Non-Gaussian Dynamics of Quantum Fluctuations and Mean-Field Limit in Open Quantum Central Spin Systems

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Central spin systems, in which a *central* spin is singled out and interacts nonlocally with several *bath* spins, are paradigmatic models for nitrogen-vacancy centers and quantum dots. They show complex emergent dynamics and stationary phenomena which, despite the collective nature of their interaction, are still largely not understood. Here, we derive exact results on the emergent behavior of open quantum central spin systems. The latter crucially depends on the scaling of the interaction strength with the bath size. For scalings with the inverse square root of the bath size (typical of one-to-many interactions), the system behaves, in the thermodynamic limit, as an open quantum Jaynes-Cummings model, whose bosonic mode encodes the quantum fluctuations of the bath spins. In this case, non-Gaussian correlations are dynamically generated and persist at stationarity. For scalings with the inverse bath size, the emergent dynamics is instead of mean-field type. Our Letter provides a fundamental understanding of the different dynamical regimes of central spin systems and a simple theory for efficiently exploring their nonequilibrium behavior. Our findings may become relevant for developing fully quantum descriptions of many-body solid-state devices and their applications.

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Collective quantum systems, such as spin ensembles with infinite-range interaction, are ubiquitous in physics and naturally emerge, e.g., in cold-atom experiments [1–7]. The broad set of tools available for these systems [8–24] permits for an in-depth characterization of their emergent behavior [8–10,17,25–34], which is, in general, exactly described by a *mean-field* theory [18–24,35].

A paradigmatic class of many-body systems featuring collective interaction is that of (open quantum) central spin systems [36–46]. The latter consist of a *central* spin which couples nonlocally to N *bath* spins, with interaction strength g [cf. Figs. 1(a)–1(b)]. These systems provide quantum models for nitrogen-vacancy centers and quantum dots, and describe their applications as quantum memories or nanoscale quantum sensors [47–55]. Despite such a broad relevance, an emergent theory for central spin systems in the thermodynamic limit is still missing [50,56–59], especially within the framework of open quantum systems [41,42,60]. In this regard, a key complication arises from the fact that, even though they feature a collective interaction [cf. Fig. 1(a)], central spin systems are not always captured by a mean-field theory [60]. This observation poses the challenge of understanding why mean-field theory can fail to describe these systems in certain parameter regimes and whether there still exists an effective theory for these cases. Answering these questions can pave the way to a fully quantum description of many-body solid-state devices [50,56–59], thus enabling the analysis and the exploration of protocols for controlling quantum bath-spin many-body dynamics or for engineering correlated quantum states [44,61–70].

In this Letter, we make progress in this direction by analytically deriving the emergent dynamical theory for open quantum central spin systems [cf. Figs. 1(a)–1(b)]. For $g \sim 1/\sqrt{N}$, we show that the central spin system behaves, in the thermodynamic limit, as a one-spin one-boson system, related to the Jaynes-Cummings model [71],

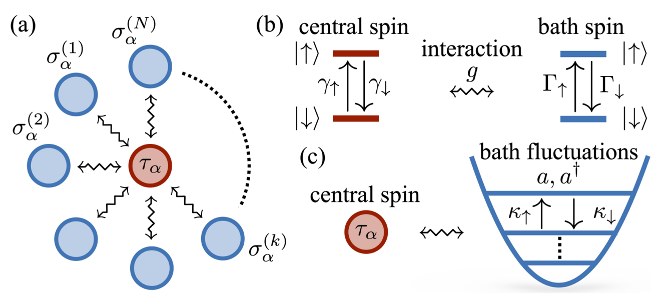

FIG. 1. Sketch of the system. (a) A central spin, described by Pauli matrices τ_α, interacts with N bath spins, denoted by the matrices $\sigma_\alpha^{(k)}$. (b) Spins are subject to decay and pump of excitations, with rates $\gamma_{\uparrow,\downarrow}$ ($\Gamma_{\uparrow,\downarrow}$) for the central spin (bath spins). The central spin interacts with the bath spins, with coupling strength g, via exchange of excitations. (c) For $g \sim 1/\sqrt{N}$ and in the limit $N \to \infty$, the central spin system behaves as an open quantum Jaynes-Cummings model. The bosonic mode accounts for the quantum fluctuations of the bath spins, which develop a strong non-Gaussian character.

which encodes the coupling of the central spin with the quantum fluctuations [23,72–74] of the bath [see illustration in Fig. 1(c)]. In this scenario, the system does not obey a mean-field theory, but rather a quantum *fluctuating-field* one, and develops strong long-lived non-Gaussian correlations which persist in the thermodynamic limit. Central spin systems are thus a promising resource for engineering complex quantum fluctuations and non-Gaussian correlations in many-body systems—which is a matter of current interest [75]. We further consider an interaction strength scaling as $g \sim 1/N$. In this case, the central spin couples to the average behavior of the bath spins and the system is described by a mean-field theory.

Our Letter delivers new insights into the dynamics of open quantum central spin systems and resolves a discrepancy between recent numerical results and mean-field prediction for these systems [60]. It further provides a clear-cut example of a quantum fluctuating-field theory in open quantum systems, whose fundamental properties may be relevant for developing an emergent theory for solid-state devices, able to account for many-body spin baths in the quantum regime [50,56–59].

Central spin system.—We focus on the system depicted in Fig. 1(a), consisting of $N+1$ spin-1/2 particles, with basis states $|\uparrow\rangle$ and $|\downarrow\rangle$. The central spin is described by the Pauli matrices τ_α while $\sigma_\alpha^{(k)}$, $k = 1, 2, \ldots N$ are those of the bath spins. The system Hamiltonian is (see below for extensions)

$$H = H_\tau + H_{\mathrm{int}}, \quad \text{with} \quad H_{\mathrm{int}} = g(\tau_+ S_- + \tau_- S_+). \quad (1)$$

Here, $H_\tau = \sum_\alpha w_\alpha \tau_\alpha$ is the Hamiltonian of the central spin only, $S_\pm = \sum_{k=1}^N \sigma_\pm^{(k)}$ and $\tau_\pm, \sigma_\pm$ are ladder operators, e.g., $\sigma_- = |\downarrow\rangle\langle\uparrow|$ and $\sigma_+ = \sigma_-^\dagger$. The interaction Hamiltonian H_{int} describes a collective excitation exchange, with coupling strength g, between the bath spins and the central one [cf. Figs. 1(a)–1(b)]. The system is also subject to irreversible processes, shown in Fig. 1(b), so that the dynamics of any system operator X is implemented by the equation $\dot{X}_t = \mathcal{L}[X_t] \coloneqq i[H, X_t] + \mathcal{D}_\tau[X_t] + \mathcal{D}_{\mathrm{bath}}[X_t]$ [76–78], where

$$\mathcal{D}_\tau[X] = \gamma_\downarrow \mathcal{W}_{\tau_-}[X] + \gamma_\uparrow \mathcal{W}_{\tau_+}[X],$$
$$\mathcal{D}_{\mathrm{bath}}[X] = \sum_{k=1}^N (\Gamma_\downarrow \mathcal{W}_{\sigma_-^{(k)}}[X] + \Gamma_\uparrow \mathcal{W}_{\sigma_+^{(k)}}[X]). \quad (2)$$

The rates $\gamma_{\uparrow,\downarrow}$ ($\Gamma_{\uparrow,\downarrow}$) are associated with irreversible pump and decay of excitations for the central spin (bath spins) and $\mathcal{W}_\nu[X] = \nu^\dagger X \nu - (\nu^\dagger \nu X + X \nu^\dagger \nu)/2$.

A particular instance of the system above was investigated in Ref. [60]. It was numerically shown that a mean-field approach, obtained by neglecting correlations among spins, does not capture the behavior of the system in the thermodynamic limit, for $g \sim 1/\sqrt{N}$. This came as quite a surprise since it is, at least at first sight, in stark contrast with what happens to structurally similar spin-boson models [15,24,32,60,79,80]. In what follows, we rigorously explain the dynamical behavior of central spin systems through exact analytical results.

Since we will work in the limit $N \to \infty$, it is useful to make a few considerations on the generator $\mathcal{L}$. Its dissipative terms describe irreversible processes occurring independently for each spin and are thus well defined for any N. The Hamiltonian H_{int} in Eq. (1) shows instead a peculiar behavior. From the viewpoint of the bath spins, it features the expected extensive character, with norm proportional to gN. However, this extensivity is problematic for the central spin. To see this, let us compute

$$\Omega([H_{\mathrm{int}}, \tau_z]) = 2gN(\tau_- \langle\sigma_+\rangle - \tau_+ \langle\sigma_-\rangle), \quad (3)$$

where Ω is the partial expectation over the bath spins, such that $\Omega(\tau_- S_+) = \tau_- \langle S_+ \rangle$, which we assume to be uncorrelated, $\Omega(\sigma_\alpha^{(k)} \sigma_\beta^{(h)}) = \Omega(\sigma_\alpha^{(k)}) \Omega(\sigma_\beta^{(h)})$, $\forall\ k \neq h$, and permutation invariant, $\Omega(\sigma_\alpha^{(k)}) = \langle \sigma_\alpha \rangle$, $\forall\ k$. In the thermodynamic limit, Eq. (3), which provides a term appearing in the time derivative of τ_z at time $t = 0$, diverges unless $\langle\sigma_\pm\rangle = 0$. Even using this assumption, the term

$$\Omega([H_{\mathrm{int}}, [H_{\mathrm{int}}, \tau_z]]) = 4g^2 N(\tau_z \langle \sigma_+ \sigma_- \rangle - \tau_+ \tau_- \langle \sigma_z \rangle), \quad (4)$$

shows that the Heisenberg equations for the central spin can diverge with N. To make the above dynamics well behaved, one has to appropriately rescale g. We first consider $g \sim 1/\sqrt{N}$ and show that this choice gives rise to an effective one-spin one-boson dynamics. Later, we turn to the case $g \sim 1/N$ which, as we demonstrate, is instead exactly described by a mean-field theory.

Local state of the bath spins.—Rescaling the coupling constant also affects the dynamics of the bath spins. Considering a generic *local* bath operator A (i.e., an operator solely acting on a finite number of bath spins [81]), we indeed have that $\|[H_{\mathrm{int}}, A]\| \sim g$ vanishes whenever g decays with N. This implies that the Hamiltonian H_{int} is irrelevant for the dynamics of local bath operators, which thus solely evolve according to $\mathcal{D}_{\mathrm{bath}}$ in the thermodynamic limit. This fact is summarized in the following Lemma, whose proof is given in Ref. [81].

Lemma 1.—For an interaction strength $g = g_0/N^z$, with $z > 0$ and g_0 an N-independent constant, we have

$$\lim_{N\to\infty} \|e^{t\mathcal{L}}[A] - e^{t\mathcal{D}_{\mathrm{bath}}}[A]\| = 0,$$

for any local bath-spin operator A.

The time evolution of local operators of the bath spins, e.g., the operators $\sigma_\alpha^{(k)}$, and of the so-called average operators $m_\alpha^N = \sum_{k=1}^N \sigma_\alpha^{(k)}/N$ as well (see Ref. [81]), is thus not affected by the presence of the central spin.

Furthermore, the dynamics generated by $\mathcal{D}_{\rm bath}$ drives the bath spins towards the permutation-invariant uncorrelated state $\Omega_{\rm SS}$, defined by the expectation values $\Omega_{\rm SS}(\sigma_z^{(k)}) = \Gamma_-/\Gamma_+$, with $\Gamma_\pm = \Gamma_\uparrow \pm \Gamma_\downarrow$, and $\Omega_{\rm SS}(\sigma_\pm^{(k)}) = 0$. Note that the latter relation, combined with the rescaling $g \sim 1/\sqrt{N}$, gives a well-defined thermodynamic limit for Eqs. (3)–(4). It is thus reasonable to assume $\Omega_{\rm SS}$ to be the "reference" (initial) state for the bath spins. As we shall see below, the bath spins nevertheless experience some dynamics. Their quantum fluctuations, described by nonlocal unbounded operators, are indeed affected by the coupling with the central spin and thus can evolve in time [21–23,87,88]. Without loss of generality, we focus on the case $\Gamma_\uparrow < \Gamma_\downarrow$ and define $\varepsilon \coloneqq -\Omega_{\rm SS}(\sigma_z^{(k)}) > 0$.

Bath-spin fluctuations.—For $g = g_0/\sqrt{N}$, the central spin couples to bath operators of the form $S_\pm/\sqrt{N}$, as clear from Eq. (1). These nonlocal unbounded operators are known as quantum fluctuation operators and behave, in the thermodynamic limit, as bosonic operators [72–74,85,89]. This can be understood by considering their commutator $[S_-, S_+]/N = -m_z^N$, which is proportional to an average operator. For product states like $\Omega_{\rm SS}$, average operators converge to their expectation value [82–84,90], essentially due to a law of large numbers. As such, we have $[S_-, S_+]/N \to \varepsilon$, which suggests the definition of the rescaled quantum fluctuations

$$a_N = \frac{S_-}{\sqrt{\varepsilon N}}, \qquad a_N^\dagger = \frac{S_+}{\sqrt{\varepsilon N}}. \tag{5}$$

The latter are such that $[a_N, a_N^\dagger] \to 1$, and thus behave as annihilation and creation operators. The quantum state of the limiting fluctuation operators $a = \lim_{N\to\infty} a_N$ and $a^\dagger = \lim_{N\to\infty} a_N^\dagger$ (where convergence is meant in a quantum central limit sense [72–74,81]) emerges from the state $\Omega_{\rm SS}$. It is a bosonic thermal state ρ_β identified by the occupation $\mathrm{Tr}(\rho_\beta a^\dagger a) = \lim_{N\to\infty}\Omega_{\rm SS}(a_N^\dagger a_N) = \Gamma_\uparrow/(\varepsilon\Gamma_+)$, as proved in the following Proposition.

Proposition 1.—The state $\Omega_{\rm SS}$ and the operators a_N, $a_N^\dagger$ give rise, in the limit $N \to \infty$, to a bosonic algebra, with operators a, $a^\dagger$ and state $\tilde{\Omega}_\beta(\cdot) = \mathrm{Tr}(\rho_\beta \cdot)$, where

$$\rho_\beta = \frac{e^{-\beta\omega a^\dagger a}}{1 - e^{-\beta\omega}}, \quad \text{and} \quad \beta\omega = -\log\frac{\Gamma_\uparrow}{\Gamma_+\varepsilon + \Gamma_\uparrow}.$$

Proof.—Following, e.g., Refs. [72,86], in order to show that the operators a_N, $a_N^\dagger$ behave, in the limit $N \to \infty$, as bosonic operators equipped with the state ρ_β, we need to show that (in the spirit of a central limit theorem)

$$\lim_{N\to\infty} \Omega_{\rm SS}\left(e^{s a_N - s^* a_N^\dagger}\right) = e^{-|s|^2/(2\varepsilon)} = \mathrm{Tr}\left(\rho_\beta e^{s a - s^* a^\dagger}\right),$$

and analogous relations for products of the above exponentials. These limits define an equivalence relation between bath-spin fluctuations and a Gaussian bosonic system. The explicit calculation is reported in Ref. [81]. ■

Mapping bath-spin fluctuations onto bosonic operators shows that the central spin system becomes, in the thermodynamic limit, a one-spin one-boson model [cf. Fig. 1(c)]. The task is now to derive its dynamics.

Emergent non-Gaussian dynamics.—The terms in the generator concerning the central spin only, i.e., H_τ and $\mathcal{D}_\tau$, are not affected by the limit $N \to \infty$. However, to identify the emergent dynamics we also have to control the action of the generator $\mathcal{D}_{\rm bath}$, and of the interaction Hamiltonian $H_{\rm int}$, on the relevant operators. The aim is then to interpret this action as that of a dynamical generator for the one-spin one-boson model formed by the central spin and the bath-spin quantum fluctuations.

For the interaction Hamiltonian, we observe that [recalling Eq. (5) and using that $g = g_0/\sqrt{N}$]

$$i[H_{\rm int}, a_N] = i g_0 \sqrt{\varepsilon}\tau_-[a_N^\dagger, a_N] \xrightarrow{N\to\infty} -i g_0\sqrt{\varepsilon}\tau_-, \tag{6}$$

which follows from the bosonic character of quantum fluctuations. Since central spin operators are not affected by the limit, we conclude that the emergent interaction is described by the Jaynes-Cummings Hamiltonian [71]

$$\tilde{H}_{\rm int} = g_0\sqrt{\varepsilon}(\tau_- a^\dagger + \tau_+ a). \tag{7}$$

The dissipator $\mathcal{D}_{\rm bath}$ is not of collective type. Still, we can understand its limiting behavior by analyzing its action on the operators a_N, $a_N^\dagger$. We observe that $\mathcal{D}_{\rm bath}[a_N] = -\Gamma_+ a_N/2$, and that

$$\mathcal{D}_{\rm bath}[a_N^\dagger a_N] = -\Gamma_+ a_N^\dagger a_N + \frac{\Gamma_\uparrow}{\varepsilon} \xrightarrow{N\to\infty} -\Gamma_+ a^\dagger a + \frac{\Gamma_\uparrow}{\varepsilon}.$$

These relations suggest that the dissipative processes are implemented on quantum fluctuations by the map

$$\tilde{\mathcal{D}}[X] = \kappa_\downarrow \mathcal{W}_a[X] + \kappa_\uparrow \mathcal{W}_{a^\dagger}[X]. \tag{8}$$

The above is a quadratic map and encodes loss and pump of bosonic excitations, with rates $\kappa_\downarrow = \Gamma_+ + \Gamma_\uparrow/\varepsilon$ and $\kappa_\uparrow = \Gamma_\uparrow/\varepsilon$ [cf. Fig. 1(c)]. Our considerations, gathered in the following Theorem, allow us to establish that the central spin system becomes an emergent spin-boson system associated with the dynamical generator

$$\tilde{\mathcal{L}}[X] = i[H_\tau + \tilde{H}_{\rm int}, X] + \mathcal{D}_\tau[X] + \tilde{\mathcal{D}}[X]. \tag{9}$$

In essence, this generator describes Jaynes-Cummings physics [71] in the presence of dissipation and of a possible Hamiltonian "driving," H_τ, on the central spin.

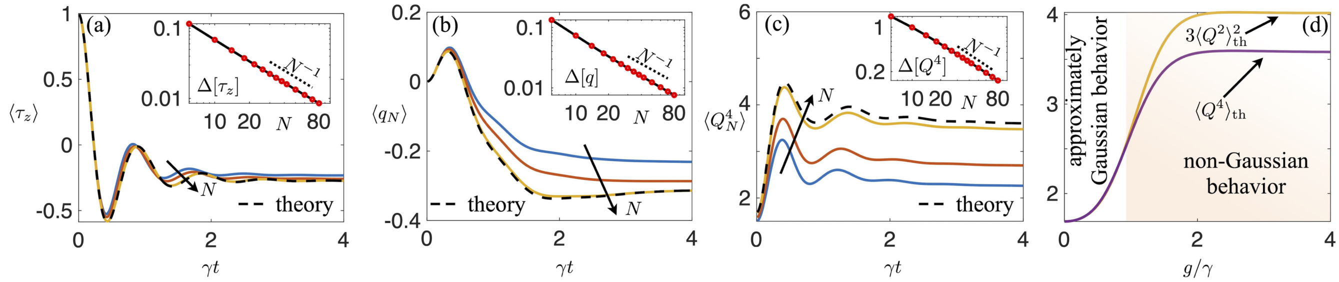

FIG. 2. Emergent dynamics and non-Gaussian fluctuations. System with $\gamma_\uparrow = 0.8\gamma$, $\gamma_\downarrow = 0.1\gamma$, $\Gamma_\uparrow = 0.2\gamma$, $\Gamma_\downarrow = \gamma$, $w_x = 2w_z = \gamma$, and γ being a reference rate. Initially, the central spin is in state $|\uparrow\rangle$ and the bath spins are described by $\Omega_{\rm SS}$. (a) Dynamics of the magnetization $\langle\tau_z\rangle$. The curves shown are for $N = 6$, 10, 80. The dashed line is the model in Eq. (9). Here, $g/\gamma = 4$. The inset shows $\Delta[\tau_z] \coloneqq \max_{\gamma t\in[0,4]}|\langle\tau_z\rangle - \langle\tau_z\rangle_{\rm th}|$, where $\langle\cdot\rangle_{\rm th}$ denotes the expectation for the model in Eq. (9). (b) Same as (a) for the "quadrature" $q_N = (a_N + a_N^\dagger)/\sqrt{2}$. The inset shows $\Delta[q] \coloneqq \max_{\gamma t\in[0,4]}|\langle q_N\rangle - \langle q\rangle_{\rm th}|$, with $q = (a + a^\dagger)/\sqrt{2}$. (c) Fourth moment of the centered quadrature $Q_N = q_N - \langle q_N\rangle$, compared with the prediction (dashed line). The inset displays $\Delta[Q^4] \coloneqq \max_{\gamma t\in[0,4]}|\langle Q_N^4\rangle - \langle Q^4\rangle_{\rm th}|$, with $Q = q - \langle q\rangle_{\rm th}$. (d) Stationary value of $\langle Q^4\rangle_{\rm th}$ compared with the Gaussian estimate $3\langle Q^2\rangle^2_{\rm th}$. The shaded region highlights the regime in which $\langle Q^4\rangle_{\rm th}$ is signaling a strongly non-Gaussian quantum state.

Theorem 1.—For $g = g_0/\sqrt{N}$, the action of $\mathcal{L}$ on monomials of bath-spin fluctuations and central spin operators gives rise, under any expectation taken with $\Omega_{\rm SS}$, to the map $\tilde{\mathcal{L}}$ on the emergent one-spin one-boson system.

Proof.—The idea is to make the above argument valid for generic monomials of the form $P_N = \tau_\alpha a_N^{\dagger\ell} a_N^k m_z^{Nh}$. Because of Proposition 1, P_N converges to $P = \tau_\alpha a^{\dagger\ell} a^k(-\varepsilon)^h$ in a "weak" sense, i.e., whenever considering expectation values constructed with the state $\Omega_{\rm SS}$ and other monomials (see Ref. [81]). This convergence provides the starting point to investigate the action of $\mathcal{L}$. It can indeed be shown that $\mathcal{L}[P_N]$ produces a linear combination of monomials of the same type of P_N, plus corrections of order $O(1/N)$ under the considered expectation. Proposition 1 thus guarantees that $\lim_{N\to\infty}\mathcal{L}[P_N]$ converges to a linear combination of monomials of the same form of P. By direct calculation, we can show that such linear combination is equal to that produced by $\tilde{\mathcal{L}}[P]$ [81]. ∎

The theorem directly implies that the dynamics of the emergent one-spin one-boson model, describing the central spin system in the thermodynamic limit, is governed by the generator in Eq. (9), under physical regularity conditions on the evolution (see details in Ref. [81]).

To concretely benchmark our derivation, we perform numerical simulations of the model in Eq. (9). We consider the dynamics of the central spin system, as described by Eqs. (1) and (2), and analyze convergence of the numerical data for finite systems [12,13,15,16] to our prediction, upon increasing the size of the bath. The convergence behavior is shown in Figs. 2(a), 2(b), and 2(c) for different observables. In the insets of Figs. 2(a), 2(b), and 2(c), we provide the maximal absolute difference between finite-N results and our prediction for the thermodynamic limit. This error measure decays as $\sim 1/N$, as anticipated in the proof of Theorem 1, thus confirming the validity of our theory. As shown in Fig. 2(d), quite remarkably, the central spin system features, in the thermodynamic limit, non-Gaussian correlations among the bath spins which persist at stationarity.

Mean-field regime.—We now turn to the case $g = g_0/N$. Here, the norms of H_τ and $H_{\rm int}$ are of the same order and the central spin couples to the (bounded) bath-spin average operators m_α^N [cf. Eq. (1)]. Thus, it is not necessary to require $\langle\sigma_\pm\rangle = 0$ for a well-defined thermodynamic limit [cf. Eq. (3)] and we can therefore consider more involved bath-spin dynamics. For concreteness, we still focus on the dissipator $\mathcal{D}_{\rm bath}$ and introduce a noninteracting Hamiltonian $H_{\rm bath} = \sum_\alpha h_\alpha \sum_{k=1}^N \sigma_\alpha^{(k)}$. Other collective dynamics [23,35] would give analogous results.

The bath-spin dynamics is not affected by the central spin [cf. Lemma 1] and the evolved average operators $e^{t\mathcal{L}}[m_\alpha^N]$ converge (weakly) to the time-dependent multiples of the identity $m_\alpha(t)$, obeying a mean-field theory. The central spin instead feels the presence of the bath spins via a coupling to their average operators. The latter thus provide time-dependent (mean) fields "modulating" the central spin Hamiltonian (see also Ref. [91]). This is the content of the next Theorem proved in Ref. [81].

Theorem 2.—Consider $g = g_0/N$, an initial bath-spin permutation-invariant uncorrelated state Ω and the generator $\mathcal{L}$, with $H \to H + H_{\rm bath}$. Under any possible expectation (that is, in the weak operator topology [82–84]), the dynamics of central spin operators is generated, for $N \to \infty$, by $\mathcal{D}_\tau$ and the time-dependent Hamiltonian

$$H_\tau^{\rm mf} = H_\tau + g_0[m_-(t)\tau_+ + m_+(t)\tau_-].$$

Here, $m_\pm(t) = [m_x(t) \pm i m_y(t)]/2$ and $m_\alpha(t)$ are the (scalar) limits of the evolved bath-spin average operators.

The exactness of a mean-field theory in many-body systems is thus not merely related to the structure of the interaction. It relies on substituting certain time-evolved operators, in a finite set, with their expectation value. For this, it is sufficient that (i) the substitution is valid for the initial state, in the thermodynamic limit [90]; (ii) the action of the generator on these operators gives a "regular" function of them, plus at most terms vanishing with N [23,24,35]. If this happens, the involved operators converge to scalars at all times [24,35,81]. Despite the collective interaction, for $g \sim 1/\sqrt{N}$, the central spin couples to the quantum fluctuations of the bath spins, which do not even converge to scalar quantities in the initial state. In this case, mean-field theory cannot be exact.

Inhomogeneous coupling and extensions.—Our approach remains valid for inhomogeneous couplings [50,53,92]. To show this, let us consider the interaction $H_{\mathrm{int}} = g\tau_+ \sum_{k=1}^{N} c_k \sigma_-^{(k)} + \mathrm{H.c.}$ and set, without loss of generality, $\sum_{k=1}^{N} |c_k|^2/N \to 1$, in the thermodynamic limit. By defining the quantum fluctuation $a_N = (1/\sqrt{\varepsilon N}) \sum_{k=1}^{N} c_k \sigma_-^{(k)}$, we have that $[a_N, a_N^\dagger] \to 1$, we can prove Proposition 1 and the results in Eqs. (6)–(9), leading to the emergent one-spin one-boson theory for $g \sim 1/\sqrt{N}$. By defining the average bath-spin operators $m_\alpha^N = (1/N) \sum_{k=1}^{N} c_k \sigma_\alpha^{(k)}$ and following Theorem 2, we can show the validity of mean-field theory for $g \sim 1/N$.

Our derivation holds for generic couplings of the form $\tau_\alpha \sum_{k=1}^{N} \sigma_{x/y}^{(k)}$ and interactions among bath spins (see Ref. [81]). It also holds for arbitrary spin particles [23] and for non-Markovian dynamics with time-dependent generators. In the latter case, time-dependent bath-spin rates would lead to a time-dependent state on local bath-spin operators, and thus to a time-dependent ε. Note that our results could not be obtained via Holstein-Primakoff approaches [93], due to the presence of local dissipation and/or inhomogeneous coupling [32]. Even when Holstein-Primakoff transformations can be applied (e.g., unitary dynamics [94,95]), our derivation inherently accounts for the state-dependent emergent commutation relation between the operators $S_\pm/\sqrt{N}$ [23,72–74,85], encoded in ε, which may be overlooked by other approaches.

Discussion.—Central spin systems can be realized with nitrogen-vacancy centers or quantum dots [41,42,52,53,92]. In these cases, the coupling between the central spin and each bath spin depends on their distance and on the angle between the two spins and the applied magnetic field [53,92]. This allows one to control the "microscopic" couplings, and even to realize the $1/\sqrt{N}$ or the $1/N$ scalings, by engineering suitable structures [e.g., (quasi) one-dimensional ones] and choosing appropriate field directions. In the case of fixed couplings, desired regimes may instead be achieved by scaling-up with N other parameters, such as the driving fields [42,60]. Coupling strengths $g \propto 1/N$, related to the mean-field limit, are accurate in regimes with delocalized central-spin wave function [50]. Still, due to the finiteness of realistic systems, bath-spin fluctuations become relevant, on long time-scales, also in these cases [39,50]. Our approach allows us to treat them in the quantum regime. Furthermore, central spin systems can be realized with Rydberg atoms [96–98], guaranteeing highly controllable couplings [97,98]. Our findings thus also provide a simple way to benchmark these quantum-simulation platforms.

We now comment on related results. References [42,60] consider Hamiltonian H_τ and/or dissipation $\mathcal{D}_\tau$ with the same extensivity of H_{int}. This case is similar to that of Theorem 2 and indeed shows mean-field behavior. Another related result is Lemma 1.5 of Ref. [22], which focuses on closed systems with Hamiltonian given by H_τ and, e.g., $H_{\mathrm{int}} = (g_0/\sqrt{N})\tau_x S_x$. There, the emergent bosonic mode reduces to a classical random variable.

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