crossing.

Excluding the spin-1 operator.—Focusing on the last part of the puzzle, we expand the G function (19) in conformal blocks in arbitrary dimensions to first few orders in z and \bar{z} . By construction, the first block that appears in

the expansion is $[\Delta^{(s)}, 0]$. The product of the OPE coefficients of the leading block is read off as $\mu^{(0,0)} \equiv S_0$. The coefficient $\mu^{(0,1)}$ of the spin-1 block $[\Delta^{(s)}, 1]$ can be obtained by matching the coefficients of the series for the order $\sim (z\bar{z})^{\Delta^{(s)}/2} (z + \bar{z})$ as

$$\frac{\mu^{(0,1)}}{S_0} = \frac{\Delta_2 + \Delta_3 - \Delta^{(t)}}{2} (1 + \mathcal{P}) - \mathcal{Q} - \frac{(\Delta^{(s)} - \Delta_1 + \Delta_2)(\Delta^{(s)} + \Delta_3 - \Delta_4)}{4\Delta^{(s)}}, \quad (20)$$

where the sums $\mathcal{P} \equiv \sum_{j \geq 1} D_{j0}/S_0$, $\mathcal{Q} \equiv \sum_{j \geq 1} j D_{j0}/S_0$ must converge for the OPE coefficient to be well defined.

This spin-1 operator cannot appear in any OPE of two identical scalar operators on general grounds. Indeed, $O(x_1)O(x_2)$ is even with respect to the interchange $x_1 \leftrightarrow x_2$, but a spin-1 operator must appear in the OPE as $\sim (x_1 - x_2) \times \partial O_{\Delta^{(s)}}((x_1 + x_2)/2)$ which is odd. Exploiting this fact, we set $O_2 \equiv O_1$ in which case $\mu^{(0,1)} = 0$, and Eq. (20) becomes a constraint on Δ 's:

$$\Delta^{(s)} + \Delta_3 - \Delta_4 + 4\mathcal{Q} = 2(\Delta_1 + \Delta_3 - \Delta^{(t)})(1 + \mathcal{P}). \quad (21)$$

In the context of MF correlators at ATs, we identify $\Delta^{(s)} \equiv \Delta_{\gamma_1 + \gamma_2}$, $\Delta^{(t)} \equiv \Delta_{\gamma_3 + \gamma_2}$. The neutrality condition $\sum_i \gamma_i = -\rho_b$ fixes $\Delta_4 = \Delta_{-\rho_b - \gamma_1 - \gamma_2 - \gamma_3} = \Delta_{\gamma_1 + \gamma_2 + \gamma_3}$. Now the continuity of MF spectra allows us to choose $\gamma_1 = \gamma_2 = \epsilon e_i$, where $e_i = (0, \dots, 1, \dots, 0)$ (unit in the i th place), with $\epsilon \ll 1$, and $\gamma_3 = \gamma$ in Eq. (21). Then we can expand in orders of ϵ [73], which gives $\mathcal{Q} = \mathcal{P} = 0$, and our main result:

The only MF spectrum Δ_γ , which satisfies generalized Abelian fusion and crossing symmetry, has the form given in Eq. (6).

Going back to Eq. (21), we substitute $\mathcal{P} = \mathcal{Q} = 0$, and the quadratic solution for Δ_γ to find that the constraint

$$2\Delta_{\gamma_1} + \Delta_{\gamma_3} - 2\Delta_{\gamma_1 + \gamma_3} - \Delta_{2\gamma_1} + \Delta_{2\gamma_1 + \gamma_3} = 0 \quad (22)$$

correctly picks out Abelian CFTs in $d \geq 2$, and thus is the appropriate generalization of the 2D Vafa-Lewellen constraint with a single exchanged Virasoro primary.

Summary and outlook.—Using conformal invariance, we have shown that any Abelian CFT in $d > 2$ must be intimately related to the Coulomb gas theory and have a quadratic spectrum. Our main assumptions, fundamentally related to each other, were the Abelian fusion (16) and the form (18) for the G function. As in the case of weakly perturbed CFTs [80], it remains to be seen if the generalized Abelian CFT defined here could be perturbed so that the derivative operators gain anomalous dimensions.

Let us discuss the implication of conformal invariance. As we have summarized earlier, perturbative analytical results in $d = 2 + \epsilon$ and numerical simulations in $d = 3, 4, 5$ have shown that the MF spectra for generic ATs are in fact, *not* parabolic [38–48]. In light of our result, it follows that conformal invariance is likely lost at ATs. The alternative scenario advocated in Refs. [67,81] is that the symmetries of the sigma models that were used to derive Abelian fusion and Weyl symmetry are spontaneously broken at the critical point. We believe this alternative to be unlikely, since it contradicts the vast body of literature on ATs, including the aforementioned numerical confirmations of the Weyl symmetry [38,39,41–45,57–64]. Thus, we propose ATs as examples of systems where scale invariance *does not* imply conformal invariance.

Perturbative MF spectra at random critical points [16,17] are also nonparabolic, suggesting lack of conformal invariance. Moreover, the authors of Ref. [82] argued that conformal invariance generically breaks down at strongly random fixed points. On the other hand, most systems where the MF spectrum is known to be parabolic are also conformally invariant [83]. These include 2D Dirac fermions in random gauge potentials [86–92], a recent proposal for the critical-point theory of the integer quantum Hall transition [37,67,93], Coulomb gas and Liouville CFTs in arbitrary dimensions [71,72], and rigorous probabilistic studies of 2D quantum gravity and Liouville CFT [30–32]. All of these results support the picture where parabolicity of MF spectra and conformal invariance go hand in hand, and that both are absent at critical points in random and disordered systems [94].

A natural extension of our Letter is to consider implications of conformal invariance for multifractality near boundaries of finite systems [38,39,60,62,95–99] using crossing symmetry and conformal bootstrap in a boundary CFT.

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