Data-Driven Determination of the Light-Quark Connected Component of the Intermediate-Window Contribution to the Muon $g-2$

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We present the first data-driven result for $a_\mu^{\mathrm{win,lqc}}$, the isospin-limit light-quark connected component of the intermediate-window Hadronic-vacuum-polarization contribution to the muon anomalous magnetic moment. Our result, $(198.8 \pm 1.1) \times 10^{-10}$, is in significant tension with eight recent mutually compatible high-precision lattice-QCD determinations, and provides enhanced evidence for a puzzling discrepancy between lattice and data-driven determinations of the intermediate-window quantity, one driven largely by a difference in the light-quark connected component.

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Since the pioneering work of Schwinger [1], and the subsequent experimental confirmation of his result [2], the study of the lepton magnetic moments, $\vec{\mu}$, has played a central role in the development of QED and, later, in that of the standard Model (SM) of particle physics. The anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$, defined in terms of the muon charge q, mass m_μ, and spin $\vec{s}$ as

$$\vec{\mu} = g_\mu \left(\frac{q}{2m_\mu}\right)\vec{s} \tag{1}$$

is, at present, one of the most accurately known quantities in physics. The new 2021 and 2023 Fermilab E989 measurements of a_μ, performed by the Muon $g-2$ Collaboration, reached a precision of 0.20 ppm [3–5], and are fully compatible with the previous Brookhaven National Laboratory E821 experiment results [6]. As a consequence, the experimental average of a_μ is known nowadays to 0.19 ppm. In order to stringently test the SM, the theory prediction for this quantity must reach a similar level of accuracy.

In 2020, the $g-2$ Theory Initiative released a white paper (WP) [7] where, based on the works of Refs. [8–31], the SM expectation for a_μ was determined to 0.37 ppm. The SM result is a sum of pure QED, electroweak and Higgs physics, hadronic vacuum polarization (HVP), and hadronic light-by-light scattering contributions. The latter two, which involve QCD, are particularly difficult to assess. The final uncertainty in the WP SM result for a_μ is strongly dominated by the HVP contribution, a_μ^{HVP}. It is thus a fundamental task to control this contribution as well as possible.

The a_μ assessment of the WP is based on data-driven evaluations of the HVP contribution [12–15]. As is well known, this result showed a 4.2σ tension with the then-current experimental average, motivating many works aimed at finding potential beyond-the-SM explanations for this disagreement. Many developments, however, have taken place since the publication of the WP assessment. In particular, the Budapest-Marseille-Wupertal (BMW) Collaboration has published a complete, subpercent lattice-QCD evaluation of a_μ^{HVP} [32]. If this result is used in the SM assessment of a_μ, the outcome is compatible with the experimental average at the 1.5σ level. In this situation, in order to conclude whether the discrepancy between the SM assessment(s) of a_μ and experimental results is due to beyond-the-SM effects, it is, first of all, crucial to understand the discrepancy between data-driven and lattice-QCD–based HVP results.

The complete computation of the HVP in lattice QCD is a very challenging task and, as of today, only the BMW result is sufficiently precise to allow for a detailed comparison with data-driven HVP determinations. This has motivated the introduction of the window quantities by the

RBC-UKQCD Collaboration [33]. The so-called intermediate window, which cuts out short- and long-distance regions of the $a_\mu^{\rm HVP}$ integral on the lattice-QCD side, is of particular importance because it significantly suppresses the associated systematic lattice uncertainties related to continuum limit extrapolation and finite volume effects. The intermediate-window contribution to $a_\mu^{\rm HVP}$, henceforth $a_\mu^{\rm win}$, is now known, with excellent precision and very good agreement between results from different groups, from four different lattice-QCD collaborations [32,34–36]. A recent data-driven assessment of $a_\mu^{\rm win}$ using electroproduction input [37], however, shows an important tension with these lattice results (see, e.g., Ref. [38]). (This tension is reduced if, instead, one uses τ data for the two-pion contribution [39], estimating the required model-dependent isospin-breaking (IB) corrections with the models discussed in Ref. [40].)

The lattice evaluation of $a_\mu^{\rm HVP}$ is split into several building blocks with the dominant contributions arising from the isospin-limit (defined by taking $m_\pi = m_{\pi^0}$) light- and strange-quark connected and disconnected parts, with additional, smaller, contributions from charm and bottom quarks. IB effects, both of electromagnetic (EM) and strong (SIB) origin, are accounted for perturbatively, keeping terms to first order in an expansion in the fine-structure constant α and the up-down quark-mass difference $m_u - m_d$. The light-quark connected (lqc) contribution to $a_\mu^{\rm win}$ in the isospin symmetric limit, denoted $a_\mu^{\rm win,lqc}$, is known now with very good precision from eight different lattice determinations (see the blue data points in Fig. 1). These eight determinations are all in excellent agreement and have small relative errors, ranging from 0.3% to 1.1%. Since this contribution gives about 87% of $a_\mu^{\rm win}$, and appears to be under good control, given the agreement among the eight different lattice determinations, it is highly desirable to obtain a precise data-driven estimate of $a_\mu^{\rm win,lqc}$ in order to further scrutinize the discrepancy between lattice-QCD and data-driven determinations of $a_\mu^{\rm HVP}$. It is the aim of this Letter to present this estimate.

We turn now to a short review of the theoretical framework for our data-driven determination of $a_\mu^{\rm win,lqc}$. To be able to compute $a_\mu^{\rm HVP}$ on the lattice-QCD side, one determines the Euclidean-time zero-momentum two-point correlation function given by

$$\begin{aligned} C(t) &= \frac{1}{3}\sum_{i=1}^{3}\int d^3x \langle j_i^{\rm EM}(\vec{x},t) j_i^{\rm EM}(0)\rangle \\ &= \frac{1}{2}\int_{m_\pi^2}^{\infty} ds \sqrt{s} e^{-\sqrt{s}t}\rho_{\rm EM}(s) \qquad (t>0), \end{aligned} \tag{2}$$

where m_π is the neutral pion mass, $j_\mu^{\rm EM}(x)$ is the EM current, and $\rho_{\rm EM}$ is the associated inclusive hadronic spectral function. In terms of $C(t)$, the intermediate-window contribution to $a_\mu^{\rm HVP}$ is given by

$$a_\mu^{\rm win} = 2\int_0^\infty dt\, w(t) W_{\rm win}(t) C(t), \tag{3}$$

where the function $w(t)$ can be obtained from its counterpart in s space [45] and $W_{\rm win}(t)$ is the weight function associated with the RBC-UKQCD intermediate-window [33], defined as

$$W_{\rm win}(t) = \frac{1}{2}\left(\tanh\frac{t-t_0}{\Delta} - \tanh\frac{t-t_1}{\Delta}\right), \tag{4}$$

with $t_0 = 0.4$ fm, $t_1 = 1.0$ fm, and $\Delta = 0.15$ fm. The corresponding expression for $a_\mu^{\rm HVP}$ is obtained by removing the factor $W_{\rm win}(t)$ from Eq. (3). Because of the presence of $W_{\rm win}(t)$ in Eq. (3) the short- and long-distance contributions to the integral are strongly suppressed.

Since we are concerned with the data-driven determination of $a_\mu^{\rm win,lqc}$, we need the data-driven (or dispersive) counterpart to Eq. (3), which is

$$a_\mu^{\rm win} = \frac{4\alpha^2 m_\mu^2}{3}\int_{m_\pi^2}^{\infty} ds \frac{\hat{K}(s)}{s^2}\tilde{W}_{\rm win}(s)\rho_{\rm EM}(s), \tag{5}$$

where $\hat{K}(s)$ is a well-known, slowly varying kernel function [46,47] (see Ref. [7] for the explicit expression) and $\tilde{W}_{\rm win}(s)$ is the s-space representation of $W_{\rm win}(t)$,

$$\tilde{W}_{\rm win}(s) = \frac{\int_0^\infty dt\, W_{\rm win}(t) w(t) e^{-\sqrt{s}t}}{\int_0^\infty dt\, w(t) e^{-\sqrt{s}t}}. \tag{6}$$

Here, when evaluating the different contributions to Eq. (5) in the exclusive-mode region, we employ the data compilation of Refs. [13,15] (KNT19 in what follows).

Our goal is to isolate the lqc contribution to the full result of Eq. (5). This can be achieved employing an idea first implemented with sufficient precision in Refs. [48,49], where the foundations of our method are laid out in detail. We start from the usual decomposition of the three-flavor EM current into its $I = 1$ and $I = 0$ parts, which produces analogous decompositions of $C(t)$ and $\rho_{\rm EM}$ into $I = 1$, $I = 0$ and mixed-isospin (MI) parts. In the isospin limit, the contribution associated with the $I = 1$ current contains only light-quark connected contributions and one has

$$\rho_{\rm EM}^{\rm lqc} = \frac{10}{9}\rho_{\rm EM}^{I=1}(s). \tag{7}$$

The data-driven estimate of $a_\mu^{\rm win,lqc}$ thus requires the identification of the $I = 1$ component of $\rho_{\rm EM}$. This can be accomplished, assuming isospin symmetry and using KNT19 data, on a channel-by-channel basis, in the KNT19 exclusive-mode region, $\sqrt{s} \leq 1.937$ GeV.

There are two classes of such contributions. The first, and dominant one, consists of contributions from modes with well-defined, positive G parity. Such modes have $I = 1$ and thus contribute to the lqc component in the isospin limit. Contributions from such "unambiguous" modes constitute the main ingredient in our determination of $a_\mu^{\mathrm{win,lqc}}$. The remaining contributions come from higher-threshold modes with no well-defined G parity. For these "ambiguous" modes, and especially for the dominant such channels—$K\bar{K}$ and $K\bar{K}\pi$—one resorts to external information, whenever available, in order to identify, as accurately as possible, the $I = 1$ component of the experimental $I = 0 + 1$ sum. The third ingredient is perturbative QCD supplemented with an estimate of duality violation contributions, which we use in the inclusive region, $\sqrt{s} > 1.937$ GeV. Finally, to the sum of the results thus obtained one must apply IB corrections since the experimental data, inevitably, contain IB contributions. Only after these corrections have been applied can the data-based result be compared directly with isospin-limit lattice $a_\mu^{\mathrm{win,lqc}}$ results. We now detail how we treat each of the four aforementioned contributions.

We start with the unambiguous modes, which give the dominant contribution to $a_\mu^{\mathrm{win,lqc}}$. The results are obtained with Eq. (5) using the KNT19 spectra for the different exclusive-mode contributions to $\rho_{\mathrm{EM}}(s)$. There are 13 such $I = 1$ channels, with the largest contribution, by far, arising from the $\pi^+\pi^-$ channel, which contributes $144.15(49) \times 10^{-10}$ to $[a_\mu^{\mathrm{win}}]_{I=1}$. The results for the different modes are given in Table I. The sum over all G-parity-unambiguous modes gives a total $I = 1$ contribution to a_μ^{win} of $168.24(72) \times 10^{-10}$. From Eq. (7), the final contribution of all unambiguous channels to $a_\mu^{\mathrm{win,lqc}}$ is then

$$[a_\mu^{\mathrm{win,lqc}}]_{G-\mathrm{par}} = 186.93(80) \times 10^{-10}. \tag{8}$$

TABLE I. Contributions from G-parity unambiguous modes to a_μ^{win} for $\sqrt{s} \le 1.937$ GeV obtained from KNT19 [15] exclusive-mode spectra. All entries in units of 10^{-10}.

$I = 1$ modes X	$[a_\mu^{\mathrm{win}}]_X \times 10^{10}$
Low-s $\pi^+\pi^-$	0.02(00)
$\pi^+\pi^-$	144.13(49)
$2\pi^+2\pi^-$	9.29(13)
$\pi^+\pi^-2\pi^0$	11.94(48)
$3\pi^+3\pi^-$ (no ω)	0.14(01)
$2\pi^+2\pi^-2\pi^0$ (no η)	0.83(11)
$\pi^+\pi^-4\pi^0$ (no η)	0.13(13)
$\eta\pi^+\pi^-$	0.85(03)
$\eta 2\pi^+2\pi^-$	0.05(01)
$\eta\pi^+\pi^-2\pi^0$	0.07(01)
$\omega(\to \pi^0\gamma)\pi^0$	0.53(01)
$\omega(\to \mathrm{npp})3\pi$	0.10(02)
$\omega\eta\pi^0$	0.15(03)
Total	168.24(72)

We turn next to our treatment of ambiguous-mode contributions, which follows the general strategy outlined in Sec. IV of Ref. [49]. For some of these contributions, notably those of the numerically dominant $K\bar{K}$ and $K\bar{K}\pi$ channels, external experimental information can be used in separating the desired $I = 1$ component from the experimental $I = 0 + 1$ sum. Modes for which external experimental information is not available have much smaller contributions. For these modes, one employs a maximally conservative $I = 1/0$ separation, based on the observation that the $I = 1$ part of the mode-X contribution to $\rho_{\mathrm{EM}}(s)$ must lie between 0 and the full $I = 0 + 1$ contribution obtained from the KNT19 spectrum for that mode. The contribution of ambiguous-mode X to $a_\mu^{\mathrm{win,lqc}}$ lies, therefore, in the following range:

$$[a_\mu^{\mathrm{win,lqc}}]_X = \frac{10}{9}\left(\frac{1}{2} \pm \frac{1}{2}\right)[a_\mu^{\mathrm{win}}]_X = \left(\frac{5}{9} \pm \frac{5}{9}\right)[a_\mu^{\mathrm{win}}]_X. \tag{9}$$

Let us discuss in some detail the significant ambiguous-mode contribution arising from the $K\bar{K}$ channels, K^+K^- and $K^0\bar{K}^0$. Independent experimental information on the $K\bar{K}$ contribution to the purely $I = 1$ spectral function can be obtained from the *BABAR* spectrum for the decay $\tau \to K^-K^0\nu_\tau$ [50]. Using the conserved vector current relation, these results can be used to determine the $I = 1$ $K\bar{K}$ contribution to $\rho_{\mathrm{EM}}(s)$ up to $s = 2.7556\ \mathrm{GeV}^2$, and hence the associated contribution to $[a_\mu^{\mathrm{win,lqc}}]_{K\bar{K}}$, which, using Eq. (7), is found to be $10/9 \times 0.465(29) \times 10^{-10}$. For $s > 2.7556\ \mathrm{GeV}^2$, the $I = 1$ part is found using KNT19 data and the maximally conservative treatment of Eq. (9). We find, for $s > 2.7556\ \mathrm{GeV}^2$, a contribution of $10/9 \times 0.055(55) \times 10^{-10}$ to $[a_\mu^{\mathrm{win,lqc}}]_{K\bar{K}}$. From these results one obtains, for the full exclusive-region $K\bar{K}$ contribution,

$$[a_\mu^{\mathrm{win,lqc}}]_{K\bar{K}} = 0.578(69) \times 10^{-10}. \tag{10}$$

A similar treatment of the $K\bar{K}\pi$ modes is possible thanks to the Dalitz plot analysis of *BABAR*, which provides a separation of the $I = 1$ and $I = 0$ contributions to the $K\bar{K}\pi$ cross sections [51]. Integrating the *BABAR* $I = 1$ result, we find

$$[a_\mu^{\mathrm{win,lqc}}]_{K\bar{K}\pi} = 0.521(86) \times 10^{-10}. \tag{11}$$

For the $K\bar{K}2\pi$ modes, only a small improvement is possible over the maximally conservative treatment. This is obtained by first subtracting the small $I = 0$ $\phi[\to K\bar{K}]\pi\pi$ contribution implied by *BABAR* $e^+e^- \to \phi\pi\pi$ cross sections [52], and applying the maximally conservative treatment only to the residual $I = 0 + 1$ sum. This leads to the result

$$[a_\mu^{\mathrm{win,lqc}}]_{K\bar{K}2\pi} = 0.60(60) \times 10^{-10}. \tag{12}$$

The very small (often completely negligible) contributions of the remaining ambiguous modes [$K\bar{K}3\pi$, $\omega(\to \mathrm{npp})K\bar{K}$,

$\eta(\to \mathrm{npp})K\bar{K}$ (no ϕ), $p\bar{p}$, $n\bar{n}$, and low-s $\pi^0\gamma$ and $\eta\gamma$] (npp = nonpurely pionic) are obtained from the KNT19 spectra using the maximally conservative separation of Eq. (9). The total contribution from all G-parity-ambiguous exclusive modes is, finally,

$$[a_\mu^{\mathrm{win,lqc}}]_{\mathrm{amb}} = 1.74(61) \times 10^{-10}, \qquad (13)$$

with $1.70(61) \times 10^{-10}$ from $K\bar{K}$, $K\bar{K}\pi$, and $K\bar{K}2\pi$.

In the inclusive region, $\sqrt{s} > 1.937$ GeV, we use QCD perturbation theory, which is known to $\mathcal{O}(\alpha_s^4)$, supplemented with an estimate for the $\mathcal{O}(\alpha_s^5)$ coefficient, as described in Refs. [48,49]. To this result, we add an estimate of the duality violation (DV) contribution, obtained from our previous study of the $I = 1$ hadronic spectral function in $\tau \to \mathrm{hadrons} + \nu_\tau$ [53], using the parametrization of DVs discussed in Refs. [54–56]. Since DVs represent a fundamental limitation of perturbation theory, we use the resulting central value as the total uncertainty on the perturbative contribution. This enlarges the uncertainty of the perturbative contribution without DVs by a factor of about 10, and should provide a very conservative assessment. The resulting inclusive-region contribution is then

$$[a_\mu^{\mathrm{win,lqc}}]_{\mathrm{pt.QCD+DVs}} = 11.06(16) \times 10^{-10}. \qquad (14)$$

The fourth and final ingredient in the determination of $a_\mu^{\mathrm{win,lqc}}$ is an evaluation of the EM and SIB contributions to be subtracted from the data-based results obtained above before comparison with isospin-symmetric lattice-QCD results. The general strategy employed for this subtraction is detailed in Refs. [48] and [48,49]. The main observation is that, to first order in IB, SIB is present only in the MI component of $\rho_{\mathrm{EM}}(s)$. EM IB, on the other hand, occurs in all of the pure $I = 1/0$ and MI components. The IB correction to $a_\mu^{\mathrm{win,lqc}}$ then contains two parts. The first, which appears in the pure $I = 1$ component, is of EM origin. No breakdown of this correction into individual exclusive-mode contributions is required; an inclusive determination is sufficient. The situation for the MI contribution is different since we must estimate the MI contamination on a channel-by-channel basis, removing from the "nominally" $I = 1$ results above the component that arises from $\rho_{\mathrm{EM}}^{\mathrm{MI}}(s)$. These contributions are expected to be dominated by the 2π mode through $\rho - \omega$ mixing in the process $e^+e^- \to \omega \to \rho \to 2\pi$.

At present, given the absence of complete data-driven estimates for some potentially important components of the pure $I = 1$ EM IB contribution (see, e.g., the discussion in the Appendix of Ref. [48]), we are forced to rely on the lattice, and employ for this correction the result obtained by BMW in Ref. [32],

$$\Delta_{\mathrm{EM}} a_\mu^{\mathrm{win,lqc}} = 0.035(59) \times 10^{-10}. \qquad (15)$$

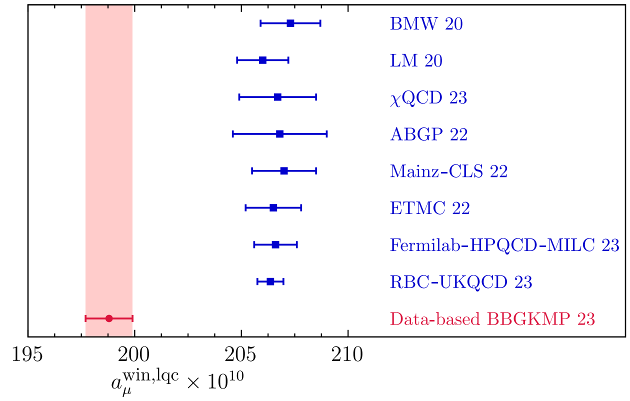

FIG. 1. Comparison of our final result (BBGKMP 23), Eq. (17), with lattice results for $a_\mu^{\mathrm{win,lqc}}$ from [32] (BMW 20), [41] (LM 20), [42] (χQCD 23), [43] (ABGP 22), [34] (Mainz-CLS 22), [35] (ETMC 22), [44] (FHM 23), and [36] (RBC-UKQCD 23).

This correction is very small, given the size of other uncertainties, and we will neglect it in what follows.

The MI contamination to the 2π exclusive mode was obtained in Ref. [57] from 2π electroproduction data fitting a dispersive representation of the pion form factor that includes $\rho - \omega$ mixing. The 2π MI component is found to be $[a_\mu^{\mathrm{win}}]_{\pi\pi}^{\mathrm{MI}} = 0.83(6) \times 10^{-10}$, which is about 0.6% of the total 2π contribution to a_μ^{win}. Since the MI components of other nominally $I = 1$ modes have no analogous narrow-resonance enhancements, we consider it very safe to assume that their MI total will not exceed 1% of the sum, 25.68×10^{-10}, of their contributions. To account for the total of the non-2π-mode MI contaminations we thus add an uncertainty of 0.26×10^{-10} to the 2π results of Ref. [57]. Using Eq. (7), this leads to a MI correction of

$$\Delta^{\mathrm{MI}} a_\mu^{\mathrm{win,lqc}} = -0.92(7)_{2\pi}(29)_{\mathrm{non}-2\pi} \times 10^{-10}. \qquad (16)$$

We are now in a position to obtain our final data-driven estimate for $a_\mu^{\mathrm{win,lqc}}$. Adding the contributions from Eqs. (8), (13), (14), and applying the IB correction of Eq. (16), we find, as our final result,

$$a_\mu^{\mathrm{win,lqc}} = 198.8(1.1) \times 10^{-10}. \qquad (17)$$

In Fig. 1, we compare our data-driven estimate with the lattice-QCD results from eight different collaborations. The tension between the data-driven and lattice results is striking. Assuming, for simplicity, all errors to be Gaussian, we find tensions ranging from 3.3σ to 6.1σ. Our result indicates that the discrepancy between data-driven and lattice-QCD results for a_μ^{win} is almost entirely due to the light-quark connected contribution, which, in turn, is strongly dominated by the 2π channel, accounting for about 81% of the result of Eq. (17). Given this 2π dominance, it is relevant to note that recent CMD-3 results for the $e^+e^- \to \pi^+\pi^-$ cross sections [58], which are in significant tension with those of earlier experiments, and known to significantly increase the 2π contribution to a_μ^{HVP}, would similarly increase our result for $a_\mu^{\mathrm{win,lqc}}$, making it more compatible with lattice determinations. Since the source of the disagreements between the previously published and new CMD-3 2π results is both presently unclear and the subject of ongoing study, we refrain from addressing this issue more quantitatively for now.

We note that our final result is based on the KNT19 data compilation. An equivalent analysis using other dispersive evaluations (e.g., DHMZ data [12,14]) would be desirable. We remark, however, that for the lqc contribution to a_μ^{HVP}, which can be obtained based on publicly available results, KNT19- and DHMZ-based estimates are in very good agreement [48].

In a forthcoming publication we will present results for several other window quantities, including both the light-quark-connected and strange-quark-plus-all-disconnected contributions. The latter require the treatment of the $I = 0$ sector. The impact of new phenomenological estimates of MI IB corrections in the 3π channel [59] will be discussed and we will present a comparison between lattice-QCD and phenomenological IB corrections. A preliminary estimate of the potential impact of new CMD-3 results for $e^+e^- \to \pi^+\pi^-$ cross section [58] will also be given.

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