Entropy Bounds for Rotating AdS Black Holes

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We propose novel thermodynamic inequalities that apply to stationary asymptotically anti–de Sitter (AdS) black holes. These inequalities incorporate the thermodynamic volume and refine the reverse isoperimetric inequality. To assess the validity of our conjectures, we apply them to a wide range of analytical black hole solutions, observing compelling evidence in their favor. Intriguingly, our findings indicate that these inequalities may also apply for black holes of nonspherical horizon topology, as we show their validity as well for thin asymptotically AdS black rings.

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Introduction.—The thermal nature of black holes underpins many of the deepest insights into quantum gravity. Black hole entropy ensures the consistency of the second law of thermodynamics in our Universe and is a "smoking gun" for a microscopic description of the gravitational field [1,2]. Over the last decade, our understanding of the laws of black hole mechanics has expanded to include pressure and volume [3,4]. The study of these terms is often called extended black hole thermodynamics and has led to new perspectives on gravitational phase transitions [5], black hole heat engines [6], and holography [7–11].

Extended black hole thermodynamics centers on thermodynamic volume, which has a geometric definition in terms of Komar integrals and is important for rendering consistent the Smarr formula for anti–de Sitter (AdS) black holes [3]. If one allows for variations in the cosmological constant, then the thermodynamic volume appears in the first law as its conjugate quantity.

A particularly interesting early result in extended thermodynamics is the reverse isoperimetric inequality (RII) [4]. The RII conjectures that for a black hole in D dimensions with horizon area A and thermodynamic volume V, the ratio

$$\mathcal{R} \equiv \left(\frac{V}{\mathcal{V}_0}\right)^{1/(D-1)} \left(\frac{\mathcal{A}_0}{A}\right)^{1/(D-2)} \tag{1}$$

satisfies $\mathcal{R} \geq 1$ [12]. Physically, the RII is the idea that, for a black hole with a fixed thermodynamic volume, there is a maximum possible entropy. The maximum entropy is achieved for the Schwarzschild-AdS black hole, which saturates the inequality. This allows for the following alternate interpretation of the RII: The entropy of a black hole of thermodynamic volume V is no more than the entropy of the Schwarzschild-AdS black hole with the same volume, or

$$A(V) \leq A_{\mathrm{Schw}}(V). \tag{2}$$

Support for the RII is robust. For instance, in [4] the RII was found to hold for a wide variety of asymptotically AdS black holes in four and higher dimensions. Further corroboration was given in [13], where it was found to hold for the topologically nontrivial higher-dimensional AdS black rings. In [14], it was found that the conjecture applies to black hole and cosmological horizons in asymptotically de Sitter spacetimes, while [15] showed the RII is amenable to the inclusion of conical deficits. Intriguingly, the results of [10] point toward a "quantum" RII when semiclassical effects are accounted for. Despite this progress, no general mathematical proof of the inequality or a precise statement of its necessary or sufficient conditions has been established, except for a few specific cases. For example, in [16], researchers proved the RII for static black holes with planar horizons, assuming an empirically motivated formula for the thermodynamic volume and the null energy condition.

In addition to intrinsic interest as a well-supported geometric and thermodynamic inequality, the RII has led to several intriguing results. For instance, in [17], a negative heat capacity at constant volume was linked with a violation of the RII, implying thermodynamic instability. The authors of [18] investigated the thermodynamic volume from a microscopic perspective and suggested that a violation of the RII in three spacetime dimensions is related to overcounting of the field theory entropy in the Cardy formula. Additionally, in [8], the authors established a connection between thermodynamic volume and the complexity of formation in holography. They argued that the RII can be understood as a lower bound on the complexity of formation set by the entropy.

There are no known counterexamples to the RII for asymptotically AdS black hole solutions of Einstein gravity in dimension $D \geq 4$. Beyond this, there are only two cases where violations of the RII have been argued for, but neither case is completely compelling. The first example pertains to a class of "ultraspinning" black holes that arise from a singular limit of the Kerr-AdS metric [19–22]. The ultraspinning black hole is asymptotically locally AdS and possesses an event horizon that is noncompact yet has finite area. In the original Refs. [20,21], it was argued that these black holes highlight the role played by horizon topology in the statement of the RII. However, whether they provide a legitimate counterexample to the RII was subsequently called into question [23,24]. Essentially, the ultraspinning limit results in a metric of reduced cohomogeneity, which makes the thermodynamics as considered ill-defined.

The second possible violation of the RII is associated with electrically charged Bañados-Teitelboim-Zanelli (BTZ) black holes and is closely connected with the challenges of defining Komar charges in lower dimensions (see also [25]), as well as the peculiarities of the charged BTZ solution [26]. The RII is only violated when an alternative (nongeometric) definition of the thermodynamic volume is employed. If one adheres to the geometric definition of the thermodynamic volume, then the RII is satisfied and no violation occurs [26,27]. In summary, while the literature features potential violations of the RII, to date, there is no definitive counterexample to the conjecture.

Our purpose here is to present novel inequalities involving the thermodynamic volume that are generalizations of the RII. Our refined reverse isoperimetric inequalities (RRIIs) include angular momentum. They reduce to the standard RII when the angular momentum vanishes, but are otherwise strictly stronger statements. The strongest variant of the RRII is the following.

Conjecture 1 (Strong RRII).—For a stationary asymptotically AdS black hole of mass M, angular momenta J_i, and thermodynamic volume V, the following inequality holds:

$$A(M, J_i, V) \leq A_{\text{Kerr}}(M, J_i, V), \tag{3}$$

where A_{Kerr} is the area of the Kerr-AdS black hole with the same parameters.

Conjecture 1 is the statement that for fixed values of (M, J_i, V), the Kerr-AdS black hole (if it exists) has maximum entropy [28]. Any deformation of the solution, e.g., through the incorporation of additional charges or matter fields, leads to a decrease in the black hole entropy. In the limit $J_i \to 0$, the Kerr-AdS area reduces to the Schwarzschild-AdS area and the RII (1) is recovered.

The conjecture takes inspiration from the Penrose inequality and its stronger generalizations that incorporate conserved charges [29]. Restricted to stationary spacetimes, the Penrose inequality provides a bound on the mass in terms of the area of the horizon, holding as an equality for slices of the asymptotically flat Schwarzschild black hole and as an inequality for other stationary, asymptotically flat black holes. Analogously, there exists a stronger form of the Penrose inequality that includes angular momentum. This version holds as an equality for slices of the Kerr solution and as an inequality for more general solutions [30,31] (see also [32–37]).

The RII (1) holds as an equality for the Schwarzschild-AdS black hole, while it is an inequality under more general circumstances. In the same spirit as the stronger version of the Penrose inequality, we sought to find a generalization of the RII that holds as an equality for the Kerr-AdS black holes, and then investigate whether this relation holds more generally as an inequality. This led us to the RRII given in (3). Below, we will provide evidence in favor of this conjecture by examining a large number of examples. We also present conjectures weaker than (3), but stronger than (1).

Evidence for the strong RRII.—We will now present the evidence we have accumulated in favor of the conjecture (3). To streamline the discussion, the form of the metrics and relevant thermodynamic parameters have been presented in the Supplemental Material [38].

Consider first the Kerr-Newman-AdS black hole, for which the extended thermodynamics was first studied in [39,40]. For this case, the following identity holds among the thermodynamic parameters:

$$36\pi M^2 V^2 - M^2 A^3 - 64\pi^3 J^4 = 16\pi^2 Q^2 J^2 A. \tag{4}$$

In particular, note that when the charge $Q = 0$, the metric reduces to the Kerr-AdS$_4$ solution and the left-hand side vanishes identically. To check the validity of the RRII, we hold fixed M, J, and V and study how A changes as the charge Q is varied. To satisfy the RRII conjecture requires that the right-hand side is non-negative. This is manifestly so, and therefore the conjecture (3) holds for the Kerr-Newman-AdS black hole.

For our next example, we examine the charged, rotating AdS C-metric. We validate the inequality using the Christodoulou-Ruffini mass formula from [15], specifically referencing (17) from that study, which yields

$$\frac{4\pi M^2}{S} \leq \left(\frac{3\pi M V}{2S^2} - \frac{2C^2}{x^2}\right)^2 - 4\left(\frac{\pi J}{S}\right)^4, \tag{5}$$

while the combination of (11) and (13) of [15] gives

$$\frac{3\pi M V}{2S^2} - \frac{2C^2}{x^2} > 0. \tag{6}$$

Combining these relations and replacing $S = A/4$, we obtain

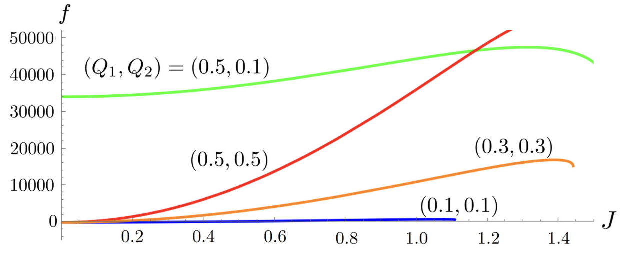

FIG. 1. For the pairwise-equal charge black holes of $D = 4$ gauged supergravity, we plot $f \equiv 36\pi M^2 V^2 - M^2 A^3 - 64\pi^3 J^4$ against several charge values, keeping $m = 1$ and $l = 5$ constant. The RRII is valid when $f \geq 0$. The curve end point for $(Q_1, Q_2) = (0.5, 0.1)$ represents an extremal black hole.

$$36\pi M^2 V^2 - M^2 A^3 - 64\pi^3 J^4 \geq 0. \tag{7}$$

Therefore, the RRII holds for this solution.

We next consider the rotating pairwise-equal charge black holes in $D = 4$ gauged supergravity. These, characterized by two $U(1)$ charges, were first detailed in [41] and later analyzed thermodynamically in [4,42]. Using *Mathematica*, it is possible to directly prove that (3) holds. For illustrative purposes, we present a few representative examples in Fig. 1.

We now turn to higher dimensions. In these cases, it is not possible to proceed analytically, and we resort to numerical exploration of the parameter space. To test the RRII in five dimensions, we examine a charged and rotating solution of minimal gauged supergravity presented in [54]. This solution is defined by five parameters —(r_+, a, b, g, q)—that represent the horizon radius, spin parameters, cosmological length scale, and charge, respectively. Our testing approach involves generating random values for these parameters and verifying that they correspond to a physically reasonable black hole by ensuring a nonsingular exterior metric. Next, we compute the conserved charges associated with these parameter values, then we determine whether one (or more) Kerr-AdS_5 black hole exists with the same volume and conserved charges, and compute the associated parameters $(\tilde{r}_+, \tilde{a}, \tilde{b}, \tilde{g})$. Finally, we compare the areas to confirm the validity of the conjecture (3).

For the solution of [54], we have carried out this procedure for approximately 100 000 randomly sampled parameter values, and in no case have we found a violation of the RRII (3). For all parameter values we have explored, if there exists a corresponding Kerr-AdS_5 black hole with the same volume and conserved charges, it has a larger area than the corresponding supergravity solution. Our results provide strong numerical support for the validity of (3).

In $D = 7$ gauged supergravity, there are exact rotating black hole solutions with three independent angular momenta and equal $U(1)$ charges [43]. We study their extended thermodynamics for the first time in the Supplemental Material [38]. While studying the RRII for this black hole, computational constraints limited our verification to a few hundred random parameter sets. Despite this, no counterexamples emerged, even in solutions with pathologies like closed timelike curves.

We can make further analytical progress by noting a useful corollary of the strong RRII (3) that follows when the black hole of interest reduces to the Kerr-AdS black hole as some parameter $w \to 0$ [55]. If we further assume that the area is an analytic function of the parameter w, we can expand (3) in the vicinity of the Kerr solution. The RRII implies in this limit that the first nonvanishing signed derivative of A with respect to the parameter w must be negative,

$$\lim_{w\to 0} \mathrm{sign}(w)^{n_\star} \left(\frac{\partial^{n_\star} A}{\partial w^{n_\star}}\right)_{M, J_i, V} \leq 0, \tag{8}$$

where $n_\star$ denotes the order of the first nonvanishing derivative. This inequality implies the RRII in some small neighborhood of the Kerr solution where higher-order terms in a Taylor expansion can be neglected. As such, this is a necessary condition for the validity of conjecture (3), but it is not a sufficient condition. One advantage of (8) is that it can be applied directly to a particular black hole of interest and does not require a direct comparison with the Kerr solution.

For all black holes studied, we have proven that (8) holds. In every instance, the first derivative vanishes and the second is strictly negative. This reveals the Kerr-AdS black hole as a local entropy maximum, and confirms the RRII near the Kerr-AdS solution.

Bounds on the isoperimetric ratio.—It would be of interest to write (3) as an explicit correction to the bound satisfied by the isoperimetric ratio $\mathcal{R}$. By making reference to the physical parameters of only a single black hole, such an inequality would apply in cases where a Kerr-AdS black hole may not exist for the specified values of (M, J_i, V).

The first step toward constructing such an inequality would be to find a relationship among the physical parameters of the Kerr-AdS black hole of interest. Since the conjecture (3) involves only A, M, J_i, and V, the necessary relationship would be a function of these parameters $f(A, M, J_i, V)$ such that $f(A, M, J_i, V) = 0$. In high dimensions, obtaining such a relationship would generically require solving a polynomial of degree greater than 4. This prevents us from presenting in the general case RRII that explicitly involves the isoperimetric ratio $\mathcal{R}$. However, below we will present weaker versions of (3) that are similar to (1).

One exception is the four-dimensional case. There, it is straightforward to obtain a function of the relevant parameters that vanishes for Kerr-AdS black hole,

$$f(M, A, V, J) \equiv 36\pi M^2 V^2 - M^2 A^3 - 64\pi^3 J^4, \tag{9}$$

which we have implicitly made use of earlier in (4). In terms of this function, the inequality (3) becomes the statement $f(A, M, V, J) \geq 0$. Some algebra allows this inequality to be expressed directly in terms of the isoperimetric ratio. The simplest way to do so yields

$$\mathcal{R} \geq \left[1 - \left(\frac{4\pi J^2}{3MV}\right)^2\right]^{-1/6}. \tag{10}$$

The inequality is saturated for the Kerr-AdS black hole, and obviously reduces to (1) when $J \to 0$. In four dimensions, (10) is completely equivalent to (3).

In higher dimensions, it is not in general possible to do the same, because obtaining the identity $A_{\mathrm{Kerr}} = A_{\mathrm{Kerr}}(M, J_i, V)$ requires solving a polynomial of high degree. For these cases, we present reverse isoperimetric inequalities of intermediate strength, that is, strictly stronger than (1) but weaker than (3). The advantage of these intermediate inequalities is that they involve only quantities defined for one solution. Therefore, the intermediate inequalities have a potentially larger domain of applicability compared to the RRII.

Because of differences in rotating black holes across even and odd dimensions, the intermediate inequalities also vary. We have the following in even and odd dimensions.

Conjecture 2-1 (Intermediate RRII: even D).—For a black hole of mass M, angular momenta J_i, area A, and thermodynamic volume V, the following inequality holds:

$$\mathcal{R}^{D-1} \geq \left[1 - \left\{\frac{2\pi(D-2)J_{\min}^2}{(D-1)MV}\right\}^2\right]^{-1/2}, \tag{11}$$

where $J_{\min} \equiv \min\{|J_i|\}$ and $\mathcal{R}$ is defined in (1). If it happens that there is only a single nonzero angular momentum, call it J, then we can further say

$$\mathcal{R}^{D-1} \geq \left[1 - \left\{\frac{8\pi}{(D-2)(D-1)}\frac{J^2}{MV}\right\}^2\right]^{-1/2}. \tag{12}$$

Conjecture 2-2 (Intermediate RRII: odd D).—For a black hole of mass M, angular momenta J_i, area A, and thermodynamic volume V, the following inequality holds:

$$\mathcal{R}^{D-1} \geq \left(1 - \frac{2\pi J_{\min}^2}{MV}\right)^{-(D-3)/[2(D-2)]} \times \left(1 + \frac{2\pi(D-3)}{(D-1)}\frac{J_{\min}^2}{MV}\right)^{-(D-1)/[2(D-2)]}, \tag{13}$$

where $J_{\min} \equiv \min\{|J_i|\}$ and $\mathcal{R}$ is defined in (1). If it happens that there is only a single nonzero angular momentum, then we can further say

$$\mathcal{R}^{D-1} \geq \left[1 - \frac{4\pi}{(D-1)(D-2)}\frac{J^2}{MV}\right]^{-(D-3)/[2(D-2)]} \times \left[1 + \frac{4\pi(D-3)}{(D-2)(D-1)^2}\frac{J^2}{MV}\right]^{-(D-1)/[2(D-2)]}. \tag{14}$$

Conjecture 2-1 is saturated for the equal-spinning Kerr-AdS black holes in even dimensions, while Conjecture 2-2 is saturated for the odd-dimensional Schwarzschild-AdS black holes. That they hold as inequalities for the general Kerr-AdS solutions is proven in the Supplemental Material [38].

Numerically checking Conjectures 2-1 and 2-2 is more efficient than verifying the stronger (3). For the five-dimensional charged and rotating solution of minimal gauged supergravity, Conjecture 2-2 has been confirmed for approximately 10^7 parameter sets. Similarly, in the seven-dimensional case with equal charges, we have confirmed it for approximately 10^5 sets, providing strong evidence toward the validity of the conjecture.

Intriguingly, Conjectures 2-1 and 2-2 appear to hold beyond black holes with spherical horizon topology. We have checked these conjectures against the thin AdS black ring in all dimensions $D \geq 5$ [44]. The thin black ring has horizon topology $\mathbb{S}^1 \times \mathbb{S}^{D-3}$ with the $\mathbb{S}^1$ characterized by the radius R and the $\mathbb{S}^{D-3}$ characterized by the radius r_0. The ring is thin in the sense that $r_0 \ll \min\{R, \ell\}$, where ℓ is the AdS curvature radius. In particular, this means that the ratio $r_0/R \ll 1$ always. The thermodynamics of the thin black ring was explored in [44] and its extended thermodynamics in [13]. The latter showed the RII's validity for the ring, with the isoperimetric ratio (1) greatly exceeding 1 due to $r_0/R \ll 1$; cf. Sec. 6 of that work.

Here we can prove analytically that the Conjectures 2-1 and 2-2 hold for the thin black ring. The solution has a single nonvanishing angular momentum, and so in each case it is the second inequality that applies. The key detail is the expression for the ratio

$$\frac{J^2}{MV} = \frac{(D-1)[1+(D-2)\mathsf{R}^2][D-3+(D-2)\mathsf{R}^2]}{8\pi(D-2)(1+\mathsf{R}^2)^2}, \tag{15}$$

where we have introduced the notation $\mathsf{R} \equiv R/\ell$. This ratio is monotonically increasing, ranging from $J^2/(MV) = (D^2-4D+3)/[8\pi(D-2)]$ for $\mathsf{R} = 0$ to $J^2/(MV) = (D^2-3D+2)/(8\pi)$ in the limit $\mathsf{R} \to \infty$. This ratio is bounded and is order-one as a function of R/r_0. This fact ensures the intermediate RRII always holds (see Supplemental Material [38]).

Conclusions.—The original reverse isoperimetric inequality appears to be part of a hierarchy reminiscent of the nesting of Penrose inequalities for rotating and charged black holes. Within this hierarchy, the original RII is the least restrictive, and we have presented strong evidence for a more stringent version applicable to rotating black holes. This hints at an intricate link between thermodynamic volume and black hole entropy, revitalizing the initial conjecture and opening up fresh avenues of inquiry.

Here we have focused on the case of asymptotically AdS black holes in $D \geq 4$ dimensions, but similar questions could be explored in a variety of other contexts. For example, we anticipate an extension of this result to hold for de Sitter black holes and cosmological horizons, along the lines of [14]. Furthermore, it is known that Misner strings possess thermodynamic volume [56], and it may be possible to formulate a version of the (R)RII that applies to spacetimes with NUT charge. In all cases, it is natural to expect further possible extensions of the RII that incorporate angular momentum, charge, or possibly both charge and angular momentum. Our Letter has utilized examples of black holes with additional conserved charges. Analyzing the validity of the new conjecture for hairy black holes, similar to [57], would be an important step. Finally, understanding the holographic interpretation of both the RII and RRII would be worth further study. Using the framework of [9,11], it should be possible to address this question in concrete terms.

Another question concerns uniqueness. While both the Schwarzschild-AdS and Reissner-Nordström-AdS black holes saturate the RII, in $D = 4$ only the Kerr-AdS black hole saturates the RRII. This hints that saturation may occur only for Kerr-AdS black holes.

Ultimately, since the RRII is stronger than the original RII, finding a counterexample might be simpler. Such a counterexample would clarify the conditions for the (R)RII's validity and strengthen its mathematical foundation.

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