Universal Platform of Point-Gap Topological Phases from Topological Materials

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Whereas point-gap topological phases are responsible for exceptional phenomena intrinsic to non-Hermitian systems, their realization in quantum materials is still elusive. Here, we propose a simple and universal platform of point-gap topological phases constructed from Hermitian topological insulators and superconductors. We show that $(d-1)$-dimensional point-gap topological phases are realized by making a boundary in d-dimensional topological insulators and superconductors dissipative. A crucial observation of the proposal is that adding a decay constant to boundary modes in d-dimensional topological insulators and superconductors is topologically equivalent to attaching a $(d-1)$-dimensional point-gap topological phase to the boundary. We furthermore establish the proposal from the extended version of the Nielsen-Ninomiya theorem, relating dissipative gapless modes to point-gap topological numbers. From the bulk-boundary correspondence of the point-gap topological phases, the resultant point-gap topological phases exhibit exceptional boundary states or in-gap higher-order non-Hermitian skin effects.

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Introduction.—There is increasing interest in the studies of non-Hermitian physics [1–3]. Among them, a recent trend is the study of topology in non-Hermitian systems [4–6]. The prime motivation for such a research direction is that both non-Hermitian and topological systems exhibit characteristic boundary phenomena [7–13]. Certain non-Hermitian systems show a boundary phenomenon called the non-Hermitian skin effect [12,14], where a macroscopic number of bulk states are localized at the boundary. On the other hand, the bulk-boundary correspondence [15], in which bulk topological invariants count the number of gapless boundary states, is one of the most notable concepts in topological systems [16,17]. In topological systems with the non-Hermiticity, both the non-Hermitian skin effect and the topological boundary states can coexist, and the bulk-boundary correspondence should hold in an unconventional manner [12,13,18–21].

The generalization of the topological classification to non-Hermitian systems is also of interest. In the original classification of Hermitian topological insulators and superconductors [22–25], the gapped topology is mathematically characterized by the absence of the energy eigenstates of the Hamiltonian at the Fermi energy, $E \neq E_{\mathrm{F}}$. The natural extension to non-Hermitian systems is real line-gap topology defined by $\mathrm{Re}(E - E_{\mathrm{F}}) \neq 0$ [9,26]. Mathematically, the real line-gapped Hamiltonians are smoothly deformed into Hermitian-gapped Hamiltonians without closing the real-line gap [4,9,26]. Therefore, the physical consequence of the real line-gapped topology is the bulk-boundary correspondence, as in the case of the Hermitian topological phases. More generally, the line-gapped spectrum is defined as a spectrum that does not cross a specific line in the complex plane [26]. For instance, if one chooses the real axis in the complex energy plane as the reference line, such a spectrum defines the imaginary line-gap topology, which is adiabatically connected to anti-Hermitian topological phases [26].

Remarkably, the non-Hermiticity enables another extension of topology, the point-gap topology defined by $E \neq E_{\mathrm{P}}$ [26,27]. Typically, a spectrum with nontrivial point-gap topology surrounds the reference point E_{P} in the complex energy plane, and thus the point-gap topology is distinct from any Hermitian-like topology. The topological classification of point-gapped Hamiltonians has been established [26,28], and the physical consequences of the point-gap topological phases have been explored [21,27,29–62]. In particular, it has been shown that the non-Hermitian skin effects originate from one-dimensional point-gap topological numbers, i.e., the spectral winding number [31,32] or the $\mathbb{Z}_2$ number [31]. Also, higher-dimensional point-gap topological phases may support non-Hermitian skin modes localized at topological defects, mimicking the anomaly-induced catastrophes [6,31,33,34,57]. Depending on the dimension and symmetry of the system, higher-dimensional point-gap topological phases may also host boundary modes [21,35–38,57], and one of such point-gap topological phases is called an exceptional topological insulator [37]. Furthermore, for the fundamental symmetry classes called AZ† classes (see below), the correspondence between d-dimensional point-gap topological phases and $(d-1)$-dimensional anomalous gapless modes was suggested [39], and later proved as the extended Nielsen-Ninomiya theorem [41].

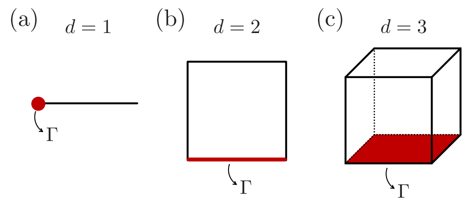

FIG. 1. Universal platforms of point-gap topological phases. d-dimensional topological insulators and superconductors are coupled to the environment at $x_d = 1$.

Whereas point-gap topological phases are responsible for these exceptional phenomena intrinsic to non-Hermitian systems, their realization in quantum materials is still elusive. This Letter proposes a simple and universal platform of point-gap topological phases in quantum materials. As illustrated in Fig. 1, the platform consists of a d-dimensional topological insulator or superconductor where one of the boundaries is coupled to the environment and thus dissipative. We show that the dissipation-induced decay constant of the topological boundary modes results in a $(d-1)$-dimensional nontrivial point-gap topological number, i.e., a $(d-1)$-dimensional point-gap topological phase. We also predict exceptional boundary states or in-gap higher-order non-Hermitian skin effects based on the bulk-boundary correspondence for point-gap topological phases [21,37].

Nontrivial topology from decay constant.—Let us start with a Chern insulator with the periodic boundary condition in the x direction and the open boundary conditions at $y = 1$, L_y in the y direction; see Fig. 2(a). The system supports a chiral edge mode at $y = 1$ and an antichiral edge mode at $y = L_y$. If we couple one of the open boundaries, say $y = 1$, to the environment, the chiral edge mode at $y = 1$ gets the decay constant in addition to the linear spectrum,

$$h_{\mathrm{edge}}(k_x) = vk_x - i\Gamma, \tag{1}$$

where k_x is the momentum in the x direction, $v(>0)$ is the group velocity, and $\Gamma(>0)$ is the decay constant. At first sight, the decay constant seems not to change the topology of the system, but it does change, as we see below.

To see the hidden topology due to the decay constant, we show that the complex spectrum in Eq. (1) is equivalently obtained by attaching a one-dimensional point-gap topological phase to the original Hermitian chiral edge state: the effective Hamiltonian of the attached system is

$$h_{\mathrm{attach}}(k_x) = \begin{pmatrix} vk_x & \Delta \\ \Delta^* & -v\sin k_x + i(\Gamma/2)\cos k_x - i(\Gamma/2) \end{pmatrix}, \tag{2}$$

where the diagonal components describe the chiral edge state and the one-dimensional point-gap topological phase, respectively, and Δ $(|\Delta| \ll 1)$ is the coupling between them. The one-dimensional point-gap topological phase is the tight-binding lattice model with asymmetric hoppings (Hatano-Nelson model [63–65]) and supports a nonzero spectral winding number. Remarkably, around $k_x = 0$, the attached system shows the spectrum $E_0(k_x) = \pm\sqrt{(vk_x)^2 + |\Delta|^2}$, and thus the original edge state has a gap in the real part of the spectrum, but there appears another chiral mode around $k_x = \pi$, $E_\pi(k_x) = v(k_x - \pi) - i\Gamma$. Therefore, by shifting the origin of the momentum space, the attached system reproduces the complex spectrum in Eq. (1). Since the attached system in Eq. (2) has a nonzero spectral winding number, the dissipative chiral edge state in Eq. (1) also should have the same nonzero winding number.

Whereas the above argument is rather heuristic, we also have a convincing discussion on the nontrivial topology: for a rigorous discussion, we assume that the decay constant induced on the boundary is uniform and thus retains the lattice translation symmetry along the edge (namely the x direction) [66]. Then, from the Bloch theorem, any energy eigenstate in the present model should be labeled by k_x, and we have the 2π periodicity in k_x for the energy eigenstates. This means that the chiral edge state at $y = 1$ and the antichiral edge one at $y = L_y$ should be exchanged when changing k_x by 2π since they cannot go back to themselves after the one period. As a result, they make a loop in the complex energy plane, as illustrated in Fig. 2. In the absence of dissipation, the loop sticks to the real axis of the complex energy plane, as in Fig. 2(a), which implies that the spectral winding number is zero. However, once the chiral edge mode at $y = 1$ has the imaginary part of the energy due to dissipation, the loop is extended to the imaginary energy direction so it immediately gets a nonzero spectral winding number as shown in Fig. 2(c). Therefore, the decay constant of the chiral edge mode

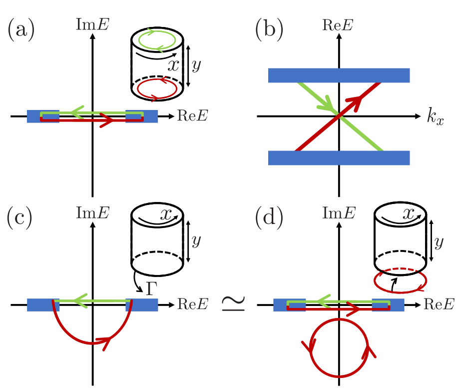

FIG. 2. (a),(b) Chern insulator under the periodic boundary condition in the x direction and the open boundary condition in the y direction. A chiral edge mode (red) at the bottom boundary and an antichiral edge mode (green) at the top boundary form a loop on the real axis in the complex energy plane. (c) Chern insulator (a) with the decay constant term Γ at the bottom boundary. (d) Chern insulator (a) attached the Hatano-Nelson model at the bottom boundary.

results in the nontrivial point-gap topology. We can also confirm that the decay of the chiral edge mode is topologically the same as the attachment of the one-dimensional Hatano-Nelson model to the boundary: the Hatano-Nelson model gives a loop spectrum in the complex energy plane in Fig. 2(d). Then through the reconnection of the spectra between the Hatano-Nelson model and the chiral edge mode, we smoothly obtain the complex spectrum in Fig. 2(c) [66].

We can make a concrete prediction due to the nontrivial spectral winding number from the decay constant. Namely, the system exhibits the non-Hermitian skin effect under the open boundary condition in the x direction. To check the prediction, we consider the Chern insulator modeled by Qi, Wu, and Zhang (QWZ) [70],

$$H_{\mathrm{QWZ}}(\boldsymbol{k}) = \sin k_x\sigma_x + \sin k_y\sigma_y + (m + \cos k_x + \cos k_y)\sigma_z, \quad (3)$$

which has the Chern number 1 for $-2 < m < 0$. Here, $\sigma_{\mu=0,x,y,z}$ are the Pauli matrices. When we impose the open boundary condition on the y direction and put the imaginary on-site potential $-i\Gamma\sigma_0$ along the boundary at $y = 1$, the chiral edge state gets a finite lifetime, as shown in Figs. 3(a) and 3(b). Then, if we further impose the open boundary condition on the x direction, the system shows the non-Hermitian skin effect; see Fig. 3(c). We have $O(L)$ corner skin modes shown in Fig. 3(d). This behavior entirely agrees with the prediction [66].

Universal platform of point-gap topological phases.—So far, we have considered a two-dimensional Chern insulator and have shown how to realize a one-dimensional point-gap topological phase from the Chern insulator. Now we generalize the idea to other topological insulators and superconductors. Let us consider a d-dimensional topological insulator or superconductor. Under the open boundary conditions at $x_d = 1$, L_d in the x_d direction, we have topological gapless boundary modes with an opposite topological charge at opposite boundaries. Then, if we couple one of the boundaries, say $x_d = 1$, to the environment, the topological gapless modes at $x_d = 1$ get a finite lifetime, namely the imaginary part of the spectrum, due to dissipation. As we will see shortly, this configuration realizes a $(d-1)$-dimensional point-gap topological phase.

To prove the above statement, we first clarify the symmetry of the system. The fundamental on-site symmetries for topological insulators and superconductors are time-reversal (TRS), particle-hole (PHS), and their combination, chiral symmetry (CS). These Altland-Zirnbauer (AZ) symmetries [69] protect topological gapless boundary modes of topological insulators and superconductors [22–25]. In the presence of the coupling to the environment, however, we cannot retain these symmetries in their original form. Causal fermionic theories require other on-site symmetries intrinsic to non-Hermitian systems [71,72], which we call AZ† symmetries [26]. The AZ† symmetries consist of TRS†, $H = U_T H^T U_T^\dagger$, $U_T U_T^* = \pm 1$; PHS†, $H = -U_C H^* U_C^\dagger$, $U_C U_C^* = \pm 1$; CS†, $H = -U_\Gamma H^\dagger U_\Gamma^\dagger$, $U_\Gamma U_\Gamma = 1$, where H is the Hamiltonian, and U_T, U_C, and U_Γ are unitary matrices [26]. For a Hermitian H, the AZ† symmetries coincide with the original AZ symmetries. The presence and absence of the AZ† symmetries define the tenfold AZ† classes [26]; see Table I. One can easily check that the on-site decay constant term $-i\Gamma\mathbf{1}$ due to dissipation respects the AZ† symmetries [66].

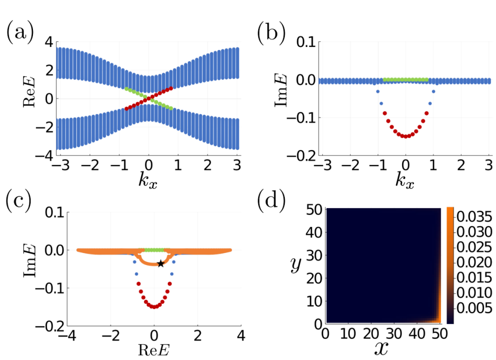

FIG. 3. The QWZ model ($m = -1.5$, $L_x = L_y = 50$) in Eq. (3) with $-i\Gamma\sigma_0$ term ($\Gamma = 0.2$) at the $y = 1$ boundary. (a),(b) The real and imaginary parts of the spectrum under the periodic boundary condition in the x direction. (c) Comparison of the complex energy spectra under the open boundary condition (orange) and the periodic boundary condition (other colors) in the x direction. (d) The skin mode with the energy $E = 0.30 - 0.04i$ [the star symbol in (c)].

TABLE I. Point-gap topological table for insulators and superconductors with dissipation at $x_d = 1$. Here, $\delta = (d-1) - D$, where d is the spatial dimension of the topological insulators and superconductors and D is the dimension of a sphere surrounding a topological defect. The topological defect goes through the x_d direction. The superscripts SE and BS indicate the topological numbers predicting non-Hermitian skin effects and boundary states, respectively; see also Ref. [21]. For $D = 0$, this table reproduces that for $(d-1)$-dimensional point-gap phases in AZ† classes in [26].

AZ† class	TRS†	PHS†	CS	$\delta=0$	$\delta=1$	$\delta=2$
A	0	0	0	0	$\mathbb{Z}^{\mathrm{SE}}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}^{\mathrm{BS}}$
AI†	+1	0	0	0	0	0
BDI†	+1	+1	1	$\mathbb{Z}$	0	0
D†	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}^{\mathrm{SE}}$	0
DIII†	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\mathrm{SE}}$	$(2\mathbb{Z}+1)^{\mathrm{SE}}$ $2\mathbb{Z}^{\mathrm{BS}}$
AII†	−1	0	0	0	$\mathbb{Z}_2^{\mathrm{SE}}$	$\mathbb{Z}_2^{\mathrm{SE}}$
CII†	−1	−1	1	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\mathrm{BS}}$
C†	0	−1	0	0	$2\mathbb{Z}^{\mathrm{SE}}$	0
CI†	+1	−1	1	0	0	$2\mathbb{Z}^{\mathrm{BS}}$

A critical mathematical result for our theory is the extended Nielsen-Ninomiya theorem [41], which holds for systems in the AZ† classes. The theorem relates the bulk gapless points at the energies E_α with the topological charges ν_α to the point-gap topological number n at the reference energy E_P,

$$n(E_\mathrm{P}) = \sum_{\mathrm{Im}(E_\alpha - E_\mathrm{P})>0} \nu_\alpha = -\sum_{\mathrm{Im}(E_\alpha - E_\mathrm{P})<0} \nu_\alpha, \tag{4}$$

where the index α labels the gapless bulk states. This theorem implies that if topological gapless states have different lifetimes, there exists a region of E_P where n is nonzero, namely we have a point-gap topological phase characterized by n.

Now we come back to our system, i.e., a d-dimensional topological insulator or superconductor with the open boundary condition in the x_d direction. Our system belongs to an AZ† symmetry class when the system is coupled to the environment at $x_d = 1$. Regarding the site index in the x_d direction as an internal degree of freedom, we can identify the system as a $(d-1)$-dimensional system with "bulk" gapless states with the internal index $x_d = 1$, L_d. Then, by the coupling to the environment at $x_d = 1$, the $(d-1)$-dimensional bulk gapless states at $x_d = 1$ have a different decay constant than those at $x_d = L_d$. Therefore, from the extended Nielsen-Ninomiya theorem in Eq. (4), there exists a region of E_P where the $(d-1)$-dimensional point-gap topological number n becomes nonzero. The nonzero value of n is given by the d-dimensional bulk topological number of the original topological insulator or superconductor since the total topological charge of the gapless states at $x_d = 1$ coincides with it up to sign.

Predictions.—The nonzero $(d-1)$-dimensional point-gap topological number n gives rise to several consequences in the physical properties of the system. First, it predicts the appearance of $(d-2)$-dimensional boundary modes or skin modes when imposing the additional open boundary condition on a different direction than x_d, say the x_{d-1} direction [21]. For $d = 2$, the nonzero n predicts a second-order non-Hermitian skin effect like the Chern insulator case in Fig. 3. The second-order skin modes form a generalized Kramers pair [26,73] when the original system has fermionic time-reversal symmetry. For $d = 3$, an odd n in class DIII† (time-reversal invariant topological superconductor) and a nontrivial n in class AII† (topological insulator) imply the in-gap non-Hermitian skin effects. Still, other nonzero ns predict boundary modes (see Table I with $D = 0$ introduced below).

Second, the proposed system also may have similar localized modes in the presence of topological defects. The topological defects should go through the x_d direction since our theory treats the site index in the x_d direction as an internal degree of freedom. Then, we can obtain the point-gap topological table in the presence of such topological defects by generalizing the argument by Teo and Kane [74] to point-gap topological phases; see Table I. In the new table, the space dimension $d-1$ of the point-gap topological phases is replaced by $\delta = (d-1) - D$ (≥ 0), where D $(\leq d-1)$ is the dimension of a sphere surrounding the topological defect. In particular, if we insert the π flux $(D = 1)$ in the x_3 direction through a three-dimensional time-reversal invariant superconductor with an odd number of the three-dimensional winding number or a three-dimensional time-reversal invariant topological insulator, we have a nontrivial $\mathbb{Z}_2^{\mathrm{SE}}$ number for class DIII† or class AII† with $\delta = 1$, in the presence of dissipation at $z = 1$. Thus, we obtain the non-Hermitian skin modes localized on the π flux.

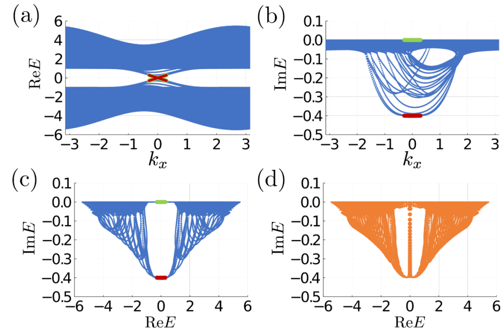

FIG. 4. 3D chiral symmetric topological insulator ($L_x = 300$, $L_y = L_z = 20$) in Eq. (5) with $-i\Gamma\sigma_0\tau_0$ $(\Gamma = 0.2)$ term at the $z = 1$ boundary. (a)–(c) The real and imaginary parts of the spectrum under the periodic boundary conditions in the x and y directions. (d) The complex spectrum under the periodic boundary condition in the x direction and the open boundary condition in the y direction. There appear in-gap boundary modes.

Finally, the point-gap topology also stabilizes the original topological boundary modes of the topological insulator or superconductor. In general, the topological boundary modes of the topological insulator or superconductor have tiny gaps because of the mixing with those at an opposite boundary. Therefore, for finite L_d, they are not always gapless and do not have well-defined topological charges in a mathematically rigorous sense. In contrast, when the point-gap topological number n becomes nontrivial, the extended Nielsen-Ninomiya theorem in Eq. (4) ensures the well-defined topological charges ν_α, which implies that the mixing disappears and the tiny gaps close. Indeed, the gapless modes at $x_d = 1$ and those at $x_d = L_d$ have different imaginary parts of the energy, and thus they do not mix. The gap closing of the boundary modes should be observed in high-resolution spectrum-sensitive experiments and sharpens the topological phenomena of the boundary modes.

Examples.—We check the validity of our scheme in various topological materials. For $d = 1$, the dissipation effect for a superconducting nanowire has been discussed in literature [33,75–78]. The coupling of a Majorana end state to the environment is shown to give a nontrivial zero-dimensional point-gap $\mathbb{Z}_2$ number [33]. It has been also demonstrated that the dissipation stabilizes the Majorana end state [78]. For $d = 2$, we have already shown above that our theory for a Chern insulator gives the second-order non-Hermitian skin effect. For $d = 3$, exactly the same scheme was discussed for a three-dimensional time-reversal invariant topological insulator [6], which showed that non-Hermitian skin modes appear in the π flux [see Fig. 5(b) in Ref. [6]]. In Fig. 4, we also show the result for the three-dimensional chiral symmetric topological insulator (CSTI),

$$H_{\mathrm{CSTI}}(\boldsymbol{k}) = [\sin k_x + (1 - \cos k_y)]\sigma_x\tau_x + \sin k_y\sigma_y\tau_x + \sin k_z\sigma_z\tau_x + \left(-2 + \sum_{i=x,y,z} \cos k_i\right)\sigma_0\tau_y, \quad (5)$$

where σ_μ and τ_μ are the Pauli matrices, CS is $U_\Gamma = \sigma_0\tau_z$, and the on-site dissipation term $-i\Gamma\sigma_0\tau_0$ is placed at $z = 1$. The system realizes the $\delta = 2$ ($d = 3$, $D = 0$) AIII class with $\mathbb{Z}^{\mathrm{BS}} = 1$. Under the open boundary conditions in both z and y directions, the system hosts a boundary state inside the point gap, as expected [66].

Summary.—We propose a universal platform for point-gap topological phases constructed from topological insulators and superconductors. Using various independent arguments, we establish that dissipation on a boundary of d-dimensional topological materials results in $(d-1)$-dimensional point-gap topological phases. We also confirm the validity of our proposal for various topological materials.

Our scheme applies to any topological materials in the original topological periodic table [22–25]. For instance, by connecting a metal to an edge of a quantum Hall state in graphene, we can realize a point-gap topological phase similar to Fig. 3. The resulting non-Hermitian skin effect can be observed as the chiral tunneling effect [79]. Another candidate is a topological superconducting nanowire with a Zeeman field [80]. By coupling a lead to one of the ends of the nanowire, the system displays a point-gap topological phase in 0D class $\mathrm{D}^\dagger$. We can also use the topological insulator $\mathrm{Bi_2Se_3}$ and the variants [81] to similarly realize a point-gap topological phase in 2D class $\mathrm{AII}^\dagger$.

Note added.— A part of the present work was reported in [82]. We are aware of related works [83,84] after the completion of this work.

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