Exact Large-Scale Fluctuations of the Phase Field in the Sine-Gordon Model

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We present the first exact theory and analytical formulas for the large-scale phase fluctuations in the sine-Gordon model, valid in all regimes of the field theory, for arbitrary temperatures and interaction strengths. Our result is based on the ballistic fluctuation theory combined with generalized hydrodynamics, and can be seen as an exact "dressing" of the phenomenological soliton-gas picture first introduced by Sachdev and Young [Phys. Rev. Lett. **78**, 2220 (1997)], to the modes of generalized hydrodynamics. The resulting physics of phase fluctuations in the sine-Gordon model is qualitatively different, as the stable quasiparticles of integrability give coherent ballistic propagation instead of diffusive spreading. We provide extensive numerical checks of our analytical predictions within the classical regime of the field theory by using Monte Carlo methods. We discuss how our results are of ready applicability to experiments on tunnel-coupled quasicondensates.

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Introduction.—Understanding correlations and fluctuations in quantum and classical interacting many-body systems is a crucial problem of theoretical physics. Needless to say, in strongly interacting models this is a daunting task, too complicated to be carried out in the most general setting. However, at large scales *universality* emerges [1–5]: microscopic details are unimportant and information is carried only by slowly decaying modes, coupled to the local conservation laws of the underlying Hamiltonian. This is the hallmark of hydrodynamics.

The most significant correlations of a given observable are due to its coupling with hydrodynamic modes (sound, heat, etc.) associated with conservation laws [6]. Along the velocities of such modes, power-law behavior is observed instead of exponential decay. But some observables do not couple to hydrodynamic modes, such as those sensitive to topological excitations, because of the intrinsically nonlocal nature of the latter. For instance, correlation functions of order parameters often show exponential decay throughout space-time. Is there a general theory for understanding such behavior? How do hydrodynamic modes interact with order parameters and what information can be extracted from their correlation functions? Even with the mathematical tools of integrability, computing correlations of order parameters from a microscopic analysis is challenging in noninteracting cases [7–17] and unpractical in the presence of interactions [18–21]. This calls for a more universal hydrodynamic approach. The relation between order parameters and hydrodynamic modes was recently addressed [22,23] in the XX quantum chain using free-fermionic techniques; and a general, but phenomenological, picture for the influence of topological excitations on correlations was proposed by Sachdev and Young (SY) [24]. However, to our knowledge, there are no results beyond free excitations or extremely dilute gases.

A paradigmatic model where these questions are of central relevance is the sine-Gordon model

$$H = \int \mathrm{d}x \left[\frac{c^2 g^2}{2} \Pi^2 + \frac{1}{2g^2} (\partial_x \phi)^2 - \frac{c^2 m^2}{g^2} \cos(\phi) \right], \qquad (1)$$

that manifests itself in the most diverse contexts [3,25–29]. Above, the field $\Pi(x)$ is conjugated to the phase $\phi(x)$, g tunes the interactions, c is the "velocity of light," and m is a mass scale. In the following, we measure lengths in units of $[mc]^{-1}$. The low-energy sector of many systems is well described by the sine-Gordon model, as perturbations can induce Berezinskii-Kosterlitz-Thouless transitions in the ubiquitous Luttinger liquid [3] ($m = 0$) field theory, but it has many applications in high-energy physics as well [30]. Notably, the sine-Gordon model is a paradigmatic example of *integrable field theory* [31], hence it is amenable to nonperturbative analysis, and shows peculiar thermalization [32] and transport [33,34] properties. In this model, fluctuations create topological excitations of the phase field ϕ: phase-slips of 2π interpolating between the degenerate ground states $\phi \in 2\pi\mathbb{Z}$. The nonlocality of these excitations with respect to the model's order parameter ϕ places it in the general category we outlined, leading us to the central question of this Letter: can we build a general framework able to capture the large scale fluctuations of the phase field?.

In addition to being a long-standing unsolved problem of mathematical physics, this question is of central experimental relevance. Correlation functions of vertex operators $e^{i\lambda\phi}$ capture response functions at low energy in certain materials [35–38] and in multispecies cold atomic gases [3], and order-parameter correlations functions in spin chains [29,39]. Moreover, recent experimental advances probe fluctuations of phase differences $\Delta\phi(t,x) = \phi(t,x) - \phi(0,0)$ themselves. This is possible in tabletop quantum simulators of the sine-Gordon model realized by tunnel-coupled condensates [40,41]. Matter-wave interferometry gives access to projective measurements of phase differences in both equilibrium [42,43] and nonequilibrium settings [44]: any analytical insight would be of utmost interest not only from a theoretical point of view, but also in very concrete experiments. In this Letter, we solve this problem.

We show that the probability distribution of the phase differences, $P\{[\Delta\phi(t,x)/2\pi] = \delta\}$, obeys a large deviation principle $P[\delta] \asymp e^{-\ell I_\alpha(\delta)}$, where $\ell = \sqrt{x^2 + c^2t^2}$ and the large-deviation function (LDF) I_α is fully determined by hydrodynamic modes and depends only on the "ray" $x/(ct) = \tan\alpha$. Our theory fully corrects the SY picture [45–49] by accounting for quantum distributions and coherence, and gives *qualitatively different, analytical, and exact* results that are valid in the scaling limit of large ℓ, and are applicable at arbitrary interactions, finite temperature and even on generalized Gibbs ensembles [32].

The sine-Gordon field theory.—The sine-Gordon model is integrable both in the classical [50] and quantum [51] regimes. The fundamental excitations are relativistic topological solitons interpolating between the valleys of the periodic potential and parametrized by their rapidity θ. Hereafter, we refer to "kinks" ("antikinks") when they cause a positive (negative) phase slip $+2\pi$ from left to right. A kink-antikink pair can form a stable bound state called breather [50,51]. In the quantum case, breathers are absent in the repulsive regime $cg^2 > 4\pi$; for smaller interactions, breathers appear in the spectrum, with masses $m_n = 2M\sin[(\pi/2)\xi n]$ for integers $n < \xi^{-1} = [(8\pi)/(cg^2)] - 1$. The classical soliton mass $M = 8m/(cg^2)$ is renormalized upon quantization [52]. At weak interactions $g \to 0$, the breather masses collapse to a continuum, and classical physics is recovered [53].

The Hilbert space is described in terms of the asymptotic scattering states of these stable excitations. In integrable models, the interactions are fully encoded within the two-body scattering matrix [31]. For example, two wave packets of colliding breathers are transmitted through each other,

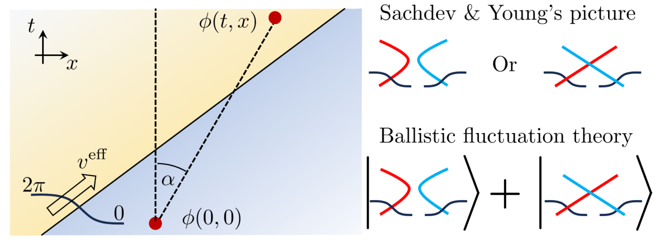

FIG. 1. Phenomenology of phase fluctuations and topological solitons. Pictorial representation of the phase fluctuation induced by a traveling soliton, moving with a velocity v^{eff}. Whenever the (anti)kink worldline intersects the segment connecting (t,x) with the origin, the phase difference jumps $\pm 2\pi$. Ballistic fluctuation theory exactly captures coherence that causes ballistic fluctuations and exponential decay of vertex operator correlations, neglected by the SY phenomenological approach which instead gives diffusion around the space-time ray at $x = 0$.

experiencing in the meanwhile a nontrivial displacement (see e.g., [54]) akin to classical soliton gases [55]. Kink-kink and breather-kink scattering behaves similarly, but reflection is generally possible in kink-antikink scattering (except for the reflectionless points $\xi^{-1} \in \mathbb{N}$). The task of diagonalizing this quantum process in terms of appropriate coherent combinations of scattering states, is accomplished by the thermodynamic Bethe ansatz (TBA) [56–61] and generalized hydrodynamics (GHD) [34,62–64], worked out in the sine-Gordon model in [48,65]. We summarize some aspects in the Supplemental Material (SM) [66].

In the SY phenomenological approach applied to sine-Gordon [45–49], phase fluctuations are assumed to come solely from a dilute gas of (anti)kinks with Maxwell-Boltzmann statistics, justified at low temperatures. Whenever the trajectory of an (anti)kink intersects the ray connecting (t,x) with the origin, it causes the phase difference $\phi(t,x) - \phi(0,0)$ to jump, see Fig. 1: hence, the statistics of phase differences is intimately connected with that of traveling solitons. Damle and Sachdev [45] considered the repulsive regime, where only (anti)kinks are present, and assumed a fully reflective scattering, as justified by the universal low-energy limit of the scattering matrix, leading to a diffusive behavior of the vertex operator correlation function in space-time [45]. At finite temperature, transmission is possible, but a hybrid semiclassical picture of incoherent processes with finite transmission probability is still diffusive as reflection eventually dominates. We find that these conclusions do not hold in the sine-Gordon theory, where integrability plays a pivotal role in preserving coherent scattering, giving rise to ballistic transport *at all temperatures and coupling strengths* and exponential decay of correlation functions everywhere in space-time.

Large scale correlation functions and full counting statistics.—The topological charge is defined as $Q_{\mathrm{top}} = \int \mathrm{d}x q_{\mathrm{top}}(x)$, where $q_{\mathrm{top}} = (1/2\pi)\partial_x\phi$ is its density. It is an

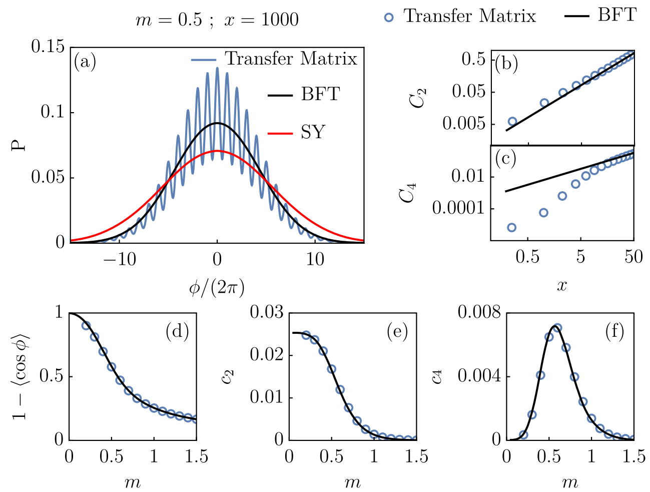

FIG. 2. Equal-time probability and cumulants. We compare analytic predictions (BFT) (black line) in the classical regime of sine-Gordon with numerical results from transfer matrix (blue line and symbols), and with predictions from the SY picture of a gas of (anti)kinks with Maxwell-Boltzmann statistics (red line), in order to illustrate the neglected dressing effects in this picture, in the chosen regime of parameters. The bare mass m is tuned while keeping $\beta = g = c = 1$. (a) The probability of phase differences is reported for a typical mass scale and distance, showing the convergence to the scaling behavior. (b),(c) The convergence of cumulants upon increasing the relative distance is shown. (d)–(f) We scan different values of the bare mass: the vertex operator (d) helps to identify the strongly interacting regimes away from the massless limit $\langle\cos\phi\rangle \simeq 1$ and the large-mass noninteracting regime $1 - \langle\cos\phi\rangle \simeq (4m)^{-1}$ [53]. (e),(f) The large-distance scaling of the second and fourth cumulants is shown, clearly non-Gaussian and in perfect agreement with numerics.

extensive conserved quantity: since the cosine-potential does not confine the field, $\phi(x) - \phi(0)$ can grow indefinitely. The associated continuity equation is $\partial_t q_{\rm top} + \partial_x j_{\rm top} = 0$, with the current $j_{\rm top} = -(1/2\pi)\partial_t\phi$. Integrating $q_{\rm top}(x)$ on a finite interval, we recover the difference of phases at the interval's endpoints. The ballistic fluctuation theorey (BFT) provides general formulas for the full counting statistics (FCS) of total charges and currents on large intervals of space-time solely from hydrodynamic data [67,68]. For a generic density $q(x,t)$ and current $j(x,t)$, and a thermal or Generalized Gibbs Ensemble (GGE) [32] $\langle\cdots\rangle$, the theory predicts that for $(x, ct) = (\ell\sin\alpha, \ell\cos\alpha)$ and large ℓ one has

$$\langle e^{\lambda\int_0^1 \mathrm{d}s[\dot t_s j(x_s,t_s) - \dot x_s q(x_s,t_s)]}\rangle \asymp e^{\ell F_\alpha(\lambda)}, \tag{2}$$

where $s \mapsto (x_s, t_s)$ is a path in space-time from $(x_0, t_0) = (0,0)$ to $(x_1, t_1) = (x, t)$. The FCS $F_\alpha(\lambda)$, a "dynamical specific free energy," is the main result of the BFT. It is expressed in terms of the current $\mathtt{j}_\lambda = \langle j\rangle_\lambda$ and density $\mathtt{q}_\lambda = \langle q\rangle_\lambda$ evaluated in a λ-dependent GGE $\langle\cdots\rangle_\lambda$ as

$$F_\alpha(\lambda) = \int_0^\lambda \mathrm{d}\lambda'(c^{-1}\mathtt{j}_{\lambda'}\cos\alpha - \mathtt{q}_{\lambda'}\sin\alpha). \tag{3}$$

The λ-dependent GGE is fixed by a *flow equation* from $\langle\cdots\rangle$ at $\lambda = 0$, that describes the deformation of the state by the exponential operator on the left-hand side of Eq. (2).

Taking $Q = Q_{\rm top}$ we have $\int_0^1 \mathrm{d}s[\dot t_s j_{\rm top}(x_s,t_s) - \dot x_s q_{\rm top}(x_s,t_s)] = -\{[\Delta\phi(x,t)]/2\pi\}$, thus the left-hand side of Eq. (2) is $\int \mathrm{d}\delta P\{[\Delta\phi(t,x)/2\pi] = \delta\}e^{-\lambda\delta}$. The theory predicts that all cumulants $C_n = \langle[(\Delta\phi)/(2\pi)]^n\rangle_c$ of phase differences scale extensively $C_n \sim \ell c_n$ as $\ell\to\infty$, with $F_\alpha(\lambda) = \sum_{n=1}^\infty c_n(-\lambda)^n/n!$ the Legendre-Fenchel transform of $I_\alpha(\delta)$. In the SM [66] we review the BFT and apply it to the topological charge of the sine-Gordon model. Remarkably simple is the closed expression, valid at reflectionless points and in the classical regime, for the second cumulant whenever the average topological charge is zero,

$$c_2(\alpha) = 2\int \mathrm{d}\theta\rho_K(\theta)f(\theta)|c^{-1}v_K^{\rm eff}(\theta)\cos\alpha - \sin\alpha|. \tag{4}$$

Here kinks and antikinks have the same GGE distribution $\rho_K(\theta)$ and (dressed) velocity $v_K^{\rm eff}(\theta)$, and $f(\theta)$ is a state dependent statistical factor ($f \to 1$ in the semiclassical limit). In practice, $c_2(\alpha)$ is the scaled variance for the number of solitons whose wordline intersects the segment connecting (t,x) and the origin, see Fig. 1. All the terms in Eq. (4) are exactly known from TBA and GHD. The full second c_2 and fourth c_4 cumulants are reported in the SM

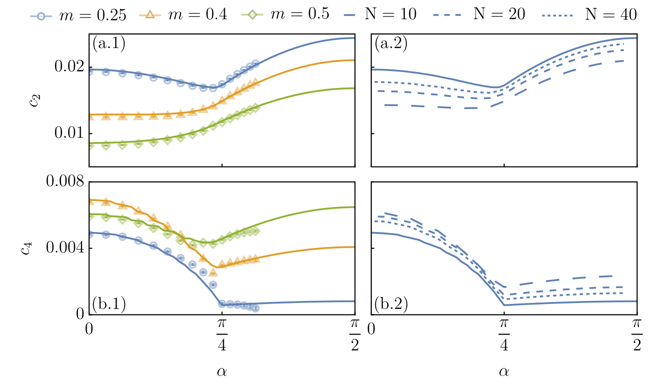

FIG. 3. Cumulants with space-time separation. Scaling behavior of the second (a.1) and fourth (b.1) cumulant as function of the ray $x/(ct) = \tan\alpha$ for representative choices of the mass scale m ($\beta = g = c = 1$) in the classical regime. Numerical values obtained with Monte Carlo (symbols) closely follow the analytic BFT prediction (solid lines). (a.2), (b.2) Approach of the quantum prediction (dashed lines) to the classical limit (solid line) for the c_2 and c_4, respectively. We take $m = 0.25$ and increase the number of breathers N, while tuning the quantum soliton mass according to the semiclassical limit [53,66].

[66] at the reflectionless points and classical regime, and can be obtained for arbitrary coupling from the sine-Gordon TBA [48,65].

The BFT results should be contrasted with the SY picture [45–49]. Clearly, the latter picture neglects dressing effects by setting $\rho_K(\theta)=[(Mc\cosh\theta)/(2\pi)]\exp(-\beta Mc^2\cosh\theta)$ in (4) [see Fig. 2(a)]. But most importantly, at unequal times, the resulting physics is qualitatively different: fully reflective scattering makes the topological charge an isolated hydrodynamic mode with zero velocity, and indeed Ref. [45] predicts a diffusive, instead of ballistic, behavior, with power-law instead of exponential decay of vertex operator correlations at $\alpha=0$. The BFT captures the resulting coherent scattering and shows that a ballistic behavior and exponential decay is generic in the sine-Gordon model. Note that taking purely transmissive scattering in the SY picture [45], one obtains the correct ballistic low-temperature behavior at reflectionless points [66].

The semiclassical limit and numerical benchmarks.—Strongly interacting systems at finite temperature are extremely challenging to simulate [69]. Hence, we now focus on the classical regime, which is amenable to efficient numerical benchmarks [66].

The exact thermodynamics of the classical sine-Gordon model has recently been developed in Ref. [53] building on classical limits [70–74] of quantum integrability; we apply these to the BFT framework [66]. In equilibrium, one can set interaction, temperature, and velocity to 1 upon a length scale renormalization: we opt for this choice and use the mass m as a tunable parameter. In Fig. 2, we compare equal-time phase fluctuations derived from (i) our result, (ii) SY classical picture (see the SM [66]), and (iii) numerical results obtained with the transfer matrix method [66,75,76]. In Fig. 2(a) we show the full distribution of the phase for a typical example. Notably, numerics shows "spikes," reminiscent of the fact that the number of solitons comprised in an interval $[0,x]$ is an integer number; for lower temperature (larger mass scales) the spikes are more peaked. The BFT prediction, substantially different from SY, captures the smoothed probability distribution: convergence at large separation holds in a weak sense. The BFT scaling is clearer for the cumulants, see Figs. 2(b) and 2(c); it becomes slower for higher cumulants. In Figs. 2(d)–2(f), we scan a wide parametric regime finding excellent agreement between our analytical result and numerics. In Fig. 3, we analyze unequal-times phase fluctuations: we observed spikes (not shown), but the cumulants quickly reach their asymptotic scaling. For representative values of the mass, we compare the ray-dependent growth of cumulants predicted by BFT against Monte Carlo simulation [77,78] with good agreement, and show the convergence of quantum predictions at the reflectionless points to the semiclassical ones. Further analysis is left to the SM [66].

Experimental feasibility.—A versatile tabletop simulator of the sine-Gordon model is realized by the experimental group in Vienna via two tunnel-coupled quasicondensates [40], see Fig. 4(a); phase fluctuations are probed by matter-wave interferometry measurements [79–81]. Our result can arguably give quantitative predictions for such experimental data and may be useful in state characterization, both in equilibrium [42] and nonequilibrium setups [82]. However, imperfections and finite resolution may undermine a correct phase measurement: to show that faithful phase tomography is within the reach of current experimental capability, we analyze a toy model of the measurement process. Because of the weak interactions of the atoms, sine-Gordon is realized close to its semiclassical regime [83,84] and Monte Carlo accounts for experimental observations [42,85]. We use typical experimental parameters, see the SM [66]. Atoms are trapped in a smoothed box potential of length ∼160 μm, and the transverse trap frequency [86] tunes interactions. The inhomogeneous density profile $n(x)$ is well described by the Thomas-Fermi approximation and causes weak inhomogeneities in the sine-Gordon coupling [87]. The mass scale, which is changed by adjusting the strength of the tunneling between the tubes, affects the spatial extent of the kinks, and the overall population. We choose a temperature 60 nK and a

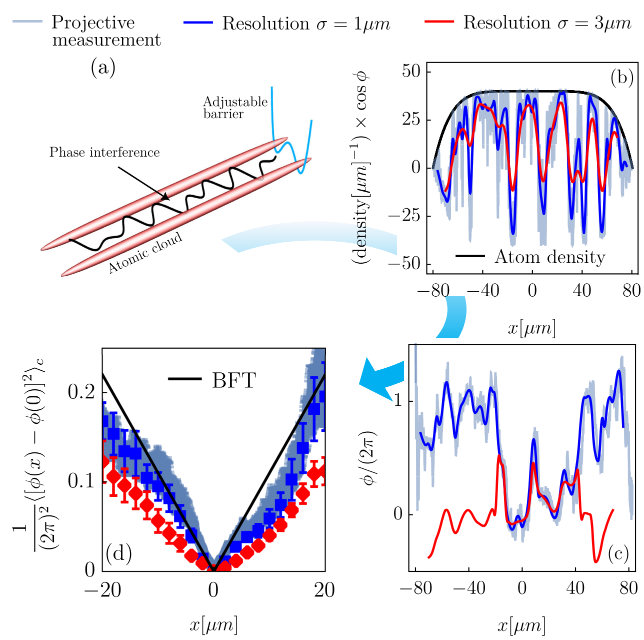

FIG. 4. The sine-Gordon model from coupled condensates. (a) Sketch of the experimental setup. (b), (c) Example of phase-reconstruction protocol from the outcome of a single projective measurement for different pixel resolutions σ (see main text). (d) Statistics built on 100 samples already shows the scaling behavior of the equal-time second cumulant stemming from the center of the trap. The effect of a low resolution σ is to "miss" kinks [see also (c)] and underestimate phase fluctuations. A good-quality measurement is already obtained with $\sigma = 1$ μm. See main text for discussion of parameters, and the SM [66] for further details and data.

bulk atom density 40 atm/μm to retain an appreciable kink density, and a mass scale such that $\langle \cos\phi \rangle \simeq 0.32$ in the bulk. Further discussion is left to the SM [66].

Matter-wave interferometry gives access to spatially resolved projective measurements of trigonometric functions of the phase $n(x)\cos\phi(x)$ and $n(x)\sin\phi(x)$ [88]. In an ideal scenario, the phase itself can then be recovered, but the finite imaging resolution causes a detrimental coarse graining [see Fig. 4(b)]. The latter is modeled by convolving $n(x)\cos[\phi(x)]$ with Gaussians with standard deviation σ and the phase is then reconstructed from these coarse grained data [66,89]. Depending on the resolution, the phase profile may be correctly recovered or kinks may be washed out by the local coarse graining, see Fig. 4(c). Finally, by building statistics over many measurements, phase correlations are obtained. In Fig. 4(d), we show the outcome of 100 independent samples: a resolution $\sigma = 1$ μm (a slight improvement on the current experimental resolution $\sigma \simeq 3$ μm [89]) is enough to capture microscopic phase fluctuations that compare well with analytical results. Large fluctuations of the second cumulant are due to the relatively small number of samples: we used 100 as a typical experimental situation.

Conclusion and outlook.—Exact results on correlation functions in interacting field theories are scarce: we give analytical predictions for the large scale phase fluctuations in the sine-Gordon model, valid at any temperature and interactions, and in generalized Gibbs ensembles. We discuss how our results are of ready applicability in experiments on coupled condensates, where equal-time correlations are accessed. Unequal-time phase differences may be accessible by locally exciting the topological charge via Raman coupling [90]. The main appeal of our results is its applicability to the quantum regime: sine-Gordon simulators in the quantum regime may be within reach of quantum gas microscopes [91,92]. One can analyze the full range of couplings using [48,65] and integrability-breaking perturbations [93–96] within the BFT. It will be important to include diffusive corrections [97,98]: a possible scenario at low temperatures is that the diffusive behavior predicted by Damle and Sachdev [45] is seen at early times, with a slow exponential decay at later times as predicted by the BFT. In contrast, if integrability is broken, isolated hydrodynamic modes are present and the diffusive SY picture should hold at all times and temperatures [49]. Studying the timescales of the various crossovers implied is of utmost interest for future studies.

Raw data and working codes are available from Zenodo [99].

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