Complete Hilbert-Space Ergodicity in Quantum Dynamics of Generalized Fibonacci Drives

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Ergodicity of quantum dynamics is often defined through statistical properties of energy eigenstates, as exemplified by Berry's conjecture in single-particle quantum chaos and the eigenstate thermalization hypothesis in many-body settings. In this work, we investigate whether quantum systems can exhibit a stronger form of ergodicity, wherein any time-evolved state uniformly visits the entire Hilbert space over time. We call such a phenomenon *complete Hilbert-space ergodicity* (CHSE), which is more akin to the intuitive notion of ergodicity as an inherently dynamical concept. CHSE cannot hold for time-independent or even time-periodic Hamiltonian dynamics, owing to the existence of (quasi)energy eigenstates which precludes exploration of the full Hilbert space. However, we find that there exists a family of aperiodic, yet deterministic drives with minimal symbolic complexity—generated by the Fibonacci word and its generalizations—for which CHSE can be proven to occur. Our results provide a basis for understanding thermalization in general time-dependent quantum systems.

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One of the cornerstones of statistical physics is the concept of ergodicity: it provides a mechanism by which a generic physical system achieves equilibrium, allowing for a simple statistical description of otherwise complex dynamics. First put forth by Boltzmann for classical systems, ergodicity refers to the property wherein all available states are explored over time, irrespective of the initial configuration [1]. For quantum systems, such a dynamical formulation of ergodicity is, however, incongruous with the existence of stationary solutions to Schrödinger's equation, such as energy eigenstates. Instead, quantum ergodicity is often defined through the statistical properties of stationary states. In single-particle chaotic systems, Berry's conjecture states that highly excited energy eigenstates are locally indistinguishable from a superposition of random plane waves [2]; and in interacting many-body systems, the eigenstate thermalization hypothesis (ETH) states that most eigenstates behave like random vectors at the level of local observables [3–7].

This formulation of quantum ergodicity, centered around stationary states of dynamics, is not fully satisfactory. For quantum systems governed by general time-dependent Hamiltonians, stationary states are typically not well defined [8,9]. In fact, energy eigenstates are guaranteed to exist only for systems with time-independent, or time-periodic Hamiltonians [10]. This immediately begs the following questions: is it possible to define a notion of quantum ergodicity that is suitable for a closed quantum system with general time dependence? Do such systems equilibrate or thermalize, and if so, what is the underlying mechanism?

In this Letter, we report progress towards understanding quantum ergodicity in general time-dependent Hamiltonian dynamics, leveraging concepts from quantum information theory. We propose a stronger form of quantum ergodicity that better captures Boltzmann's original notion of ergodicity as a dynamical property, without reference to stationary states. Furthermore, we provide a class of simple physical systems where such ergodicity holds.

We define *complete Hilbert-space ergodicity* (CHSE) as a property of quantum dynamics for which the evolution of *any* initial state uniformly explores every point of its Hilbert space over time, as depicted in Fig. 1. Equivalently, it is the condition that a time-evolved wave function is statistically indistinguishable from random vectors sampled uniformly

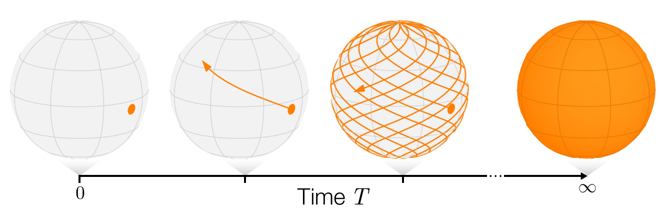

FIG. 1. Main idea of complete Hilbert-space ergodicity, illustrated for the case of a single qubit: *any* initial state (orange disk) explores the Bloch sphere uniformly over time.

from the Hilbert space. Intuitively, one expects dynamics under a random sequence of unitaries to exhibit CHSE, since any wave function will almost surely cover the entire Hilbert space over time. This is analogous to saying Brownian motion is ergodic [11]. A more interesting question is whether the same level of ergodicity can be achieved in systems with simple time dependence. Intriguingly, we find the answer in the affirmative.

We prove that CHSE holds in a large family of aperiodic yet deterministic drives, obtained by sequentially applying one of two fixed unitaries in a certain order. This order is determined by simple concatenation rules that generate the well-known Fibonacci word and its variants. Importantly, the complexity of the time dependence is the minimal possible for CHSE, in a quantifiable way, in contrast to the maximally complex random drives.

Our work has several physical and conceptual implications. For a many-body system displaying CHSE, it follows that a local subsystem necessarily achieves thermalization to infinite temperatures for almost all late times [12]. In fact, it also implies a stronger form of universality that has been recently introduced, called deep thermalization [13–22]. Furthermore, we speculate that quantum systems exhibiting CHSE constitute arenas where the growth of circuit complexity may be rigorously investigated, a subject of much interest in the quantum information community [28,29].

Complete Hilbert-space ergodicity.—Consider a quantum state $|\psi(0)\rangle$ in a finite d-dimensional Hilbert space, undergoing dynamics $|\psi(0)\rangle \mapsto |\psi(t)\rangle$ by some Hamiltonian $H(t)$ or sequence of unitary gates. Our formalism below is agnostic of the underlying structure or basis of the Hilbert space. It could be, for example, a fermionic or bosonic Fock space with a fixed number of particles, or a tensor product of many qubits or spins. We would like to inquire if the state "explores" its ambient space equally likely in time. Mathematically, this can be captured by asking whether the infinite-time average of any integrable function f of the time-evolved state equals its uniform average over states in the Hilbert space,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\, f(|\psi(t)\rangle\langle\psi(t)|) = \int \mathrm{d}\phi\, f(|\phi\rangle\langle\phi|), \quad (1)$$

where $\mathrm{d}\phi$ is the unique unitarily invariant measure on the Hilbert space, induced by the Haar measure on the unitary group $\mathrm{SU}(d)$. If Eq. (1) is true for any initial state $|\psi(0)\rangle$, then we deem the quantum dynamics as completely Hilbert-space ergodic. Equation (1) is reminiscent of the celebrated Birkhoff's ergodic theorem, which is generally applied to classical dynamical systems [30]. We are proposing to adopt Eq. (1) as a definition of ergodicity for closed quantum dynamics.

We systematically characterize CHSE by considering polynomial functions of degree k. For example, the quadratic function $f_O(|\psi(t)\rangle\langle\psi(t)|) = \langle\psi(t)|O|\psi(t)\rangle^2$ gives us information about the temporal fluctuation of an observable O. Equality of Eq. (1) for any polynomial of degree k amounts to probing that the kth moments of the temporal and spatial ensembles, $\rho_T^{(k)} \coloneqq (1/T)\int_0^T dt|\psi(t)\rangle\langle\psi(t)|^{\otimes k}$ and $\rho_{\mathrm{Haar}}^{(k)} \coloneqq \int d\phi|\phi\rangle\langle\phi|^{\otimes k}$, respectively, match at late times. This agreement can be quantitatively captured by the vanishing of the trace distance

$$\Delta_\infty^{(k)} \coloneqq \frac{1}{2}\|\rho_\infty^{(k)} - \rho_{\mathrm{Haar}}^{(k)}\|_1, \quad (2)$$

where $\rho_\infty^{(k)} = \lim_{T\to\infty}\rho_T^{(k)}$. CHSE is achieved if for any initial state $\Delta_\infty^{(k)} = 0$ for all k [31].

It can be readily shown that quantum dynamics in a sufficiently large Hilbert space admitting (quasi-)energy eigenstates cannot achieve CHSE. This is intuitive: the conservation of population on stationary states prevents a complete exploration of the Hilbert space. In the Supplemental Material [22], we show that for evolution under a time-independent Hamiltonian, the trace distance for $k \geq 2$ can be always lower bounded as

$$\Delta_\infty^{(k)} \geq B(d) \coloneqq (d+1)^{-1} - [2d(d+1)]^{-1/2} > 0, \quad (3)$$

irrespective of initial state. Thus, CHSE can only occur for time-dependent dynamics. A similar obstruction can be shown in time-periodic dynamics [32] or if the Hamiltonian becomes time independent after a finite time.

A simple example of a (discrete) time-dependent system that does exhibit CHSE is as follows: let A_0 and A_1 be two typical unitaries on a d-dimensional space, and generate dynamics by randomly applying A_0 or A_1 with equal probability at each time step. With probability 1, such an evolution exhibits CHSE. However, this is unsurprising and can be intuitively understood using results from complexity and quantum information theory. First, it is well known that a pair of unitaries drawn independently and uniformly from the special unitary group $\mathrm{SU}(d)$ is almost surely *quantum computationally universal* [33], i.e., any other unitary in $\mathrm{SU}(d)$ can be asymptotically reached by some long product of the pair. Second, an infinitely long random binary sequence almost surely exhibits a *symbolic complexity* (explained below) which is maximal. This implies that any possible product of A_0 and A_1 eventually appears in the sequence. Combined together, a random sequence of two typical unitaries over long timescales is therefore asymptotically equivalent to a sequence of Haar-random unitaries, which trivially achieves CHSE [22,34].

Generalized Fibonacci drive.—Here, we introduce a family of nonperiodic, yet deterministic drives, where despite being generated by simple rules, CHSE can be shown to hold. Fix a natural number m and two initial words $W_0 = 1$ and $W_1 = 0$. Then, we define a sequence of words by recursively concatenating shorter ones $W_{j+1} = (W_j)^m W_{j-1}$, where

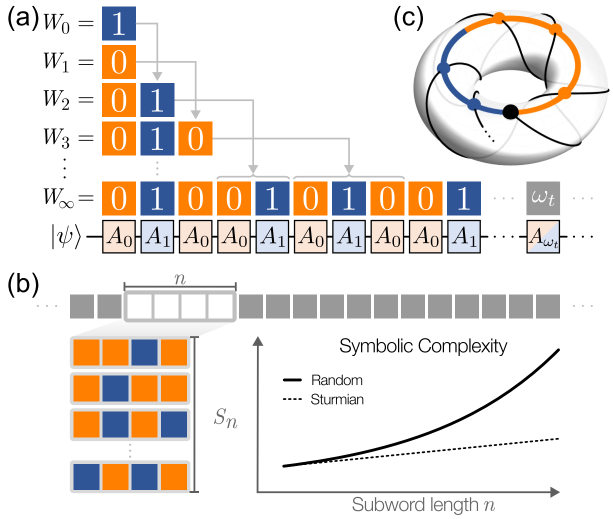

FIG. 2. (a) Sequence of Fibonacci words constructed by the concatenation rule $W_{j+1} = W_j W_{j-1}$, for order $m = 1$. The Fibonacci drive applies a pair of unitaries A_0, A_1 in the sequence prescribed by W_∞. (b) Symbolic complexity S_n counts the number of subwords of length n that appear in an infinite word. It is linear in n for Sturmian words such as W_∞ and exponential in n for random words. (c) Quasiperiodicity of the Fibonacci drive. Consider a point moving in a straight line (black) which wraps around the torus. Each time the line crosses the orange region, the unitary A_0 is applied, and when it crosses the blue region, A_1 is applied [22].

multiplication should be understood as concatenation of words. This process leads to a well defined, infinite word W_∞. The case $m = 1$ is the well-known Fibonacci word [35], $W_\infty = 0100101001001\ldots$, [see Fig. 2(a)], while the case $m = 2$ generates the so-called Pell word $W_\infty = 0010010001001\ldots$ [36]. We thus refer to W_∞ as the Fibonacci word of order m. All such sequences are examples of so-called Sturmian words [22,37,38].

Sturmian words represent the simplest possible aperiodic sequences. Specifically, they are characterized by having the minimal possible symbolic complexity S_n. Symbolic complexity measures the number of distinct contiguous subwords of length n that appear in an infinite word [39] [Fig. 2(b)]. In the case of a random binary word, $S_n = 2^n$ (exponential), which is maximal. In contrast, Sturmian words have complexity $S_n = n + 1$ (linear), which is minimal for any aperiodic word; a word with complexity $S_n < n + 1$ can be shown to be eventually repeating [37].

We use the Fibonacci words to generate discrete-time quantum dynamics on a d-level system. Given two unitaries $A_0, A_1 \in \mathrm{SU}(d)$, we apply A_{ω_t} at time $t \in \mathbb{N}$, where ω_t is the tth symbol in W_∞. Explicitly, the time evolution operator is given by $U(0) = \mathbb{1}$ and

$$U(t) = A_{\omega_t} \cdots A_{\omega_3} A_{\omega_2} A_{\omega_1} \tag{4}$$

for $t \geq 1$, so that a time-evolved state is $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. We call $U(t)$ the generalized Fibonacci drive of order m. This drive may be understood as arising from a time-quasiperiodic Hamiltonian [40–44], as its time dependence can be shown to factor through a two-dimensional torus [Fig. 2(c)] [22]. The Fibonacci drive of order 1 has gained recent attention in the quantum-dynamics community [45–48] and has been experimentally realized [49].

CHSE in the Fibonacci drives.—Because the generalized Fibonacci drives are defined in discrete time, we modify the condition for CHSE in Eq. (1) and temporal ensembles accordingly, changing the integral over continuous time to a sum over discrete time, e.g., $\rho_T^{(k)} = (1/T)\sum_{t=0}^{T-1} |\psi(t)\rangle\langle\psi(t)|^{\otimes k}$. Our aim is to show $\rho_T^{(k)}$ converges to $\rho_{\mathrm{Haar}}^{(k)}$ for large T, independent of initial state.

To show this, we note that $\rho_T^{(k)}$ can be obtained by passing the (replicated) initial state $\rho_0^{(k)} = |\psi(0)\rangle\langle\psi(0)|^{\otimes k}$ through a time-averaging quantum channel,

$$\mathcal{N}_T^{(k)}[\cdot] \coloneqq \frac{1}{T}\sum_{t=0}^{T-1} U(t)^{\otimes k}(\cdot)U(t)^{\dagger\otimes k}, \tag{5}$$

such that $\rho_\infty^{(k)} = \mathcal{N}_\infty^{(k)}[\rho_0^{(k)}]$, where $\mathcal{N}_\infty^{(k)} = \lim_{T\to\infty}\mathcal{N}_T^{(k)}$ is the infinite-time averaging channel. Here, a subtle technical point has to be noted: in general, it is not guaranteed that the limit $\mathcal{N}_\infty^{(k)}$ should exist [50]. Using Birkhoff's ergodic theorem, we proved that it exists for a class of quasiperiodic Hamiltonians related to the Fibonacci drives by an initial phase shift on the torus [22], but we could not show this for the Fibonacci drives themselves. Extensive numerics performed below, however, suggest that it does. We proceed with the assumption that $\mathcal{N}_\infty^{(k)}$ is well defined in this class of models. The question is then what this limit is. We have

Theorem 1.—For almost all pairs of unitaries A_0, $A_1 \in \mathrm{SU}(d)$, $\mathcal{N}_\infty^{(k)}$ of the Fibonacci drive of order m (assuming it exists), satisfies

$$\forall\ k \in \mathbb{N}\colon \mathcal{N}_\infty^{(k)} = \mathcal{N}_{\mathrm{Haar}}^{(k)}, \tag{6}$$

where $\mathcal{N}_{\mathrm{Haar}}^{(k)}[\cdot] = \int dU\, U^{\otimes k}(\cdot)U^{\dagger\otimes k}$ is the k-fold Haar-averaging channel.

This theorem means that for almost all A_0 and A_1 (barring exceptional scenarios such as $A_0 = A_1 = \mathbb{1}$), time-averaging under quantum dynamics of the Fibonacci drives is equivalent to a randomization under uniformly distributed unitaries: the evolution operators $\{U(t)\}_{t\in\mathbb{N}}$ are statistically indistinguishable from Haar-random unitaries.

Since $\mathcal{N}_{\mathrm{Haar}}^{(k)}[\rho_0^{(k)}] = \rho_{\mathrm{Haar}}^{(k)}$ for any initial state, this establishes our main result:

Corollary 1.—Almost surely, the Fibonacci drives of any order m exhibit complete Hilbert-space ergodicity.

Proof sketch of Theorem 1.—The proof relies on the recursive and recurrent nature of the Fibonacci drives. Below we focus on the case $m = 1$, and the more general

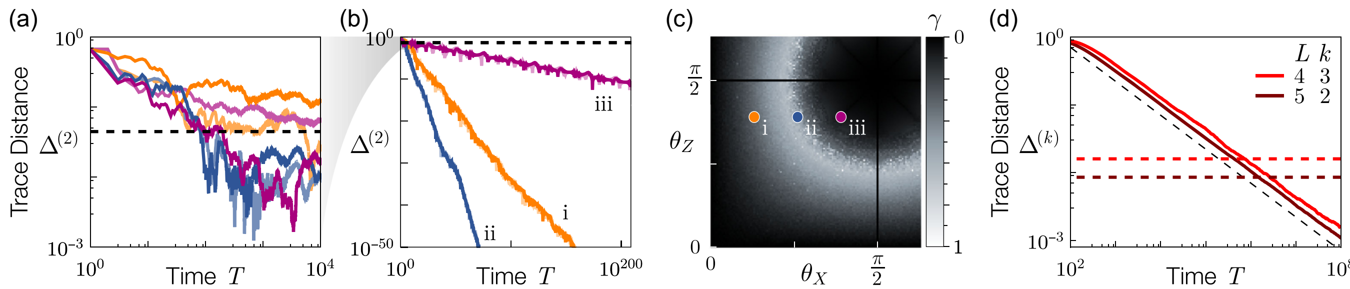

FIG. 3. Numerical analysis of complete Hilbert-space ergodicity in the Fibonacci drive of order 1. (a),(b) Trace distance $\Delta^{(k=2)}(T)$ for single qubit rotations $A_0 = e^{-i\theta_X X}$ and $A_1 = e^{-i\theta_Z Z}$, for three pairs of angles (θ_X, θ_Z): i $(0.13\pi, 0.39\pi)$ (orange), ii $(0.26\pi, 0.39\pi)$ (blue), and iii $(0.39\pi, 0.39\pi)$ (purple), and two initial states: $|0\rangle$ (darker), $|+\rangle$ (lighter). Lower bound $B(d=2)$ for time-independent evolution (black dashed). (c) Power law exponent γ as a function of (θ_X, θ_Z) obtained by averaging $\Delta^{(2)}$ over a pair of random initial states, for T equal to the Fibonacci numbers F_n up to $F_{3000} \approx 10^{626}$. (d) $\Delta^{(k)}$ for many-body Fibonacci drive over a spin-1/2 chain of length L, averaged over 10 random product states $|\psi\rangle^{\otimes L}$ (solid lines). The bound $B(2^L)$ is shown with horizontal dashed lines. A power law decay $\sim T^{-1/2}$ (black dashed) is provided as reference.

case is proven in the Supplemental Material [22]. To simplify notation, we drop the superscript k.

We note that it suffices to show that $\mathcal{N}_\infty \circ \mathcal{V} = \mathcal{N}_\infty$ with $\mathcal{V}[\cdot] \coloneqq V^{\otimes k}(\cdot)V^{\dagger\otimes k}$ for any $V \in \mathrm{SU}(d)$, where $\circ$ denotes the channel composition. This is because $\mathcal{N}_{\mathrm{Haar}}$ is uniquely determined by a left or right invariance under any unitary rotation. This invariance is shown in three steps [22].

In the first step, we derive a recursive relation satisfied by $\mathcal{N}_T$ at times equal to a Fibonacci number $T = F_n$ [51]:

$$\mathcal{N}_{F_n} = \frac{F_{n-1}}{F_n}\mathcal{N}_{F_{n-1}} + \frac{F_{n-2}}{F_n}\mathcal{N}_{F_{n-2}} \circ \mathcal{U}(F_{n-1}), \qquad (7)$$

where $\mathcal{U}(T)[\cdot] = U(T)^{\otimes k}(\cdot)U(T)^{\dagger\otimes k}$. Crucially, as long as $\mathcal{N}_\infty$ exists, one is free to choose any subsequence of times $(F_{n_\ell})_{\ell\in\mathbb{N}}$ to evaluate the limit at large T. The key idea is to take an appropriate subsequence of times such that the relation in Eq. (7) reduces to the right invariance $\mathcal{N}_\infty = \mathcal{N}_\infty \circ \mathcal{L}$ under some unitary channel $\mathcal{L}[\cdot] = L^{\otimes k}(\cdot)L^{\dagger\otimes k}$, where $L = \lim_{\ell\to\infty} U(F_{n_\ell})$.

In the second step, we show that there are at least two distinct subsequences such that $L = A_0$ and A_1, respectively. This is enabled by the Poincaré recurrence theorem applied to a measure-preserving dynamical map $\Phi\colon (V, W) \mapsto (W, VW)$ for $V, W \in \mathrm{SU}(d)$, which generates our Fibonacci drive $\Phi^n(A_1, A_0) = [U(F_n), U(F_{n+1})]$. Specifically, the theorem states that almost any initial condition eventually comes arbitrarily close back to itself under sufficiently long repeated application of Φ. Translated to our case, almost surely (A_1, A_0) is seen to be the limit of some subsequence of $[U(F_n), U(F_{n+1})]_{n\in\mathbb{N}}$. This immediately implies that $\mathcal{N}_\infty$ is right invariant not only under the unitary channels $\mathcal{A}_0$, $\mathcal{A}_1$ generated by A_0, A_1, respectively, but also under any unitary channel $\mathcal{V}$ generated by arbitrarily many products of A_0 and A_1.

In the final step, we invoke a well-known fact in quantum information, that almost any pair A_0, A_1 of unitaries chosen uniformly from the Haar-measure generates the entire unitary group $\mathrm{SU}(d)$; it is said that A_0, A_1 constitute universal "gates" [33]. Taken together, this then implies $\mathcal{N}_\infty$ is right-invariant under any unitary channel $\mathcal{V}$, as desired. ∎

Both Theorem 1 and Corollary 1 demonstrate a strong form of ergodicity which quantum systems can exhibit: they illustrate how "disorder" (complete randomness) can emerge from "order" (a structured drive). However, our analysis does not give us information about the speed at which $\rho_T^{(k)}$ converges to $\rho_{\mathrm{Haar}}^{(k)}$ for different choices of A_0, A_1 and initial state. To this end, we turn to numerical simulations.

Numerical analysis.—Below, we focus on the Fibonacci drive of order $m = 1$ and numerically compute the finite-time trace distance $\Delta^{(k)}(T) \coloneqq \frac{1}{2}\|\rho_T^{(k)} - \rho_{\mathrm{Haar}}^{(k)}\|_1$.

We first analyze the single qubit $d = 2$ case, fixing $A_0 = e^{-i\theta_X X}$ and $A_1 = e^{-i\theta_Z Z}$ to be Pauli X and Z rotations by angles θ_X and θ_Z, respectively. Figures 3(a) and 3(b) show $\Delta^{(2)}(T)$ for three choices of angles (θ_X, θ_Z) indicated in Fig. 3(c), starting from two initial states $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. We observe an empirical power-law decay $\Delta^{(2)} \sim T^{-\gamma}$ going well below the lower bound ~ 0.04 for time-independent evolution in Eq. (3). The power-law exponent γ depends on the angles (θ_X, θ_Z) as shown in Fig. 3(c) and on k (see Supplemental Material [22]), but not on the initial state. Note that $\gamma = 0$ at special points where $\theta_X = 0$, $\pi/2$ or $\theta_Z = 0$, $\pi/2$, signaling a breakdown of CHSE. In such measure-zero cases, the set $\{A_0, A_1\}$ fails to generate all possible rotations. We present an in-depth study of CHSE for a single qubit in Supplemental Material [22].

Finally, we study CHSE in a many-body system. We consider a spin-1/2 chain of length L and generate the Fibonacci evolution with $A_0 = e^{-iH_0\tau}$ and $A_1 = e^{-iH_1\tau}$, where $H_0 = \sum_{j=1}^L X_j + \sum_{j=2}^L X_{j-1}X_j + X_L/10$ and $H_1 = \sum_{j=1}^L Z_j + \sum_{j=2}^L Z_{j-1}Z_j + Z_L/10$ for initial product states and $\tau = 1$. In Fig. 3(d) we see again a power-law decay $\Delta^{(k)} \sim T^{-\gamma}$ with an exponent $\gamma \sim 1/2$ for large Ls and all ks simulated, which goes below the bound in Eq. (3). Understanding the origin of this seemingly universal exponent is an interesting future direction. Further note that both H_0 and H_1 are integrable Hamiltonians: it is the introduction of time dependence that brings nontrivial dynamical behavior [52,53]: This suggests that CHSE can be achieved regardless of whether the system is integrable or non-integrable at any given fixed time.

Discussion and outlook.—There are several physical and conceptual implications of complete Hilbert-space ergodicity (CHSE). CHSE essentially asserts that wave functions at sufficiently late times behave like random vectors in the Hilbert space. This implies in turn that quantum many-body systems exhibiting CHSE can be rigorously shown to locally achieve thermalization to infinite temperature, since a typical Haar-random quantum state is highly entangled such that it locally appears maximally mixed [12]. By the same token, quantum systems with CHSE also *deep thermalize*: this is a recently developed notion of equilibration in which conditional states of a local subsystem, obtained via measurements of the complementary subsystem, achieve a maximally entropic wave function distribution (on the local Hilbert space) [13–21]. Using Theorem 2 of Ref. [18], we show that CHSE implies deep thermalization at late times [22]. We also present a numerical demonstration of deep thermalization in the Fibonacci drives.

Conceptually, CHSE has implications for the structure of solutions to time-dependent Schrödinger's equations. Under time-independent Hamiltonian dynamics, it is possible to decompose any initial state into a linear combination of energy eigenstates whose dynamics are trivial (specifically, having phases which wind in time), such that wave functions at arbitrarily late times can be immediately recovered from the decomposition. The same is true for the dynamics of Floquet systems using quasienergy eigenbases, which are guaranteed to exist by Floquet's theorem. The ability to solve the time-dependent Schrödinger equation in terms of stationary solutions is called *reducibility* [54]. In time-quasiperiodic systems, the question of reducibility is nontrivial [8,9]. Our results show that the time-quasiperiodic dynamics of the Fibonacci drives, or any other drives exhibiting CHSE, is irreducible, because the presence of quasienergy states is incompatible with the complete exploration of the Hilbert space in large-dimensional systems [32]. This opens the possibility that the computational complexity of solving quantum dynamics in such systems—i.e., the computational resources required, necessarily grows unboundedly over time.

Our results lead to a number of interesting future directions. First, systems exhibiting CHSE may yield settings in which rigorous studies on the growth of circuit complexity—the minimal number of local gates needed to prepare a given quantum state from an unentangled state—may be conducted. Second, the question of how to experimentally verify and utilize CHSE, given finite coherence times of experiments, needs to be addressed. Third, the relation between the breakdown of CHSE and integrability is an open question. We note that the presence of any conserved quantities—local or even nonlocal, such as energy—necessarily implies the breakdown of CHSE. The converse, however, calls for further investigation. To this end, it would be interesting to incorporate symmetries into the analysis. In particular, including the conservation of energy may bridge the gap between our dynamical notion of ergodicity and more conventional concepts based on the statistical properties of stationary states [55], ultimately providing a unified framework to understand quantum ergodicity.

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