# Irreversibility across a Nonreciprocal $\mathcal{PT}$-Symmetry-Breaking Phase Transition

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Nonreciprocal interactions are commonplace in continuum-level descriptions of both biological and synthetic active matter, yet studies addressing their implications for time reversibility have so far been limited to microscopic models. Here, we derive a general expression for the average rate of informational entropy production in the most generic mixture of conserved phase fields with nonreciprocal couplings and additive conservative noise. For the particular case of a binary system with Cahn-Hilliard dynamics augmented by nonreciprocal cross-diffusion terms, we observe a nontrivial scaling of the entropy production rate across a parity-time symmetry breaking phase transition. We derive a closed-form analytic expression in the weak-noise regime for the entropy production rate due to the emergence of a macroscopic dynamic phase, showing it can be written in terms of the global polar order parameter, a measure of parity-time symmetry breaking.

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Though microscopic forces respect the action-reaction principle, effective reciprocity-breaking interactions commonly arise at the mesoscopic scale. In living matter, one may even argue that reciprocity-breaking interactions are the rule rather than the exception, such as in classical predator-prey and promoter-inhibitor models [1–4]. Similarly, nonreciprocity arises in systems whose dynamics depend on information propagation as in crowds of social animals [5–8]. Furthermore, nonreciprocal interactions generically emerge from microscopic interactions mediated by a nonequilibrium medium [9–12], leading to fundamentally nonequilibrium physics [13–16].

This breaking of reciprocal-symmetry generically leads to directed motion as seen in diffusiophoretic colloidal mixtures [17–19] and binary systems of active and passive particles [20,21]. Self-propelling mesoscopic clusters emerge from an imbalance of attraction-repulsion interactions between microscopic agents [22,23]. In biology, the *chase-and-run* behavior displayed by neural crest cells and placodal cells provides a generic mechanism of coordinated cell migration, driving many fundamental morphogenetic and physiological processes [24]. In particular, the emergence of dynamical phases in systems with nonreciprocal interactions has been associated with the breaking of parity-time ($\mathcal{PT}$) symmetry [13,20], where solutions to the governing equations are no longer invariant under a joint parity-time inversion $\mathcal{PT}\colon\ r, t \mapsto -r, -t$. These represent another example of $\mathcal{PT}$-symmetry breaking transitions [25], which includes optical systems [26], directional interface growth [27–29] and polar swarm models [13,30].

While coarse graining a microscopic model is generally arduous, approaches based on conservation and symmetry principles used to model dynamic critical phenomena at equilibrium can be extended to active systems [31–34]. For example, the scalar active field theories *Active Model B* [35] and $B+$ [36] describe nonequilibrium liquid-gas phase separation phenomena [37,38]. Owing to their simplicity and generality, active field theories present an attractive starting point for analyzing the nonequilibrium thermodynamic properties of living systems. However, at the level of these continuum descriptions, the dynamics are formulated in terms of macroscopic order parameters (such as the agent density); while they generically capture accurately densities and correlations, no direct connection can be made between time-reversal-symmetry (TRS) breaking and the rate of microscopic energy dissipation, as this requires a notion of particle entity [39,40]. It was recently argued that calculating the entropy production rate for those effective field theories (even if stemming from the coarse graining of a microscopic dynamics) will not in general provide meaningful information about the work done at the scale of the microscopic particles unless Doi-Peliti field theories are used [41]. This problem has also been illustrated explicitly for nonreciprocal particle systems [42]. Despite this, the extent to which TRS is broken, given by the *informational* entropy production rate (IEPR), still provides a metric for the *distance from equilibrium* in nonequilibrium systems at the level of the macroscopic dynamics [39,43–49]. Of particular interest are the effects of phase transitions in active systems, where nontrivial scalings of the entropy production have recently been observed [50–53].

In this Letter, we elucidate the impact of nonreciprocal interactions and the resulting $\mathcal{PT}$-symmetry breaking transition on the time reversibility of dynamics in a scalar active field theory. First, we derive a general result for the average informational entropy production rate in a mixture of conserved scalar fields, then apply this to a scalar field theory of nonreciprocal interactions [54]. In the case of a binary system, we show that this entropy production rate exhibits nontrivial scaling across a transition from a static to a dynamic phase. We identify explicitly a contribution stemming from the breaking of $\mathcal{PT}$ symmetry, i.e., the emergence of macroscopic dynamics, which scales quadratically with the speed of the dynamic phase, mirroring the relation between self-propulsion speed and TRS breaking observed in active particle systems [45,46,55].

*Informational entropy production in scalar active mixtures.*—We consider a system of $N$ interacting, conserved active fields $\{\phi_i(\mathbf{r},t)\}_{i\in[1,N]}$, with governing equations

$$\dot{\phi}_i(\mathbf{r},t) = \nabla^2\mu_i(\mathbf{r},t) + \nabla\cdot\mathbf{\Lambda}_i(\mathbf{r},t), \tag{1}$$

where $\mu_i(\mathbf{r},t)$ is a chemical potential which can include passive and active contributions and $\mathbf{\Lambda}_i$ is a noise term capturing thermal fluctuations in the system. For the sake of tractability, the noise is taken to be additive as is commonly done in field theories of active phase separation [17,34–36,48,56]. In practice, the conserved noise terms appearing in Eq. (1) require careful regularization since their power spectrum is unbound in the ultraviolet, which may lead to divergences. While this is often done by regularizing the noise correlator (without affecting the conservative nature of the noise), we show importantly that divergences in the entropy production rate originate from the infinite dimensionality of the continuum field and can only be cured by imposing finite dimensionality [41]. To do so, we employ a suitable spatial discretization scheme when analyzing the dynamics below, effectively imposing a UV cutoff (see Ref. [57] for an extended discussion). Note that we keep notation pertaining to continuous space here for readability [59].

For our system, the extent of TRS breaking is quantified by the Kullback-Leibler divergence per unit time [44,47,60]:

$$\dot{\mathcal{S}} = \lim_{\tau\to\infty}\frac{1}{\tau}\left\langle \log\frac{\mathbb{P}_F[\{\phi_{i=\{1,\ldots,N\}}\}_{t=0}^{t=\tau}]}{\mathbb{P}_B[\{\phi_{i=\{1,\ldots,N\}}\}_{t=0}^{t=\tau}]}\right\rangle, \tag{2}$$

where $\mathbb{P}_F[\cdot]$ and $\mathbb{P}_B[\cdot]$ denote the path probability for the forward and backward paths, respectively, for the combined dynamics of the $N$ fields. The average of the log-ratio is taken over realizations of the noise terms $\mathbf{\Lambda}_{i=\{1,\ldots,N\}}$. In a thermodynamically consistent microscopic theory, $\dot{\mathcal{S}}$ would correspond to the total rate at which entropy is produced. However, here it shall be understood only in the informational sense, i.e., as a measure of TRS breaking in the dynamics, as argued.

Employing the usual approach for the treatment of stochastic field theories, the log-ratio of these two path-probabilities [Eq. (2)] can be written as the difference of two dynamical actions which take the form of Onsager-Machlup functionals [45,47,59]. Suppose that the noise terms $\mathbf{\Lambda}_i$ in Eq. (1) have a diagonal correlation matrix $\Theta_{ij} = \langle\Lambda_i(\mathbf{r},t)\Lambda_j(\mathbf{r}',t')\rangle = 2D\delta_{ij}\delta(\mathbf{r}'-\mathbf{r})\delta(t'-t)$, then each of the path probabilities can be decomposed into products of independent contributions from the realizations of each noise term. As shown in [57], we treat Eq. (2) in the usual way [34,47,48,59] and write the IEPR as a sum of these individual contributions

$$\dot{\mathcal{S}} = -\lim_{\tau\to\infty}\frac{1}{D\tau}\int_0^\tau dt\int d\mathbf{r}\sum_{i=1}^N\langle\mu_i\dot{\phi}_i\rangle. \tag{3}$$

If we decompose each chemical potential $\mu_i$ into equilibrium and nonequilibrium contributions $\mu_i = \mu_i^{(\mathrm{eq})} + \mu_i^{(\mathrm{neq})}$ and define the *free-energy* functional, $\mathcal{F}[\{\phi_{i=\{1,\ldots,N\}}\}]$ such that the equilibrium contribution is written as $\mu_i^{(\mathrm{eq})} = \delta\mathcal{F}/\delta\phi_i$, then $\int d\boldsymbol{r}\dot{\phi}\mu_i^{(\mathrm{eq})} = \dot{\mathcal{F}}$ and Eq. (3) simplifies to

$$\dot{\mathcal{S}} = -\lim_{\tau\to\infty}\frac{1}{D\tau}\int_0^\tau dt\int d\mathbf{r}\sum_{i=1}^N\langle\mu_i^{(\mathrm{neq})}\dot{\phi}_i\rangle, \tag{4}$$

provided that the free energy $\mathcal{F}$ is bounded in time [34,47,57]. As expected, Eq. (4) shows that the dynamics are symmetric in time ($\dot{\mathcal{S}} = 0$) in the absence of nonequilibrium contributions to the chemical potential. Our result holds for arbitrary nonequilibrium terms, e.g., we recover the result for *Active Model B* by setting $N = 1$ and substituting in the active term $\mu_1^{(\mathrm{neq})} = \lambda|\nabla\phi_1|^2$ [47].

*Scalar active mixtures with nonreciprocal couplings.*—To study the link between TRS breaking and nonreciprocity, we consider the nonreciprocal scalar field theory introduced in [54], which extends the classical Cahn-Hilliard model [31,32] to include nonreciprocal linear couplings (cross diffusion). As such, the field theory describes phase separation in a wide class of systems where reciprocal-symmetry breaking interactions appear at the continuum level [61–64].

The governing equations for this field theory take the form [54]

$$\dot{\phi}_i(\mathbf{r},t) = \nabla^2\left[\frac{\delta\mathcal{F}}{\delta\phi_i} + \sum_j\alpha_{ij}\phi_j(\mathbf{r},t)\right] + \nabla\cdot\mathbf{\Lambda}_i(\mathbf{r},t), \tag{5}$$

where, as before, $i \in \{1, \ldots, N\}$ and we have defined the global free-energy-like functional as

$$\mathcal{F}[\{\phi_i\}] = \int d\mathbf{r}\left(\sum_i f_i(\phi_i) + \sum_{i<j} \kappa_{ij}\phi_i\phi_j\right) \quad (6)$$

and $\{\alpha_{ij}\}_{i,j\in[1,N]}$ is an antisymmetric matrix. The functional includes two contributions: the first determines how each field evolves in isolation and the second describes the (reciprocal) enthalpic interactions between fields. Henceforth, we suppose that the free energy densities take a Landau square gradient, $\phi^4$ form: $f_i(\phi_i) = \chi_i\phi_i^2/2 + \phi_i^4/12 + \gamma_i|\nabla\phi|^2/2$, where the sign of $\chi_i$ controls whether the field phase separates and $\gamma_i$ sets the effective surface tension when interfaces arise in the system [33,36]. This results in *Model B*-like dynamics augmented by nonreciprocal couplings.

*Breaking TRS through nonreciprocity.*—We call upon Eq. (4) to write an expression for the entropy production rate in our system with nonreciprocal couplings:

$$\dot{\mathcal{S}} = -\lim_{\tau\to\infty}\frac{1}{D\tau}\int_0^\tau dt \int d\mathbf{r}\sum_{i=1}^N\sum_{j\neq i}\langle\alpha_{ij}\phi_j\dot{\phi}_i\rangle. \quad (7)$$

Note that $\dot{\mathcal{S}}$ vanishes as expected when $\alpha_{ij} = 0$ for all $i$ and $j$, as this describes equilibrium dynamics. While linear-order couplings were studied in [54], our result [Eq. (4)] holds for arbitrary nonreciprocal couplings and is thus valid beyond the class of systems described by Eq. (5) (see Ref. [65] for a recent work on nonlinear nonreciprocity).

To illustrate our result, we confine our system to one spatial dimension on $[0, L)$ with periodic boundary conditions and set $N = 2$. In this case, a single parameter controls the strength of the nonreciprocal coupling. Indeed, we can write the coupling coefficients as $\kappa_{12} = \kappa_{21} = \kappa$ and $\alpha_{12} = -\alpha_{21} = \alpha$. The resulting equations governing the dynamics of the fields, which we denote $\phi_1(r,t)$ and $\phi_2(r,t)$, take the form

$$\dot{\phi}_1(r,t) = \partial_r^2\mu_1^{(\mathrm{eq})}(r,t) + \alpha\partial_r^2\phi_2(r,t) + \partial_r\Lambda_1(r,t), \quad (8a)$$

$$\dot{\phi}_2(r,t) = \partial_r^2\mu_2^{(\mathrm{eq})}(r,t) - \alpha\partial_r^2\phi_1(r,t) + \partial_r\Lambda_2(r,t), \quad (8b)$$

where the chemical potentials are again defined as the following functional derivatives $\mu_i^{(\mathrm{eq})} = (\delta\mathcal{F}/\delta\phi_i)$ and $\Lambda_i(r,t)$ are zero-mean Gaussian white noise terms with covariance matrix $\Theta_{ij}(r-r', t-t') = 2D\delta_{ij}\delta(r-r')\delta(t-t')$. From Eq. (7) we write the steady-state IEPR for this binary system as

$$\dot{\mathcal{S}} = -\lim_{\tau\to\infty}\frac{\alpha}{D\tau}\int_0^\tau dt\int_0^L dr\langle\phi_2\dot{\phi}_1 - \phi_1\dot{\phi}_2\rangle. \quad (9)$$

We first explore the entropy production rate in this system numerically, quantifying how $\dot{\mathcal{S}}$ scales with the strength of the nonreciprocal coupling. In what follows, we work in the limit of weak noise and further place ourselves in the case where (i) $\phi_1$ phase separates (by fixing $\chi_1 < 0$ and $\gamma_1 > 0$) and (ii) $\phi_2$ is purely diffusive (by imposing $\mu_2^{(\mathrm{eq})} = \chi_2\phi_2$). We also suppose that the two fields feel a weak (reciprocal) repulsion ($\kappa > 0$) [20]. We solve Eq. (8) (see details of our numerical method in [57]) then evaluate the integral in Eq. (9) numerically. As seen in Fig. 1(a), the entropy production rate initially scales as $\dot{\mathcal{S}} \propto \alpha^2$. At a critical value of the nonreciprocal coupling $\alpha = \alpha_c$, this scaling disappears and $\dot{\mathcal{S}}$ quickly increases continuously by several orders of magnitude. As $\alpha \gg \alpha_c$, we recover a quadratic scaling.

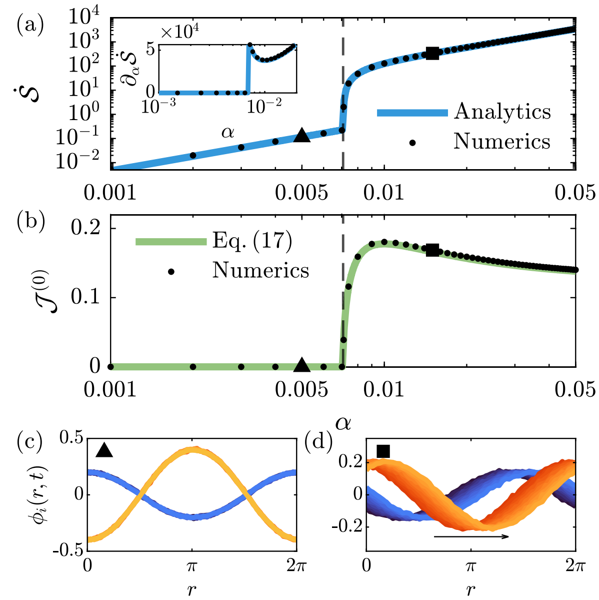


FIG. 1. Nontrivial scaling of $\dot{\mathcal{S}}$ across a $\mathcal{PT}$-breaking phase transition in a nonreciprocal system. (a) For $D = 10^{-7}$, we identify a non-trivial scaling in the entropy production rate in the binary system governed by Eq. (8), with a discontinuous derivative (inset) at a critical value of the nonreciprocal coupling $\alpha = \alpha_c$. [Numerical simulations: symbols; analytic results from Eqs. (13) and (15): solid lines.] (b) Discontinuity in the scaling of $\dot{\mathcal{S}}$ coinciding with the breaking of $\mathcal{PT}$ symmetry, characterized by the nonzero value of the polar order parameter $\mathcal{J}^{(0)}$ for the deterministic equations [i.e., $D = 0$ in Eq. (8)]. The critical nonreciprocal coupling corresponds to a transition between (c) static states and (d) dynamic states (traveling waves). The simulations were run for parameter values $L = 2\pi$, $\gamma_1 = 0.04$, $\gamma_2 = 0$, $\chi_1 = -0.05$, $\chi_2 = 0.005$, and $\kappa = 0.005$.

To explain this nontrivial scaling, we explore in more detail the dynamics of the system governed by Eq. (8) (see also [20]). In particular, we turn momentarily to the deterministic case, $D = 0$, and observe that for $\alpha < \alpha_c$, the system reaches a static stationary state, where the two fields phase separate and exhibit demixing behavior [see Fig. 1(c)]. For $\alpha > \alpha_c$, the system instead displays a traveling wave solution and we observe the emergence of a nonzero global polar order $\mathcal{J}$ [see Figs. 1(b)–1(d)], as defined by

$$\mathcal{J} = \frac{1}{\tau}\int_0^\tau dt \int_0^L dr \langle \phi_2 \partial_r \phi_1 - \phi_1 \partial_r \phi_2 \rangle \tag{10}$$

Indeed, the integrand in (10) represents the net flux through a point in space averaged over realizations of the noise and is derived as the real-space representation of the probability current for the complex-valued field $A = \phi_1 + i\phi_2$ (see Ref. [54] for details). Thus, this order parameter becomes nonzero when the stationary interacting fields are out of phase in such a way that explicitly breaks the joint $\mathcal{PT}$: $r, t \mapsto -r, -t$ symmetry. As discussed in [20], this transition to a dynamic phase is thus an example of $\mathcal{PT}$-symmetry breaking transition, where the asymmetric distribution of the fields leads to an imbalance of effective forces and thus persistent motion in the macroscopic dynamics [17–20,66]. In principle, adding noise to the system will induce a nonzero rate of reversals in the traveling wave solutions. Although these are rare in the weak noise strength regime, the thermodynamic quantities derived below are valid for the dynamics between reversal events.

*Emergence of macroscopic dynamics and entropy production.*—This second-order transition to motion controlled by $\alpha$ coincides with the nontrivial scaling of the entropy production rate as seen in Fig. 1. We now formally connect these two phenomena by identifying the contribution of the macroscopic dynamics to the steady-state entropy production rate. To do so, we consider a change of variables to rewrite Eq. (9) in the comoving frame of reference. We let $v(\alpha)$ denote the velocity of the traveling wave solution [see Fig. 2(a)] and proceed to the transformation $(r', t') = [r - v(\alpha)t, t]$. We denote the fields $\phi_i$ in this new frame of reference by $\Phi_i$, such that the time derivatives in Eq. (9) take the form $\dot{\phi}_i \to \dot{\Phi}_i - v(\alpha)\partial_{r'}\Phi_i$ and the entropy production rate now takes the form $\dot{\mathcal{S}} = \dot{\mathcal{S}}_A(\alpha) + \dot{\mathcal{S}}_B(\alpha)$ with

$$\dot{\mathcal{S}}_A = \lim_{\tau\to\infty} \frac{\alpha}{D\tau}\int_0^\tau dt' \int_0^L dr' \langle \Phi_1 \dot{\Phi}_2 - \Phi_2 \dot{\Phi}_1 \rangle, \tag{11a}$$

$$\dot{\mathcal{S}}_B = \lim_{\tau\to\infty} \frac{\alpha v(\alpha)}{D\tau}\int_0^\tau dt' \int_0^L dr' \langle \Phi_2 \partial_{r'} \Phi_1 - \Phi_1 \partial_{r'} \Phi_2 \rangle. \tag{11b}$$

As seen in Fig. 2(b), we observe that $\dot{\mathcal{S}}_A \propto \alpha^2$ independently of the traveling wave speed; it describes the entropy production due to the nonequilibrium suppression of fluctuations in a stationary phase-separated system, reminiscent of the entropy production observed in *Active Model B* [34,35], which also scaled quadratically with the nonequilibrium contribution to the dynamics [47].

The second term $\dot{\mathcal{S}}_B$ captures the contribution of the macroscopic motion to the total entropy production rate, vanishing when $v(\alpha) = 0$. Strikingly, the integral contribution to $\dot{\mathcal{S}}_B$ is exactly the global polar order parameter defined in Eq. (10), which implies $\dot{\mathcal{S}}_B = \alpha v(\alpha)\mathcal{J}/D$ and thus provides a direct link between the macroscopic dynamics and IEPR, i.e., TRS breaking.

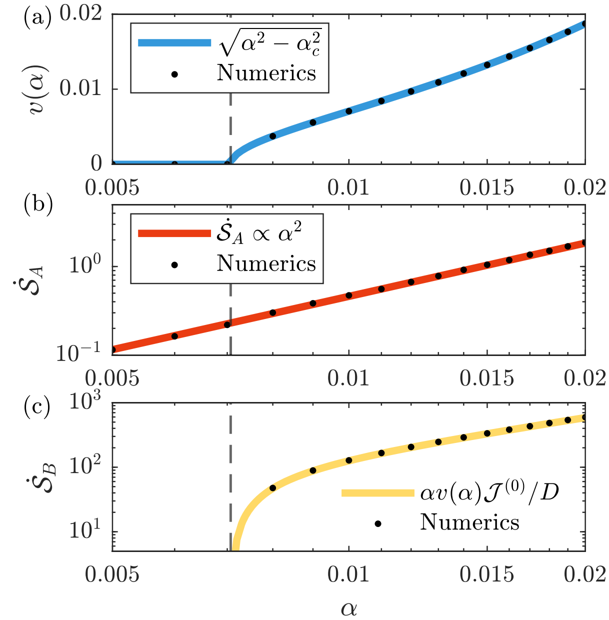


FIG. 2. Evaluation of contributions to entropy production rate $\dot{\mathcal{S}}$. (a) Transition to motion for nonreciprocal coupling strengths exceeding critical value, $\alpha_c = \sqrt{\kappa^2 + \chi_2^2}$. (b) Numerically evaluated contribution $\dot{\mathcal{S}}_A$ shown to scale quadratically with the nonreciprocal parameter $\alpha$. (c) A second contribution $\dot{\mathcal{S}}_B$ to the entropy production rate stems directly from the emergence of macroscopic dynamics as exhibited in (a).

*Weak-noise expansion.*—From Eq. (4), we expect that signatures of TRS breaking are most striking when the noise is weak. In this regime, we can make progress towards evaluating Eq. (11) analytically by expanding each of the fields perturbatively around the deterministic solution ($D = 0$):

$$\Phi_i(r', t') = \Phi_i^{(0)}(r') + \sqrt{D}\Phi_i^{(1)}(r', t') + D\Phi_i^{(2)}(r', t') + \mathcal{O}(D^{3/2}). \tag{12}$$

Taking a time derivative of Eq. (12), one can derive governing equations for the dynamics of each field $\Phi_i^{(j)}$, which are independent of $D$ [57]. The two distributions $\Phi_i^{(0)}$ are given by the deterministic solutions to Eq. (8) in the frame of reference $(r', t')$.

Substituting Eq. (12) into Eq. (11a), the terms of order $\mathcal{O}(D^{-1})$ and $\mathcal{O}(D^{-1/2})$ disappear in the expansion for $\dot{\mathcal{S}}_A$ at steady state [57]; to leading order, this contribution reads

$$\dot{\mathcal{S}}_A = \alpha I_A(\alpha) + \mathcal{O}(D^{1/2}) \tag{13}$$

with

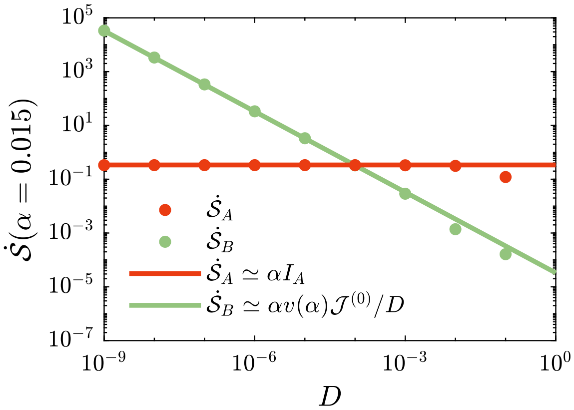


FIG. 3. Scaling of $\dot{\mathcal{S}}_A$ and $\dot{\mathcal{S}}_B$ in the weak-noise regime. Given in the regime $\alpha > \alpha_c$ to ensure that both contributions to $\dot{\mathcal{S}}$ are nonzero, we confirm through the numerical simulations (symbols) the analytic results obtained through the one-mode approximation (solid lines): $\dot{\mathcal{S}}_A \propto D^0$ [Eq. (13)] (prefactor determined by fitting numerical data) and $\dot{\mathcal{S}}_B \propto D^{-1}$ [Eq. (15)].

$$I_A(\alpha) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt \int_0^L dr \langle \Phi_1^{(1)}\dot{\Phi}_2^{(1)} - \Phi_2^{(1)}\dot{\Phi}_1^{(1)}\rangle. \quad (14)$$

Therefore, $\dot{\mathcal{S}}_A \propto D^0$ in the small $D$ regime, which we confirm against numerical results in Fig. 3. To obtain a closed analytic expression, we require the form of $\Phi_i^{(1)}$, but the governing equations for these fields (derived in [57]) do not generally admit analytic solutions.

To leading order, we further write that

$$\dot{\mathcal{S}}_B = \frac{\alpha v(\alpha)}{D}\mathcal{J}^{(0)} + \mathcal{O}(D^{-1/2}), \quad (15)$$

where $\mathcal{J}^{(0)}$ denotes the global polar order parameter evaluated in the deterministic limit

$$\mathcal{J}^{(0)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt' \int_0^L dr' \left(\Phi_2^{(0)}\partial_{r'}\Phi_1^{(0)} - \Phi_1^{(0)}\partial_{r'}\Phi_2^{(0)}\right), \quad (16)$$

Note that we have discarded the average over noise realizations as—by construction of the expansion in Eq. (12)—we only require analytic expressions for the deterministic solutions $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ to Eq. (8) to write the leading order contribution to $\dot{\mathcal{S}}_B$.

When $|\chi_1/\gamma_1| \gtrsim (2\pi/L)^2$, one can argue that only the lowest allowed wave number mode $q = 2\pi/L$ is linearly unstable and it is expected to dominate the structure of the stationary distributions in a Fourier series expansion [57], which is not expected in general [67]. We can then proceed to a one-mode approximation for the stationary distributions, in effect writing $\Phi_i^{(0)}(r') \propto \cos(2\pi r'/L - \theta_i)$, where $\theta_1$ can be set to zero by translational invariance and $\theta_2$ sets the difference in phase of the distributions [20].

In the case where $L = 2\pi$, we find under this one-mode approximation that

$$\mathcal{J}^{(0)} = \frac{8\pi v(\alpha)(\chi_1 + \chi_2 + \gamma_1)}{(\kappa - \alpha)}, \quad (17)$$

giving us a closed-form analytic expression for $\dot{\mathcal{S}}_B$ to leading order [57]. Overall, we conclude that $\dot{\mathcal{S}}_B \propto D^{-1}$ in the small $D$ regime which agrees with our numerical results. Interestingly, the parameter space corresponding to the existence of a traveling wave solution is a subset of the space required for $\dot{\mathcal{S}}_B > 0$, guaranteeing that the second law of thermodynamics is strictly satisfied. Strikingly, we also recover an expression relating the entropy production rate, the traveling wave velocity and diffusion coefficient: $\dot{\mathcal{S}}_B \propto v^2(\alpha)/D$, which becomes more accurate when $\alpha \gg \kappa$. This exact scaling relation appears when studying the entropy production rate for a self-propelled particle with propulsion speed $v$ and diffusion coefficient $D$ [45,46].

*Discussion and outlook.*—We have studied the TRS breaking implications of nonreciprocal couplings in active field theories. Using our results for generic mixtures of conserved active fields with additive and independent noise terms, we quantified TRS breaking in a binary mixture of nonreciprocally coupled scalar fields. We observed nontrivial scaling in the IEPR across the transition from a static to dynamic phase, driven by the breaking of $\mathcal{PT}$ symmetry. We quantify precisely the scaling of the informational entropy production across the exceptional point, a previously unexplored perspective in addressing phase transitions for scalar active matter models, where perturbative renormalization group approaches have been employed [68,69]. Similar nontrivial scalings have been observed in self-propelled microscopic systems at the transition to collective motion [50,51], suggesting a more general link between the thermodynamic properties of nonequilibrium systems and $\mathcal{PT}$-symmetry breaking transitions. We elucidate this in the nonreciprocal Cahn-Hilliard (NRCH) model, identifying a contribution to the IEPR that depends explicitly on the order parameter quantifying the emergence of motion at the macroscale. The deterministic NRCH model has also been shown to capture the universal amplitude dynamics near a conserved-Hopf instability, ubiquitous in active systems with two conservation laws [73]. We note that a recent study obtained independently results consistent with the theory we present in this Letter, see Refs. [70–72]. As we focused on the weak-noise limit, a regime which displays model-independent universality, we thus believe that our analytical results may hold for other out-of-equilibrium pattern-forming systems in biology.

Future studies will extend this work to a broader class of active mixtures, including, for instance, the presence of nonconservative dynamics, such as Markovian switching modeling chemical reactions [74–76] or microscopic changes of state [38,77–79]. Extending the methodology to systems with correlated or multiplicative noise remains a challenge; the corresponding problem for isolated active fields was recently studied [80–83]. Altogether, we expect that establishing a framework to analyze the nonequilibrium thermodynamic properties of complex active and living systems will quickly draw considerable attention.

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