Fluctuation-Dissipation Relation in a Nonequilibrium Quantum Fluid

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(Received 14 May 2023; revised 20 July 2023; accepted 16 November 2023; published 13 December 2023)

There is no simple fluctuation-dissipation theorem (FDT) for nonequilibrium systems. We show that for a fluid in a nonequilibrium steady state (NESS) characterized by a constant temperature gradient there is a generalized FDT that relates commutator correlation functions to the bilinear response of products of observables. This allows for experimental probes of the long-range correlations in such a system, quantum or classical, via response experiments. We also show that the correlations are not tied to thermal fluctuations but are intrinsic to the NESS and reflect a generalized rigidity.

DOI: 10.1103/PhysRevLett.131.244001

A fundamental result of equilibrium statistical mechanics is the fluctuation-dissipation theorem, which states that the system's linear response to an external perturbation is related to the fluctuations in the equilibrium state [1,2]. Specifically, the response functions are proportional to the corresponding correlation functions, and in a classical system the proportionality factor is simply the inverse temperature [3]. In a nonequilibrium system, this is no longer true. There is a substantial body of work on the properties of fluctuations far from equilibrium (see, e.g., Refs. [4,5] and references therein), on linear response in nonequilibrium systems [6], and on aspects of relations between the two [5,7,8], but there is no general prescription for probing fluctuations via the response to external perturbations. Consequently, fluctuations can be observed only via scattering experiments, which become increasingly difficult as the temperature is lowered, but not via the linear response to a macroscopic perturbation. One of the main results reported in this Letter is that for a fluid, classical or quantum, in a nonequilibrium steady state (NESS) characterized by a fixed temperature gradient there still is a relation between fluctuations and response, but it is not linear. Rather, the nonequilibrium parts of the correlation functions determine the bilinear response of products of observables to external perturbations, as we will demonstrate in Eqs. (14) and (18) below. Since perturbations can be controlled experimentally independent of other parameters, this opens new avenues for observing correlations, especially in quantum fluids, where thermal fluctuations are weak due to the low temperature. In addition, we elucidate several other aspects of fluids in such a NESS, especially in the quantum limit.

To motivate our investigations, we recall that classical simple fluids subject to a fixed temperature gradient harbor correlations that are extraordinarily long ranged [9–11]. For instance, the equal-time temperature-temperature correlation function diverges for small wave numbers k as $1/k^4$. In real space, this corresponds to correlations that extend over the entire width of the sample and decay on the same scale [11]. These correlations are generic in the sense that they do not require any fine-tuning of parameters, and they are not related to any broken symmetry. Rather, they are the result of the coupling of the temperature fluctuations to the diffusive shear velocity. This surprising result has been confirmed theoretically by means of a variety of techniques [11–13], and it has been observed by many light scattering experiments; see Ref. [14] and references therein.

Despite being well established and confirmed, this phenomenon raises several questions that historically have not been emphasized. One is the fact, mentioned above, that the relevant correlation functions in a NESS are not in any obvious way related to response functions. Another question is whether the long-ranged correlations are tied to thermal fluctuation effects, or whether they are more generic and reflect some type of generalized rigidity [15] that is present even in the zero-temperature limit and also manifests itself in the response of the system to external perturbations. Recent work on classical fluids has suggested the latter [16], but a relation between correlation functions and response theory has been lacking. Part of the purpose of this Letter is to provide such a relation.

The missing correlation-response relation discussed above is equally relevant for classical and quantum fluids, but it poses a particularly significant problem for the latter [17]. Direct measurements of the correlation functions via light scattering are difficult even in the classical case because of the very small scattering angles required. With decreasing temperature the fluctuations become smaller, which makes it even more desirable to observe the effect via response experiments, if feasible. Specifically, there are two obvious types of correlation functions: symmetrized, or anticommutator correlation functions that we denote by $S^{\rm sym}$, and antisymmetrized, or commutator ones that we denote by χ'' (this is the customary notation for the commutator correlation function [3], with the double prime indicating that this is the spectrum, or spectral density, of a causal function). As functions of the wave vector $\boldsymbol{k}$ and the frequency ω, they are defined by [3,23]

$$\frac{1}{2}\left\langle\left[\delta\hat{A}(\boldsymbol{k}_1,\omega_1),\delta\hat{B}(\boldsymbol{k}_2,\omega_2)\right]_+\right\rangle = V\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}2\pi\delta(\omega_1+\omega_2)S^{\mathrm{sym}}_{AB}(\boldsymbol{k}_1,\omega_1), \tag{1a}$$

$$\frac{1}{2\hbar}\left\langle\left[\delta\hat{A}(\boldsymbol{k}_1,\omega_1),\delta\hat{B}(\boldsymbol{k}_2,\omega_2)\right]_-\right\rangle = V\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}2\pi\delta(\omega_1+\omega_2)\chi''_{AB}(\boldsymbol{k}_1,\omega_1). \tag{1b}$$

Here, $\delta\hat{A}$, $\delta\hat{B}$ are operator-valued fluctuations of observables, $[,]_+$ and $[,]_-$ denote anticommutators and commutators, respectively, $\langle\ldots\rangle$ denotes a quantum mechanical expectation value plus a statistical mechanics average, and V is the system volume. We use carets to denote operator-valued quantities (and below also to denote unit vectors; this should not lead to confusion); the corresponding quantities without carets are number-valued classical objects. The two types of correlation functions are related by

$$S^{\mathrm{sym}}_{AB}(\boldsymbol{k},\omega) = \chi''_{AB}(\boldsymbol{k},\omega)\hbar\coth(\hbar\omega/2T). \tag{2}$$

Here, and throughout the Letter, we put $k_B = 1$; i.e., we measure the temperature in units of energy. We note that the exact relation between S^{sym} and χ'' is more complicated in systems that are not spatially homogeneous. It reduces to Eq. (2) if the local temperature is replaced by its spatial average. The corresponding static correlation functions are

$$S^{\mathrm{sym}}_{AB}(\boldsymbol{k}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S^{\mathrm{sym}}_{AB}(\boldsymbol{k},\omega), \tag{3a}$$

$$\chi_{AB}(\boldsymbol{k}) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\chi''_{AB}(\boldsymbol{k},\omega)/\omega. \tag{3b}$$

The symmetrized correlation functions S^{sym}_{AB} are observable by means of scattering experiments [3]. The physical meaning of the antisymmetrized correlation functions χ''_{AB} is *a priori* less obvious. In an equilibrium system, where the correlations are generically short ranged, they describe the linear response of the system to external fields, as follows. Let h_B be the external field conjugate to $\hat{B}$. Then to linear order in the fields, one has [3]

$$\delta\langle\hat{A}\rangle(\boldsymbol{k},\omega) = \chi_{AB}(\boldsymbol{k},\omega)h_B(\boldsymbol{k},\omega), \tag{4}$$

where $\chi_{AB}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}dx\chi''_{AB}(\boldsymbol{k},x)/(x-\omega-i0)$ with $i0$ an infinitesimal imaginary part. That is, the equilibrium fluctuations determine the linear response, which to second order in the external field yields the energy dissipated by the system. This is the content of the fluctuation-dissipation theorem [1,2]. In a NESS, the relation (2) still holds (if the local temperature is replaced by its spatial average T), but the commutator correlations functions no longer describe the linear response, and the usual fluctuation-dissipation theorem breaks down.

In this Letter, we identify the quantum analogs of the classical long-ranged correlations. Our two main results are (1) the long-ranged commutator correlation functions are still related to response functions, but the relation is not a simple proportionality. Rather, the nonequilibrium contributions to the correlation functions are related to the bilinear response of products of observables; see Eqs. (14) and (18) below. (2) In a modified form, the long-ranged correlations extend to zero temperature. This shows that they are not tied to thermal fluctuations, although thermal fluctuations can be used to probe them. Rather, they are an inherent long-wavelength property of the NESS and indeed reflect a novel type of generalized rigidity that does not become weaker with decreasing temperature. We will start by explaining the second result, and then demonstrate the first one.

Consider a fluid confined between two plates that is subject to a constant temperature gradient $\boldsymbol{\nabla}T$ in the z direction; see Fig. 1. Let $\boldsymbol{k}$ be the wave vector of a temperature fluctuation, and let the wave-vector space be spanned by three mutually perpendicular unit vectors $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$, $\hat{\boldsymbol{k}}_\perp$, and $\hat{\boldsymbol{k}}_\perp^{(2)}$ such that $\hat{\boldsymbol{k}}_\perp$ is coplanar with $\boldsymbol{k}$ and $\boldsymbol{\nabla}T$. For our purposes, the temperature gradient will appear in the combination $\nabla_\perp T \equiv \hat{\boldsymbol{k}}_\perp \cdot \boldsymbol{\nabla}T$. For definiteness, we consider a fermionic quantum fluid (e.g., conduction electrons in metals). Analogous effects must be present in bosonic fluids as well, but at asymptotically low temperatures, Bose-Einstein condensation will lead to complications that require additional investigation. Let τ

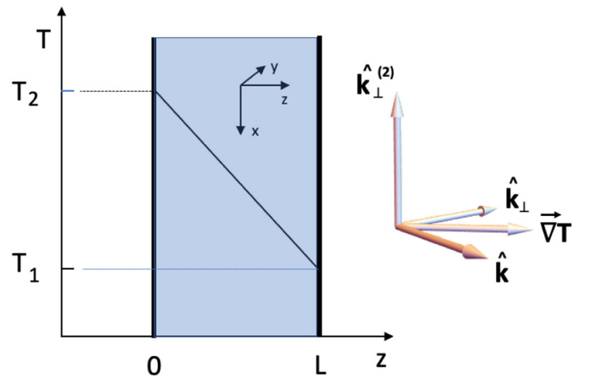

FIG. 1. A fluid subject to a constant temperature gradient in the z direction between two parallel confining plates. $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{k}}_\perp$, and $\hat{\boldsymbol{k}}_\perp^{(2)}$ are unit vectors that span the wave-number space, with the coordinate system chosen such that $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}_\perp$ are coplanar with the temperature gradient.

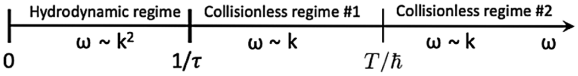

FIG. 2. Frequency regimes. The hydrodynamic regime is characterized by a diffusive shear velocity, with the frequency ω scaling as the wave number squared. In the collisionless regimes, the shear velocity is a propagating mode with the frequency scaling linearly with the wave number.

be the relaxation time, v_{F} the Fermi velocity, and ω the frequency. We need to distinguish between the hydrodynamic regime, where $\omega < 1/\tau$ or $v_{\mathrm{F}}k < 1/\tau$, and the collisionless regime where $\omega > 1/\tau$. The latter is in general subdivided into regimes where $\hbar\omega < T$ and $\hbar\omega > T$, respectively; see Fig. 2 [24]. Here, T is the spatially averaged temperature [26]. We are interested in the correlation functions for the temperature T and the component $u_\perp \equiv \hat{\boldsymbol{k}}_\perp \cdot \boldsymbol{u}_\perp$ of the fluid shear velocity $\boldsymbol{u}_\perp$. The nature of the latter changes from diffusive to propagating as one goes from the hydrodynamic regime to the collisionless one. In the latter, a hydrodynamic description no longer applies, and one has to work with a quantum kinetic theory. In what follows, we first state our results and then comment on their derivations. For details see the Supplemental Material [27].

Hydrodynamic regime.—In the hydrodynamic regime, one has $\omega < 1/\tau < T/\hbar$ [24], and the correlation functions are the same as in a classical fluid. In particular, S^{sym}_{AB} becomes a classical correlation function $\langle AB \rangle$ with the angular brackets denoting a classical statistical mechanics average, and we have [9]

$$S^{\mathrm{sym}}_{TT}(\boldsymbol{k},\omega) = \frac{2T^2}{c_V}\frac{D_{\mathrm{T}}k^2}{\omega^2 + (D_{\mathrm{T}}k^2)^2} + (\nabla_\perp T)^2 \frac{2T}{\rho}\frac{\nu k^2}{[\omega^2 + (D_{\mathrm{T}}k^2)^2)(\omega^2 + (\nu k^2)^2]}. \tag{5a}$$

The first term is the standard equilibrium contribution, and the second term is the long-ranged NESS contribution. c_V and ρ are the spatially averaged specific heat per volume and the fluid mass density, respectively, and D_{T} and ν are the spatially averaged heat diffusion coefficient and kinematic viscosity, respectively. The velocity-velocity correlation function is the same as in equilibrium,

$$S^{\mathrm{sym}}_{u_\perp u_\perp}(\boldsymbol{k},\omega) = \frac{2T}{\rho}\frac{\nu k^2}{\omega^2 + (\nu k^2)^2}, \tag{5b}$$

and the mixed symmetrized correlation functions are

$$S^{\mathrm{sym}}_{u_\perp T}(\boldsymbol{k},\omega) = S^{\mathrm{sym}}_{T u_\perp}(-\boldsymbol{k},-\omega) = \nabla_\perp T \frac{1}{\rho}\frac{\nu k^2}{\omega^2 + (\nu k^2)^2}\frac{2T}{i\omega + D_{\mathrm{T}}k^2}. \tag{5c}$$

The corresponding commutator correlation functions are given by Eq. (2) with $\hbar\coth(\hbar\omega/2T) \approx 2T/\omega$:

$$\chi''_{AB}(\boldsymbol{k},\omega) = S^{\mathrm{sym}}_{AB}(\boldsymbol{k},\omega)\omega/2T \qquad (A,B = T, u_\perp). \tag{5d}$$

The static correlation functions are

$$S^{\mathrm{sym}}_{TT}(\boldsymbol{k}) = \frac{T^2}{c_V} + \frac{T(\nabla_\perp T)^2}{\rho D_{\mathrm{T}}(\nu + D_{\mathrm{T}})k^4}, \tag{6a}$$

$$S^{\mathrm{sym}}_{u_\perp u_\perp}(\boldsymbol{k}) = T/\rho, \tag{6b}$$

$$S^{\mathrm{sym}}_{u_\perp T}(\boldsymbol{k}) = S^{\mathrm{sym}}_{T u_\perp}(-\boldsymbol{k}) = \nabla_\perp T \frac{T/\rho}{(\nu + D_{\mathrm{T}})k^2}, \tag{6c}$$

and

$$\chi_{AB}(\boldsymbol{k}) = S^{\mathrm{sym}}_{AB}(\boldsymbol{k})/T \qquad (A,B = T, u_\perp). \tag{6d}$$

$S^{\mathrm{sym}}_{TT}(\boldsymbol{k})$ in Eq. (6a) is the equal-time temperature-temperature correlation function mentioned in the introduction that diverges as $1/k^4$. This result is exactly the same as for a classical fluid, as expected in the hydrodynamic regime.

Collisionless regime.—In the collisionless regime, the expressions for the dynamic correlation functions are lengthy and are given in the Supplemental Material, together with an outline of their derivations [27]. Here we give only the results for the static temperature-temperature correlations. To leading order as $T \to 0$, we have

$$\chi_{TT}(\boldsymbol{k}) = \frac{1}{N_{\mathrm{F}}}\frac{3}{\pi^2}\left[1 + \frac{\pi^2}{12}(2\pi^2 - 3)\frac{(\nabla_\perp T)^2}{\epsilon_{\mathrm{F}}^2 k^2}\right] \tag{7}$$

everywhere in the collisionless regime. Here N_{F} is the density of states at the Fermi level, and ϵ_{F} is the Fermi energy. The symmetrized correlation function is $S^{\mathrm{sym}}_{TT}(\boldsymbol{k}) = T\chi_{TT}(\boldsymbol{k})$ in collisionless regime #1 in Fig. 2 and, apart from a factor of $O(1)$, $S^{\mathrm{sym}}_{TT}(\boldsymbol{k}) \approx v_{\mathrm{F}}k\chi_{TT}(\boldsymbol{k})$ in collisionless regime #2.

To derive these results, we note that in the hydrodynamic regime the usual Navier-Stokes equations are applicable, as they are based on very general physical principles, viz., the conservation laws for mass, momentum, and energy combined with force-balance considerations [19,28,29]. Consequently, they hold for quantum fluids as well as for classical ones. In order to calculate correlation functions, they need to be augmented by fluctuating forces [18,19]. Of the various nonlinearities, we need to keep only the crucial coupling between the temperature fluctuation and the transverse velocity, which turns into a linear term in the presence of a fixed temperature gradient. We can further ignore pressure fluctuations, which lead to sound waves (or plasmons in a charged Fermi liquid) that are much faster than the diffusive shear fluctuations. The equations for the operator-valued temperature and shear velocity then are

$$(-i\omega + \nu \boldsymbol{k}^2)\hat{u}_\perp(\boldsymbol{k},\omega) = \hat{P}_\perp(\boldsymbol{k},\omega), \tag{8a}$$

$$(-i\omega + D_T \boldsymbol{k}^2)\hat{T}(\boldsymbol{k},\omega) + (\nabla_\perp T)\hat{u}_\perp(\boldsymbol{k},\omega) = \hat{Q}(\boldsymbol{k},\omega). \tag{8b}$$

Here the transport coefficients ν and D_T are to be understood as having been spatially averaged. $\hat{P}_\perp$ and $\hat{Q}$ are operator-valued fluctuating forces that are Gaussian distributed with zero mean. Their second moments can be obtained from the somewhat more general expressions derived in Ref. [18] and are given explicitly in the Supplemental Material [27]. The $u_\perp - u_\perp$ correlation then is determined by the $P_\perp$ correlation and given by Eq. (5b). This is the same as in equilibrium since the temperature does not couple into the $u_\perp$ equation. The equilibrium part of the T-T correlation is given by the Q correlation, whereas the nonequilibrium part, as well as the mixed T-$u_\perp$ correlation, is given in terms of the $u_\perp$ fluctuation, with $\nabla_\perp T$ serving as a coupling constant. This yields the second term in Eqs. (5a) and (5c). The leading singularity in the static T-T correlation function is $1/k^4$, and $1/k^2$ in the T-$u_\perp$ correlation function, as in a classical fluid. This is because the hydrodynamic equations are the same in either case.

In order to properly describe the collisionless regime, one has to work with the fluctuating quantum kinetic theory developed in Ref. [18]; see the Supplemental Material for a derivation [27]. However, the qualitative features of the results can be obtained from simple scaling arguments as follows. In the collisionless regime, the diffusive inverse propagators of the form $\mathcal{D}^{-1}(\boldsymbol{k},\omega) = \omega + iDk^2$ that appear on the left-hand sides of Eqs. (8), where D can represent either ν or D_T, effectively turn into propagating zero modes of the form $\omega \mp ck$, with $c \approx v_\mathrm{F}$ the propagation speed. The transport coefficients thus effectively become singular functions of the wave number and scale as $D \sim v_\mathrm{F}/k$. The low-temperature result for χ_{TT}, Eq. (7), then follows from Eq. (6a) by replacing ν, $D_\mathrm{T} \to v_\mathrm{F}/k$ and using the low-temperature expression for the specific heat $c_V \propto N_\mathrm{F} T$ (prefactors, as well as issues regarding reality and signs, require a more detailed analysis). For the symmetrized correlation function one needs to consider the frequency integral in Eq. (3a) and recognize that for the equilibrium contribution it needs to be cut off at $\omega \approx D_\mathrm{T} k^2$. In the limits $D_\mathrm{T} k^2 \ll T/\hbar$ and $D_\mathrm{T} k^2 \gg T/\hbar$, and using again the effective scaling of D_T and ν explained above, one then obtains the relations between $S_{TT}^{\mathrm{sym}}(\boldsymbol{k})$ and $\chi_{TT}(\boldsymbol{k})$ given after Eq. (7).

As mentioned above, the commutator functions χ'' do not determine the linear response of the NESS to external perturbations, in contrast to an equilibrium system. In order to determine the response functions, we consider the averaged Navier-Stokes equations in the presence of a field $h_{u_\perp}$ conjugate to $u_\perp$:

$$(-i\omega + \nu k^2)u_\perp(\boldsymbol{k},\omega) = \frac{\nu}{\rho} k^2 h_{u_\perp}(\boldsymbol{k},\omega), \tag{9a}$$

$$(-i\omega + D_T k^2)\delta T(\boldsymbol{k},\omega) + (\nabla_\perp T)u_\perp(\boldsymbol{k},\omega) = \frac{1}{\rho}(\nabla_\perp T)h_{u_\perp}(\boldsymbol{k},\omega). \tag{9b}$$

To avoid misunderstandings, we stress that these are equations for averaged, classical fluctuations δT and $u_\perp$. They are the standard Navier-Stokes equations [19] except that the sound modes have been omitted since they occur on timescales much faster than the diffusive shear and temperature fluctuations. They are driven by an external field $h_{u_\perp}$ that essentially represents a shift of the velocity reference frame and can be regarded as a field conjugate to $u_\perp$ [28,30]. It can be experimentally realized by imposing a shear velocity on the system. For a derivation from kinetic theory, see Ref. [31]. The response functions $X_{AB}(\boldsymbol{k},\omega)$ are defined by

$$u_\perp(\boldsymbol{k},\omega) = X_{u_\perp u_\perp}(\boldsymbol{k},\omega)h_{u_\perp}(\boldsymbol{k},\omega), \tag{10a}$$

$$\delta T(\boldsymbol{k},\omega) = X_{Tu_\perp}(\boldsymbol{k},\omega)h_{u_\perp}(\boldsymbol{k},\omega). \tag{10b}$$

From Eqs. (9), we find

$$X_{u_\perp u_\perp}(\boldsymbol{k},\omega) = \frac{1}{\rho}\frac{\nu k^2}{-i\omega + \nu k^2}, \tag{11a}$$

$$X_{Tu_\perp}(\boldsymbol{k},\omega) = \frac{1}{\rho}(\nabla_\perp T)\frac{1}{-i\omega + D_\mathrm{T} k^2}\frac{-i\omega}{-i\omega + \nu k^2}. \tag{11b}$$

From Eqs. (5b) and (11a), we see that the spectrum [32] of $X_{u_\perp u_\perp}$, $X''_{u_\perp u_\perp}(\boldsymbol{k},\omega) = \mathrm{Im}X_{u_\perp u_\perp}(\boldsymbol{k},\omega)$ equals the commutator correlation function $\chi''(\boldsymbol{k},\omega)$, as expected. However, the spectrum of $X_{Tu_\perp}$,

$$X''_{Tu_\perp}(\boldsymbol{k},\omega) = -(\nabla_\perp T)\frac{\omega(\nu D_\mathrm{T} k^2 - \omega^2)}{(\omega^2 + D_\mathrm{T}^2 k^4)(\omega^2 + \nu^2 k^4)}, \tag{12}$$

while showing the same scaling behavior as $\chi''_{Tu_\perp}$ [see Eqs. (5c) and (5d)] is not identical to the latter. In particular, the static response function vanishes,

$$X_{Tu_\perp}(\boldsymbol{k}) = \int \frac{d\omega}{\pi}\frac{X''_{Tu_\perp}(\boldsymbol{k},\omega)}{\omega} = 0, \tag{13}$$

while the static correlation function is nonzero; see Eq. (6c). Nonetheless, the response functions still provide a way to directly measure the commutator correlation functions, without relying on their symmetrized counterparts that are suppressed at low temperatures. Specifically, considering Eqs. (10), (11), (5c), and (5d), we have

$$\delta T(\boldsymbol{k},\omega)u_\perp(-\boldsymbol{k},-\omega) = \frac{i}{\rho}\chi''_{Tu_\perp}(\boldsymbol{k},\omega)|h_{u_\perp}(\boldsymbol{k},\omega)|^2. \quad (14)$$

That is, the commutator T-$u_\perp$ correlation function describes the bilinear response of the product of the temperature and the shear velocity to the field conjugate to $u_\perp$. Similarly, the nonequilibrium part of the commutator T-T correlation function can be expressed as a bilinear response to the field $h_{u_\perp}$. We define an observable

$$\tilde{T}(\boldsymbol{k},\omega) = T(\boldsymbol{k},\omega) - \frac{1}{\rho}(\nabla_\perp T)\frac{1}{-i\omega + D_\mathrm{T}k^2}h_{u_\perp}(\boldsymbol{k},\omega) \quad (15)$$

that obeys the equation

$$(-i\omega + D_T k^2)\delta\tilde{T}(\boldsymbol{k},\omega) + (\nabla_\perp T)u_\perp(\boldsymbol{k},\omega) = 0. \quad (16)$$

The physical interpretation of Eq. (16) is the heat equation with a streaming term that contains the absolute shear velocity, whereas Eq. (9b) contains the shear velocity relative to the field $h_{u_\perp}$. The response of $\tilde{T}$ to the field $h_{u_\perp}$ is given by a response function

$$X_{\tilde{T}u_\perp}(\boldsymbol{k},\omega) = \frac{1}{\rho}(\nabla_\perp T)\frac{-1}{-i\omega + D_\mathrm{T}k^2}\frac{\nu k^2}{-i\omega + \nu k^2}. \quad (17)$$

From Eqs. (17), (11b), (5a), and (5d), we find that the nonequilibrium part of the commutator T-T correlation function describes the bilinear response of the product of $\tilde{T}$ and T to the field $h_{u_\perp}$:

$$\delta\tilde{T}(\boldsymbol{k},\omega)\delta T(-\boldsymbol{k},-\omega) = \frac{i}{\rho}\chi''^{\mathrm{neq}}_{TT}(\boldsymbol{k},\omega)|h_{u_\perp}(\boldsymbol{k},\omega)|^2. \quad (18)$$

Equations (14) and (18) constitute our main result. They demonstrate that in a NESS characterized by a constant temperature gradient, the commutator correlation functions for the temperature and the shear velocity are still related to the response of the system to an external shear perturbation, even though the usual fluctuation-dissipation theorem is not valid. In contrast to the situation in equilibrium, where the correlation functions are the same as the response functions, in a NESS the correlation functions are given by the bilinear response of products of observables. We note that Eqs. (14) and (18) involve the nonequilibrium parts of the correlation functions only. For $\Delta T \to 0$, $\chi''_{Tu_\perp}$ vanishes and χ''_{TT} reduces to its equilibrium part that obeys the usual fluctuation-dissipation theorem.

To summarize, we have established a relation between correlation functions and response functions for a fluid in a NESS. The resulting NESS fluctuation formulas (14) and (18) resemble the equilibrium fluctuation-dissipation theorem, Eq. (2) in the hydrodynamic limit, with the symmetrized correlation function replaced by the product of two averaged observables, and the temperature replaced by the driving field squared. The latter one has control over regardless of how low the temperature is, since the driving field is realized by an imposed shear velocity. We stress that we have derived this relation between the fluctuation functions and the response functions only for the special case of a constant temperature gradient. Their structure suggests that they might hold for more general nonequilibrium states as well, but whether or not that is true is an open question. We also have determined the quantum analogs of the long-ranged correlations known to exist in a classical fluid in a NESS. In the latter context, we note that in a fermionic quantum fluid there potentially (depending on the values of the Landau Fermi-liquid parameters) are many other zero modes that also display $\omega \sim k$ scaling, in addition to the shear velocity. These can change the prefactor of the singularity, but not the scaling behavior. Also, in a charged Fermi liquid (e.g., conduction electrons in metals) the first-sound mode turns into the massive plasmon, so our approximation of neglecting pressure fluctuations is valid *a fortiori*.

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[27] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.244001 for the correlations of the fluctuating forces, and for a derivation of the correlation functions in the collisionless regime.

[28] P. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University, Cambridge, England, 1995).

[29] For a derivation of the linearized Navier-Stokes equations from quantum kinetic theory, see Ref. [18].

[30] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, Rev. Mod. Phys. **49**, 435 (1977).

[31] T. R. Kirkpatrick and D. Belitz, Velocity-dependent forces and non-hydrodynamic initial conditions in quantum and classical fluids, arXiv:2304.06157.

[32] Some authors, e.g., Pines and Noziéres [20], refer to X'' as the “spectral density” of X.