Unifying Emergent Hydrodynamics and Lindbladian Low-Energy Spectra across Symmetries, Constraints, and Long-Range Interactions

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We identify emergent hydrodynamics governing charge transport in Brownian random time evolution with various symmetries, constraints, and ranges of interactions. This is accomplished via a mapping between the averaged dynamics and the low-energy spectrum of a Lindblad operator, which acts as an effective Hamiltonian in a doubled Hilbert space. By explicitly constructing dispersive excited states of this effective Hamiltonian using a single-mode approximation, we provide a comprehensive understanding of diffusive, subdiffusive, and superdiffusive relaxation in many-body systems with conserved multipole moments and variable interaction ranges. Our approach further allows us to identify exotic Krylov-space-resolved diffusive relaxation despite the presence of dipole conservation, which we verify numerically. Therefore, we provide a general and versatile framework to qualitatively understand the dynamics of conserved operators under random unitary time evolution.

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Introduction.—Recent years have seen a surge of interest in the nonequilibrium dynamics of quantum many-body systems, driven by rapid advancements in quantum simulation capabilities across diverse physical platforms. In particular, significant attention has been devoted to understanding the thermalization process of interacting many-body systems [1–7]. A vital theoretical tool that provides key insights into the dynamics of thermalizing quantum systems is the study of random unitary time evolution. While retaining analytical tractability, such methods can successfully capture universal properties of nonintegrable many-body dynamics such as transport, operator spreading, or entanglement growth [8–16]. In particular, the application of methods based on random unitary evolution has highlighted the importance of symmetries and constraints in many-body dynamics, unveiling a rich phenomenology of emergent hydrodynamics at late times. Recent results range from transport in long-range interacting systems [17–20] to anomalously slow subdiffusion [21–32] or even localization due to Hilbert space fragmentation in models with kinetic constraints [33–44].

In this Letter, we introduce a simple, yet powerful method to understand the qualitative behavior of late-time hydrodynamics based on Brownian Hamiltonian evolution, which can be modeled by Markovian dynamics and thus captured by a Lindblad equation [45–50]. Our approach successfully reproduces results reported in previous literature and allows us to uncover novel, unconventional hydrodynamic relaxation in constrained many-body systems. The key technical step relates dynamical properties such as the autocorrelation of conserved operators to the low-energy spectrum of an emergent effective Hamiltonian in a *doubled Hilbert space* [51,52]. The low-energy excitation spectrum of the latter thus dictates the longtime dynamics of such correlations. Accordingly, this mapping allows us to utilize well-established techniques in condensed matter physics, such as the single-mode approximation, to analyze our problem. Here, we apply this method to various scenarios: We show that systems

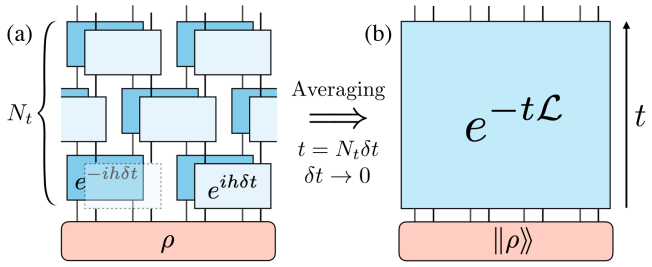

FIG. 1. Brownian circuit and effective Hamiltonian. Mapping (a) random operator dynamics to (b) imaginary-time evolution by an effective Hamiltonian $\mathcal{L}$ in a doubled Hilbert space. On the left, an operator ρ is evolved by a local Hamiltonian $H_t \equiv \sum_i h_i dB_{i,t}$ with Brownian random variable dB. Overlapping blocks for forward (backward) evolution [dark (light)] share the same Brownian variable, but all other Brownian variables are independently drawn from Gaussian distributions. On the right, we average over random variables while taking time steps to zero; this produces imaginary-time Schrödinger evolution by a Lindbladian operator.

conserving U(1) global charge as well as higher multipole moments exhibit diverse hydrodynamic relaxation depending on their symmetries and ranges of interactions. Then, we extend our approach to understand Krylov-subspace-resolved hydrodynamics, where we uncover general conditions under which relaxation is diffusive despite the presence of dipole conservation. We verify this diffusive relaxation numerically in lattice models in both one and two spatial dimensions.

Brownian circuits.—We consider time evolution by a time-dependent Hamiltonian $H_t \equiv \sum_i h_i\, dB_{i,t}$, defined via interaction terms h_i with $h_{\boldsymbol{x},\lambda} = h_{\boldsymbol{x}+\mathbf{1},\lambda}$ and Brownian random variables $dB_{t,i}$ at each time slice $[t, t+\delta)$. Here, the label $i = (\boldsymbol{x}, \lambda)$ encodes both the spatial support and operator type of h_i. The random variables have vanishing mean $\mathbb{E}[dB] = 0$ and finite variance $\mathbb{E}[dB^2] = 1/\delta$.

Under this time evolution, a density matrix $\rho(t)$ evolves as $\rho(t+\delta) \equiv e^{-iH_t\delta}\rho(t)e^{iH_t\delta}$. Averaging the infinitesimal time evolution over the random variables, the leading order operator evolution becomes (Ref. [53], Sec. B)

$$\mathbb{E}[\partial_t \rho] = -\frac{1}{2}\sum_i (h_i^2\rho - 2h_i\rho h_i + \rho h_i^2) = \mathcal{L}[\rho], \quad (1)$$

where $\mathcal{L}$ is a superoperator called the Lindblandian.

We now construct an alternative description of the operator dynamics Eq. (1) by employing the Choi isomorphism, a mapping from an operator acting on the Hilbert space $\mathcal{H}$ to a state defined on the doubled Hilbert space $\mathcal{H}_u \otimes \mathcal{H}_l$, where subscripts u, l are introduced to distinguish two copies of $\mathcal{H}$. For a given operator O, the mapping reads $O \mapsto \|O\rangle\!\rangle \equiv \sum_i |i\rangle \otimes (O|i\rangle)$, where the summation is over all basis states of the original Hilbert space (Ref. [53], Sec. A). Under this mapping, the Lindbladian superoperator $\mathcal{L}$ maps to a linear operator $\hat{\mathcal{H}}_{\mathcal{L}}$ acting on the doubled Hilbert space:

$$\hat{\mathcal{H}}_{\mathcal{L}} = \frac{1}{2}\sum_i |h_i^T \otimes \mathbb{I} - \mathbb{I} \otimes h_i|^2 =: \frac{1}{2}\sum_{\boldsymbol{x},\lambda} \mathcal{O}^\dagger_{\boldsymbol{x},\lambda}\mathcal{O}_{\boldsymbol{x},\lambda}, \quad (2)$$

where $|\cdots|^2$ should be understood as $(\cdots)^\dagger(\cdots)$, and $\mathcal{O}_{\boldsymbol{x},\lambda} = (h^T_{\boldsymbol{x},\lambda} \otimes \mathbb{I} - \mathbb{I} \otimes h_{\boldsymbol{x},\lambda})$. The average dynamics in Eq. (1) can then be recast into an imaginary time evolution generated by the effective Hamiltonian $\hat{\mathcal{H}}_{\mathcal{L}}$:

$$\partial_t \|O\rangle\!\rangle = -\hat{\mathcal{H}}_{\mathcal{L}}\|O\rangle\!\rangle \Rightarrow \|O(t)\rangle\!\rangle = e^{-t\hat{\mathcal{H}}_{\mathcal{L}}}\|O_0\rangle\!\rangle. \quad (3)$$

The dual forms of evolution can be seen diagramatically in Fig. 1. We are interested in the dynamics of a local operator O under Brownian evolution, which we characterize by the averaged autocorrelation function $\mathbb{E}\langle O_{\boldsymbol{y}}(0)O_{\boldsymbol{x}}(t)\rangle_\rho$ [54] with respect to the maximally mixed state $\rho = (1/D)\mathbb{I}$, where D is the dimension of the many-body Hilbert space.

Note that Eq. (2) inherits translation invariance from the interaction terms, $h_{\boldsymbol{x},\lambda} = h_{\boldsymbol{x}+\mathbf{1},\lambda}$. Therefore, we can label the eigenstates of $\hat{\mathcal{H}}_{\mathcal{L}}$ by their momentum; let $\|\boldsymbol{k},\nu\rangle\!\rangle$ be the eigenstates of $\hat{\mathcal{H}}_{\mathcal{L}}$ with energy $E_{\boldsymbol{k},\nu}$, carrying momentum $\boldsymbol{k}$ and an additional label ν. Inserting a completeness relation, we obtain

$$\begin{aligned}\mathbb{E}\langle O_{\boldsymbol{y}}(0)O_{\boldsymbol{x}}(t)\rangle_\rho &= \frac{1}{D}\langle\!\langle O_{\boldsymbol{y}}(0)\|e^{-t\hat{\mathcal{H}}_{\mathcal{L}}}\|O_{\boldsymbol{x}}(0)\rangle\!\rangle \\ &= \frac{1}{D}\sum_{\boldsymbol{k},\nu} e^{-tE_{\boldsymbol{k},\nu}}e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}|\langle\!\langle \boldsymbol{k},\nu\|O_{\boldsymbol{x}}\rangle\!\rangle|^2. \quad (4)\end{aligned}$$

Consider a d-dimensional system. Assuming a gapless dispersion $\min_\nu\{E_{\boldsymbol{k},\nu}\} \sim k^n$ at low momentum $\boldsymbol{k} \to 0$, as well as a finite overlap $|\langle\!\langle \boldsymbol{k},\nu\|O_x\rangle\!\rangle|^2$ of the operator of interest $\|O_{\boldsymbol{x}}\rangle\!\rangle$ with these gapless modes [55], the autocorrelation at $x = y$ decays algebraically as

$$\mathbb{E}\langle O_{\boldsymbol{x}}(t)O_{\boldsymbol{x}}(0)\rangle_\rho \underset{t\to\infty}{\sim} \int_k e^{-tk^n}\mathrm{d}^d\boldsymbol{k} \sim t^{-d/n}, \quad (5)$$

implying that the dynamical exponent $z = n$. Therefore, the study of late-time operator dynamics in the Brownian evolution reduces to the identification of gapless dispersing states in the effective Hamiltonian $\widehat{\mathcal{H}_{\mathcal{L}}}$.

Charge conservation.—We now assume that each h_i in the original Hamiltonian exhibits a U(1) charge conservation symmetry. In the doubled Hilbert space, the symmetry is doubled as well, and the effective Hamiltonian $\hat{\mathcal{H}}_{\mathcal{L}}$ in Eq. (2) must be symmetric under $G = \mathrm{U}(1)_u \times \mathrm{U}(1)_l$. We denote by G_{diag} and G_{off} the diagonal and off-diagonal subgroups of G, generated by $g_{\mathrm{diag/off}} = \hat{Q}_u \otimes \mathbb{I} \mp \mathbb{I} \otimes \hat{Q}_l$, where $\hat{Q}$ is the total charge operator (Ref. [53], Sec. A).

First, we examine the ground states of $\hat{\mathcal{H}}_{\mathcal{L}}$, which is positive semidefinite. The Choi state of the identity operator $\|\mathbb{I}\rangle\!\rangle$ satisfies $\hat{\mathcal{H}}_{\mathcal{L}}\|\mathbb{I}\rangle\!\rangle = 0$ and is thus a ground state of $\hat{\mathcal{H}}_{\mathcal{L}}$. Because of U(1) symmetry, $\mathbb{I}$ decomposes into the summation over projectors onto different charge sectors: $\mathbb{I} = \sum_m \mathcal{P}_m$, where $\mathcal{P}_m$ is the projector onto a U(1) sector of charge m. For a system with $N = L^d$ sites and local Hilbert space dimension M, $m \in \{0, 1, \ldots, ML^d\}$. We denote $\|m\rangle\!\rangle$ as the Choi state of $\mathcal{P}_m$. As such, $\|m\rangle\!\rangle$ is also a ground state of $\hat{\mathcal{H}}_{\mathcal{L}}$ with vanishing G_{diag} charge and a G_{off} charge of $2m$. Note that $\langle\!\langle m\|m\rangle\!\rangle = \dim[\mathcal{H}_m]$, the dimensionality of the charge-m sector. Moving forward, we renormalize $\|m\rangle\!\rangle$ to $\langle\!\langle m\|m\rangle\!\rangle = 1$.

The degenerate ground state manifold with different G_{off} charges implies spontaneous symmetry breaking of G_{off}. This can be shown explicitly by constructing a ground state $\|\theta\rangle\!\rangle \equiv \sum_m f(m)e^{im\theta}\|m\rangle\!\rangle$ such that under the rotation by G_{off} generator, $e^{i\alpha g_{\mathrm{off}}}\|\theta\rangle\!\rangle = \|\theta+\alpha\rangle\!\rangle \neq \|\theta\rangle\!\rangle$. The low-energy excitations of $\widehat{\mathcal{H}_{\mathcal{L}}}$ must be given by the Nambu-Goldstone modes for the broken continuous symmetry. A standard approach for constructing Goldstone modes is to apply G_{off} density modulations with momentum k on the ground state $\|m\rangle\!\rangle$. The variational ansatz for such a state is

defined as

$$\|m_{\boldsymbol{k}}\rangle\!\rangle \equiv \frac{1}{\sqrt{\mathcal{N}_{\boldsymbol{k}}}}\hat{\rho}_{\boldsymbol{k}}\|m\rangle\!\rangle, \qquad \hat{\rho}_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{x}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{L^{d/2}}(\hat{\rho}_{\boldsymbol{x},u} + \hat{\rho}_{\boldsymbol{x},l}), \tag{6}$$

where $\hat{\rho}_{\boldsymbol{x},u/l}$ measures U(1) charge in the layer u or l at position $\boldsymbol{x}$, and $\mathcal{N}_{\boldsymbol{k}} \equiv \langle\!\langle m\|\hat{\rho}^{\dagger}_{\boldsymbol{k}}\hat{\rho}_{\boldsymbol{k}}\|m\rangle\!\rangle$ is a static structural factor with $\hat{\rho}^{\dagger}_{\boldsymbol{k}} = \hat{\rho}_{-\boldsymbol{k}}$. It is straightforward to show that $\|m_{\boldsymbol{k}}\rangle\!\rangle$ carries a well-defined momentum $\boldsymbol{k}$ and thus $\langle\!\langle m_{\boldsymbol{k}}\|m_{\boldsymbol{k}'}\rangle\!\rangle = \delta_{\boldsymbol{k},\boldsymbol{k}'}$ (Ref. [53], Sec. E). We remark that since $(\hat{\rho}_{\boldsymbol{x},u} + \hat{\rho}_{\boldsymbol{x},l})$ measures a local G_{off} charge, the constructed mode corresponds to the density fluctuations of the G_{off} charge.

What is the energy of this variational state? With orthogonality between $\|m_{\boldsymbol{k}}\rangle\!\rangle$ for different momenta, the variational expected energy provides an upper bound for the low-energy dispersion of Eq. (2):

$$\langle\!\langle m_{\boldsymbol{k}}\|\hat{\mathcal{H}}_{\mathcal{L}}\|m_{\boldsymbol{k}}\rangle\!\rangle = \frac{1}{\mathcal{N}_{\boldsymbol{k}}}\sum_{\boldsymbol{x},\lambda}\langle\!\langle m\|[\mathcal{O}_{\boldsymbol{x},\lambda},\hat{\rho}_{\boldsymbol{k}}]^{\dagger}[\mathcal{O}_{\boldsymbol{x},\lambda},\hat{\rho}_{\boldsymbol{k}}]\|m\rangle\!\rangle, \tag{7}$$

where we used $\mathcal{O}_{\boldsymbol{x},\lambda}\|m\rangle\!\rangle = 0$. By using U(1) symmetry, the commutator in Eq. (7) can be recast as

$$[\mathcal{O}_{\boldsymbol{x},\lambda},\hat{\rho}_{\boldsymbol{k}}] = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sum_{\boldsymbol{y}\in\mathcal{S}_{\boldsymbol{x}}}\sum_{n=1}^{\infty}\left[\mathcal{O}_{\boldsymbol{x},\lambda}, \frac{[i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]^n}{n!}\hat{\rho}_{\boldsymbol{y}}\right], \tag{8}$$

where we used $[\mathcal{O}_{\boldsymbol{x},\lambda}, \sum_{\boldsymbol{y}}\hat{\rho}_{\boldsymbol{y}}] = 0$, and $\mathcal{S}_{\boldsymbol{x}}$ is the *local* support of the operator $\mathcal{O}_{\boldsymbol{x},\lambda}$ (thus warranting the expansion of $e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}$ for small $\boldsymbol{k}$). Generally, assuming a finite expectation value of the local dipole fluctuations $\langle\!\langle m\|[\mathcal{O}_{\boldsymbol{x},\lambda}, \sum_{\boldsymbol{y}} y_i\hat{\rho}_{\boldsymbol{y}}]|^2\|m\rangle\!\rangle$, the expansion Eq. (8) does not vanish at $n = 1$, giving rise to a leading order contribution proportional to k:

$$[\mathcal{O}_{\boldsymbol{x},\lambda},\hat{\rho}_{\boldsymbol{k}}] \propto k \Rightarrow \langle\!\langle m_{\boldsymbol{k}}\|\hat{\mathcal{H}}_{\mathcal{L}}\|m_{\boldsymbol{k}}\rangle\!\rangle \propto k^2. \tag{9}$$

Here, we focus on isotropic systems for simplicity; however, dynamical exponents can be obtained similarly for nonisotropic systems. Furthermore, $\mathcal{N}_{\boldsymbol{k}}$ is a constant, independent of k (Ref. [53], Sec. E). Therefore, $\|m_{\boldsymbol{k}}\rangle\!\rangle$ generically exhibits a quadratic ($E_{\boldsymbol{k}} \propto k^2$) dispersion, regardless of the details of the effective Hamiltonian. Note the similarity of our approach to the single-mode approximation in superfluid or quantum Hall states [56–58], where the Feynman-Bijl ansatz (Ref. [53], Sec. D) provides variational states that capture the dispersion of density fluctuation excitations.

Long-range interactions.—We extend our preceding analysis to charge-conserving systems with long-range interactions. Specifically, we consider the effects of long-range terms in our Hamiltonian of the form $h_{\boldsymbol{x},\boldsymbol{x}'} = |\boldsymbol{x}-\boldsymbol{x}'|^{-\alpha}(\hat{S}^{+}_{\boldsymbol{x}}\hat{S}^{-}_{\boldsymbol{x}'} + \text{H.c.})$, where $\hat{S}^{\pm}_{\boldsymbol{x}}$ are raising and lowering operators for the charge $\hat{\rho}_{\boldsymbol{x}}$ at site $\boldsymbol{x}$ and $\hat{Q} = \sum_{\boldsymbol{x}}\hat{\rho}_{\boldsymbol{x}}$ is conserved. The effective Hamiltonian reads $\hat{\mathcal{H}}_{\mathcal{L}} = \sum_{\boldsymbol{x},\boldsymbol{x}'}\mathcal{O}^{\dagger}_{\boldsymbol{x},\boldsymbol{x}'}\mathcal{O}_{\boldsymbol{x},\boldsymbol{x}'}$ and the commutator entering Eq. (7) becomes

$$[\mathcal{O}_{\boldsymbol{x},\boldsymbol{x}'},\hat{\rho}_{\boldsymbol{k}}] = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{(1-e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})})}{|\boldsymbol{x}-\boldsymbol{x}'|^{\alpha}}[\tilde{\mathcal{O}}_{\boldsymbol{x},\boldsymbol{x}'},\hat{\rho}_{\boldsymbol{x}}], \tag{10}$$

where $\tilde{\mathcal{O}}_{\boldsymbol{x},\boldsymbol{x}'} \coloneqq \mathcal{O}_{\boldsymbol{x},\boldsymbol{x}'}|\boldsymbol{x}-\boldsymbol{x}'|^{\alpha}$ is now distance independent. Assuming $\alpha > d/2$ and a finite expectation value for the square of the commutator on the rhs of Eq. (10), the variational energy of $\|m_{\boldsymbol{k}}\rangle\!\rangle$ is (Ref. [53], Sec. F)

$$\langle\!\langle m_{\boldsymbol{k}}\|\hat{\mathcal{H}}_{\mathcal{L}}\|m_{\boldsymbol{k}}\rangle\!\rangle \underset{k\to 0}{\propto} C_1(\alpha)|k|^{2\alpha-d} + C_2(\alpha)k^2. \tag{11}$$

Thus, for $\alpha < 1 + d/2$, the system relaxes superdiffusively with $z = 2\alpha - d$, successfully reproducing previous works on long-range interacting systems [17,18,59]. Alternatively, for $\alpha \le d/2$ the prefactors $C_1(\alpha)$ and $C_2(\alpha)$ exhibit divergences and the associated modes become gapped (Ref. [53], Sec. F); accordingly, the operator decays exponentially fast [17], entering an effectively nonlocal "all-to-all" interacting regime.

Dipole conservation.—The method outlined above also applies to systems with conserved quantities beyond U(1) charges. Let us focus on one-dimensional models with charge multipole symmetries, as relevant to fracton systems [60–68], generated by $Q^{(n)} \equiv \sum_x x^n\hat{\rho}_x = \sum_x x^n(\hat{\rho}_{x,u} + \hat{\rho}_{x,l})$. Concretely, we consider Brownian time evolution conserving the first two multipole moments $n = 0$ and $n = 1$; i.e., $[h_i, Q^{(0)}] = [h_i, Q^{(1)}] = 0$. This combination of charge and dipole symmetries generally leads to Hilbert space fragmentation [33,34,69]: For a given symmetry sector $Q^{(0)}, Q^{(1)}$ labeled by the different charge and dipole values, there are numerous distinct Krylov sectors $\mathcal{K}$ connected by the Hamiltonian evolution. Our goal is to understand the associated *Krylov-space-resolved hydrodynamics* in such systems. For this purpose, we introduce the operator $\mathcal{P}_{\mathcal{K}}$ projecting onto an individual Krylov sector $\mathcal{K}$ and its Choi state $\|\mathcal{K}\rangle\!\rangle$, which we define to be normalized. In the doubled Hilbert space formalism, we thus define new excited states, $\|\mathcal{K}_k\rangle\!\rangle = \hat{\rho}_k\|\mathcal{K}\rangle\!\rangle/(\mathcal{N}^{\mathcal{K}}_k)^{1/2}$, where $\hat{\mathcal{H}}_{\mathcal{L}}\|\mathcal{K}\rangle\!\rangle = 0$ and $\mathcal{N}^{\mathcal{K}}_k \equiv \langle\!\langle\mathcal{K}\|\hat{\rho}^{\dagger}_k\hat{\rho}_k\|\mathcal{K}\rangle\!\rangle$ is the Krylov-resolved structure factor.

In the presence of both charge and dipole conservation symmetries, the commutator in Eq. (8) now vanishes at $n = 1$, and takes a finite value only at order $n \ge 2$. Accordingly, the excited modes $\|\mathcal{K}_k\rangle\!\rangle$ carry an energy $E_k = \langle\!\langle\mathcal{K}_k\|\hat{\mathcal{H}}_{\mathcal{L}}\|\mathcal{K}_k\rangle\!\rangle \propto (1/\mathcal{N}^{\mathcal{K}}_k)k^4$. For generic dipole-conserving systems featuring *weak* fragmentation, the largest Krylov sector $\mathcal{K}_0$ makes up a finite portion of the full Hilbert space (up to a prefactor algebraic in system size). As a consequence, its static structure factor $\mathcal{N}^{\mathcal{K}_0}_k \to \mathcal{O}(1)$

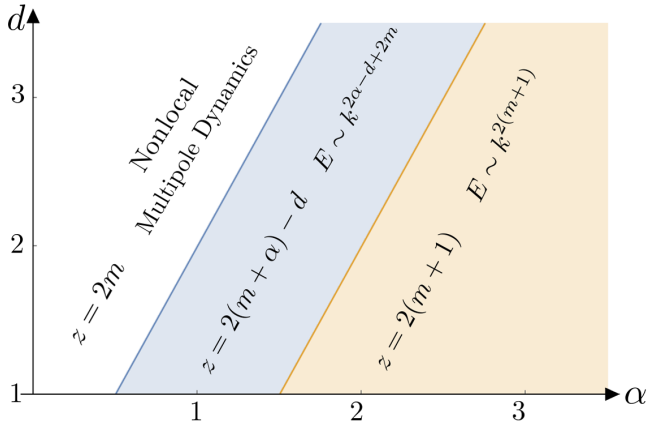

FIG. 2. Relaxation dynamics in multipole-conserving systems with long-range interactions. Systems with $(1/r^{\alpha})$ power-law decaying hopping of local multipoles of order m exhibit three distinct dynamical regimes. When $\alpha > (d/2) + 1$ (orange), the dynamics is (sub)diffusive with dynamical exponent $z = 2(m+1)$. For $(d/2)+1 > \alpha > (d/2)$ (blue), the dynamics is faster, with dynamical exponent $z = 2(m+\alpha) - d$. When $\alpha \leq (d/2)$, the system is effectively nonlocal; thus, relaxation occurs from individual mth multipole creation or annihilation operators, which are hoppings of $(m-1)$th multipole charges. This results in (sub)diffusive transport with $z = 2(m-1)+2 = 2m$.

remains finite as $k \to 0$. We thus obtain subdiffusive relaxation with dynamical exponent $z = 4$. The generalization of this result to systems conserving $\{Q^{(0)}, \ldots, Q^{(m)}\}$ multipoles is straightforward: The commutator in Eq. (8) now vanishes up to order $n = m$, giving rise to a dispersion proportional to $k^{2(m+1)}$ and dynamical exponent $z = 2(m+1)$, in accordance with previous results [22–25,70,71].

Similar to the charge-conserving case, these results can be extended to long-range interacting systems in arbitrary dimensions. For example, consider power-law decaying dipole hopping terms $h_{\boldsymbol{x}\boldsymbol{x}'} \sim (1/|\boldsymbol{x}-\boldsymbol{x}'|^{\alpha})(D_{\boldsymbol{x}}^{\dagger} D_{\boldsymbol{x}'} + \text{H.c.})$, where $D_{\boldsymbol{x}}$ is a *local* operator lowering the dipole moment. When $\alpha > (d/2)$, we determine the dispersion to be $E_{\boldsymbol{k}} \sim C_1(\alpha) k^{2\alpha+2-d} + C_2(\alpha) k^4$ (Ref. [53], Sec. F). Therefore, if $\alpha < 1 + d/2$, charge spreads faster than the subdiffusive transport $z = 4$ of short-range systems. For $\alpha < (d/2)$, dipole hopping becomes highly nonlocal, and charge transport effectively arises from individual local dipole creation or annihilation terms, analogous to systems with conventional charge conservation. In our framework, after renormalizing the single-mode dispersion to be bounded (Ref. [53], Sec. F), we obtain $E_{\boldsymbol{k}} \sim k^2$. We provide a summary of the dynamical exponents emerging in multipole-conserving systems with such long-range hopping of local moments in Fig. 2.

Constrained dynamics.—Returning to short-range models with dipole conservation, we may ask whether relaxation differing from the subdiffusive behavior $z = 4$ can emerge in specific Krylov sectors. The presence of the structure factor in the dispersion of Eq. (7) suggests this may be the case in Krylov sectors where charge fluctuations follow a *subvolume* law with vanishing $\lim_{k\to 0} \mathcal{N}_k^{\mathcal{K}} = 0$. We demonstrate this effect in concrete examples below.

Let us first consider a one-dimensional chain with charge and dipole conservation and introduce bond variables $\hat{e}_x$ defined via $\hat{\rho}_x = \hat{e}_x - \hat{e}_{x-1}$; i.e., $\hat{e}_x = \sum_{i=0}^{x} \hat{\rho}_i$. For convenience, we define the charge density $\hat{\rho}_x$ relative to its average value within $\mathcal{K}$; i.e., $\sum_x \langle \hat{\rho}_x \rangle_{\mathcal{K}} = 0$. We note that the $\hat{e}_i$ can be understood as a local dipole density, with $\sum_x \hat{e}_x = Q^{(1)}$ [27,72,73]. Let us now assume that a sector $\mathcal{K}$ exhibits *bounded* fluctuations of these bond variables. Formally, $\lim_{L\to\infty} \langle \hat{e}_k \hat{e}_{-k} \rangle_{\mathcal{K}} \xrightarrow{k\to 0} \sigma_1^2 < \infty$, where $\hat{e}_k = (1/\sqrt{L}) \sum_x e^{ikx} \hat{e}_x$ and σ_1 corresponds to the average fluctuation of the local dipole density. Since $\hat{e}_x = \sum_{i=0}^{x} \hat{\rho}_i$, the finiteness of $\hat{e}_x$ implies area-law fluctuations of the total charge within any given region. Using that $\hat{\rho}_k = (1 - e^{-ik}) \hat{e}_k$ for $k \neq 0$, the structure factor for small k becomes

$$\mathcal{N}_k^{\mathcal{K}} = \langle \hat{\rho}_k \hat{\rho}_{-k} \rangle_{\mathcal{K}} = k^2 \langle \hat{e}_k \hat{e}_{-k} \rangle \to \sigma_1^2 k^2. \tag{12}$$

Therefore, for Krylov sectors satisfying Eq. (12), the energy of the excited mode $\|\mathcal{K}_k\rangle\rangle$ scales as $E_k \propto k^2$ and we expect *diffusive* relaxation, despite the presence of dipole conservation. To interpret this result, note that the $\hat{e}_x$ constitute a conserved local density with an effectively finite local state space due to their bounded fluctuations. If $\hat{e}_x$ is bounded, these local dipoles move without additional kinetic constraints and are thus expected to relax diffusively; see also Ref. [72]. Generalization to systems conserving $\{Q^{(0)}, \ldots, Q^{(m)}\}$ is again straightforward: Krylov sectors with bounded multipole densities up to order $p \leq m$ have $\mathcal{N}_k \to \sigma_p^2 k^{2p}$, leading to a dispersion $\propto k^{2(m-p+1)}$ in short-range systems.

As a concrete example of Eq. (12), we consider random Brownian evolution in a $S = 1$ spin chain with local dipole-conserving terms $h_i = \hat{S}_i^{+} (\hat{S}_{i+1}^{-})^2 \hat{S}_{i+2}^{+} + \text{H.c.}$ Although these terms induce a strong fragmentation of the Hilbert space, there exist exponentially large, delocalized Krylov sectors [33,35]. We label the local charge density by $\hat{\rho}_x = S_x^z \in \{0, \pm\}$ and consider the Krylov sector containing the initial state $|\psi_0\rangle = |\ldots 00 + 00 \ldots\rangle$. In terms of the variables $\hat{e}_x$ introduced above, $|\psi_0\rangle = |\ldots 00111 \ldots\rangle$. corresponds to a domain wall, and the $\hat{e}_x \in \{0, 1\}$ can be shown to take values in a bounded range [33], thus satisfying our condition Eq. (12). Diffusive relaxation of this state has indeed been found in Ref. [72], and $\mathbb{E}\langle S_{x=L/2}^z(t) \rangle \sim t^{-1/2}$ can be verified numerically using random classical time evolution, as illustrated in Sec. G of Ref. [53].

To illustrate the generality of the condition Eq. (12), we consider systems beyond 1D. In analogy to $d = 1$, for $d > 1$ we write $\hat{\rho}(\boldsymbol{x}) = \boldsymbol{\nabla} \cdot \hat{\boldsymbol{e}}(\boldsymbol{x})$, where $\hat{\boldsymbol{e}}(\boldsymbol{x}) = [\hat{e}_1(\boldsymbol{x}), \ldots, \hat{e}_d(\boldsymbol{x})]$

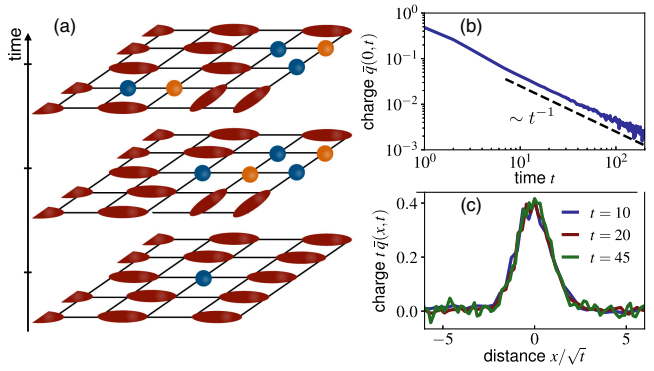

FIG. 3. Relaxation dynamics in a dipole-conserving dimer model. (a) We numerically consider a classical, discrete random time evolution in a dimer model with hard-core constraint, i.e., maximally one dimer attached to each site in the square lattice. This model can be mapped onto a U(1) link model following Refs. [74–76]. Under this mapping, vacancies, i.e., sites without attached dimer, carry positive (blue spheres) or negative charge (orange spheres), depending on their sublattice. We explicitly incorporate preservation of the hard-core constraint, the total charge, and the dipole moment associated with these charges in the time evolution. (b) Decay of the charge density $\bar{\rho}(0,t)$ for an isolated positive charge initially placed at $\boldsymbol{x} = 0$ in the bulk of the system; see (a). The decay is consistent with diffusion in two dimensions. (c) Scaling collapse of the charge distribution at different times along $\bar{\rho}[\boldsymbol{x} = (x,0),t]$, indicating Gaussian diffusion. Numerical results were averaged over 3×10^6 runs of the random time evolution (Ref. [53], Sec. G).

is now a d-component vector. We recognize that $\hat{\boldsymbol{e}}(\boldsymbol{x})$ is not uniquely determined by the charge configuration $\hat{\rho}(\boldsymbol{x})$, and the relation between these variables takes the form of a U(1) Gauss law, where the $\hat{\boldsymbol{e}}(\boldsymbol{x})$ constitute electric field degrees of freedom. Indeed, area-law charge fluctuations arise in U(1) gauge theories if fluctuations of the electric fields $\hat{\boldsymbol{e}}(\boldsymbol{x})$ are bounded, as $\int_V dV\,\hat{\rho}(\boldsymbol{x}) = \int_{\partial V} d\boldsymbol{A}\cdot\hat{\boldsymbol{e}}(\boldsymbol{x})$. Thus, imposing global dipole conservation on U(1) link models [74–76] with a finite electric field state space gives rise to diffusive behavior through Eq. (12). To verify this prediction, we numerically simulate classical, discrete random time evolution in a hard-core dimer model on a square lattice [see Fig. 3(a)], which can be mapped to a U(1) link model [77,78]. Under this mapping, a site $\boldsymbol{x}$ without any attached dimer carries a charge $\hat{\rho}(\boldsymbol{x}) = (-1)^{x_1+x_2}$ at $\boldsymbol{x} = (x_1,x_2)$, while a site with an attached dimer carries no charge. In the dynamics carried out numerically (see Sec. G of Ref. [53] as well as Refs. [21,24,25,70] for related approaches), we then explicitly incorporate conservation of the dipole moment associated to $\hat{\rho}(\boldsymbol{x})$. Starting from an initial state with an isolated positive charge in the bulk of the system $\hat{\rho}(\boldsymbol{x}, t = 0) = \delta_{x_1,0}\delta_{x_2,0}$ [see Fig. 3(a)], we numerically find a diffusive broadening of the resulting charge distribution at late times. As the overall charge density in the system vanishes, and positive and negative charges occupy different sublattices, we consider the quantity $\bar{\rho}(x_1,t) \equiv \hat{\rho}[(x_1,0),t] + \hat{\rho}[(x_1-1,0),t]$. We show in Fig. 3(c) that $t\bar{\rho}(x_1,t)$ exhibits a scaling collapse when plotted against $x_1/\sqrt{t}$, in agreement with diffusive relaxation in two dimensions.

Conclusion and outlook.—In this Letter, we have established a comprehensive understanding of conserved operator dynamics under Brownian random unitary time evolution through a duality with the spectral properties of an associated effective Hamiltonian. Though the U(1) symmetric Brownian evolution was used for clarity of presentation, these results generalize for any dynamics conserving a continuous global symmetry governed by a Lindblad equation (Ref. [53], Sec. C). As the ground state manifold always exhibits a spontaneous symmetry breaking of a continuous symmetry, a single-mode approximation could be applied to capture the low-energy physics of this effective Hamiltonian to reproduce a number of dynamical universality classes for short- and long-range interacting systems with charge and multipole conservation laws. In addition, our formalism allowed us to study the Krylov-space-resolved hydrodynamics of dipole-conserving systems, establishing diffusive behavior in Krylov spaces with area-law charge fluctuations, in contrast to more generic dynamics in the presence of dipole conservation.

We expect that such diffusive relaxation in dipole-conserving systems is valid beyond the specific examples studied numerically here and holds whenever the time evolution proceeds within an *effective* state space (not necessarily a Krylov space) that fulfills Eq. (12). In particular, bounded fluctuations of the variables $\hat{e}_x$ can arise from energetics, for example, via a term $\sim(\hat{e}_x)^2$ in the Hamiltonian, as appears naturally in standard electromagnetism. In this context, the resulting area-law charge fluctuations can be interpreted as Coulomb repulsion, which consequently leads to diffusive relaxation in dipole-conserving systems. Furthermore, bounded charge fluctuations occur in many other interesting models: It was shown in Refs. [73,79] that area-law charge fluctuations can arise in dipole-conserving Bose-Hubbard models in low-energy Mott states whenever a finite energy gap exists for charged excitations. It would be interesting to study the relevance of our results to such systems in the future.

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[1] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A **43**, 2046 (1991).

[2] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).

[3] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature (London) **452**, 854 (2008).

[4] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. **83**, 863 (2011).

[5] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).

[6] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Quantum thermalization through entanglement in an isolated many-body system, Science **353**, 794 (2016).

[7] T. Brydges, A. Elben, P. Jurcevic, B. Vermersch, C. Maier, B. P. Lanyon, P. Zoller, R. Blatt, and C. F. Roos, Probing Rényi entanglement entropy via randomized measurements, Science **364**, 260 (2019).

[8] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Quantum entanglement growth under random unitary dynamics, Phys. Rev. X **7**, 031016 (2017).

[9] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws, Phys. Rev. X **8**, 021013 (2018).

[10] A. Hamma, S. Santra, and P. Zanardi, Quantum entanglement in random physical states, Phys. Rev. Lett. **109**, 040502 (2012).

[11] A. Hamma, S. Santra, and P. Zanardi, Ensembles of physical states and random quantum circuits on graphs, Phys. Rev. A **86**, 052324 (2012).

[12] P. Kos, M. Ljubotina, and T. Prosen, Many-body quantum chaos: Analytic connection to random matrix theory, Phys. Rev. X **8**, 021062 (2018).

[13] H. Gharibyan, M. Hanada, S. H. Shenker, and M. Tezuka, Onset of random matrix behavior in scrambling systems, J. High Energy Phys. 07 (2018) 124.

[14] S. Moudgalya, T. Devakul, C. W. von Keyserlingk, and S. L. Sondhi, Operator spreading in quantum maps, Phys. Rev. B **99**, 094312 (2019).

[15] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws, Phys. Rev. X **8**, 031057 (2018).

[16] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation, Phys. Rev. X **8**, 031058 (2018).

[17] A. Schuckert, I. Lovas, and M. Knap, Nonlocal emergent hydrodynamics in a long-range quantum spin system, Phys. Rev. B **101**, 020416(R) (2020).

[18] M. K. Joshi, F. Kranzl, A. Schuckert, I. Lovas, C. Maier, R. Blatt, M. Knap, and C. F. Roos, Observing emergent hydrodynamics in a long-range quantum magnet, Science **376**, 720 (2022).

[19] J. Richter, O. Lunt, and A. Pal, Transport and entanglement growth in long-range random Clifford circuits, Phys. Rev. Res. **5**, L012031 (2023).

[20] C. Zu, F. Machado, B. Ye, S. Choi, B. Kobrin, T. Mittiga, S. Hsieh, P. Bhattacharyya, M. Markham, D. Twitchen *et al.*, Emergent hydrodynamics in a strongly interacting dipolar spin ensemble, Nature (London) **597**, 45 (2021).

[21] J. Iaconis, S. Vijay, and R. Nandkishore, Anomalous subdiffusion from subsystem symmetries, Phys. Rev. B **100**, 214301 (2019).

[22] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and heat transport in a tilted two-dimensional Fermi-Hubbard system, Phys. Rev. X **10**, 011042 (2020).

[23] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Res. **2**, 033124 (2020).

[24] J. Feldmeier, P. Sala, G. De Tomasi, F. Pollmann, and M. Knap, Anomalous diffusion in dipole-and higher-moment-conserving systems, Phys. Rev. Lett. **125**, 245303 (2020).

[25] A. Morningstar, V. Khemani, and D. A. Huse, Kinetically constrained freezing transition in a dipole-conserving system, Phys. Rev. B **101**, 214205 (2020).

[26] P. Zhang, Subdiffusion in strongly tilted lattice systems, Phys. Rev. Res. **2**, 033129 (2020).

[27] S. Moudgalya, A. Prem, D. A. Huse, and A. Chan, Spectral statistics in constrained many-body quantum chaotic systems, Phys. Rev. Res. **3**, 023176 (2021).

[28] H. Singh, B. A. Ware, R. Vasseur, and A. J. Friedman, Subdiffusion and many-body quantum chaos with kinetic constraints, Phys. Rev. Lett. **127**, 230602 (2021).

[29] J. Feldmeier, F. Pollmann, and M. Knap, Emergent fracton dynamics in a nonplanar dimer model, Phys. Rev. B **103**, 094303 (2021).

[30] P. Sala, J. Lehmann, T. Rakovszky, and F. Pollmann, Dynamics in systems with modulated symmetries, Phys. Rev. Lett. **129**, 170601 (2022).

[31] O. Hart, A. Lucas, and R. Nandkishore, Hidden quasiconservation laws in fracton hydrodynamics, Phys. Rev. E **105**, 044103 (2022).

[32] J. Feldmeier, W. Witczak-Krempa, and M. Knap, Emergent tracer dynamics in constrained quantum systems, Phys. Rev. B **106**, 094303 (2022).

[33] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity breaking arising from Hilbert space fragmentation in dipole-conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).

[34] V. Khemani, M. Hermele, and R. Nandkishore, Localization from Hilbert space shattering: From theory to physical realizations, Phys. Rev. B **101**, 174204 (2020).

[35] T. Rakovszky, P. Sala, R. Verresen, M. Knap, and F. Pollmann, Statistical localization: From strong fragmentation to strong edge modes, Phys. Rev. B **101**, 125126 (2020).

[36] S. Moudgalya, A. Prem, R. Nandkishore, N. Regnault, and B. A. Bernevig, Thermalization and its absence within Krylov subspaces of a constrained Hamiltonian, arXiv:1910.14048.

[37] Z.-C. Yang, F. Liu, A. V. Gorshkov, and T. Iadecola, Hilbert-space fragmentation from strict confinement, Phys. Rev. Lett. **124**, 207602 (2020).

[38] G. De Tomasi, D. Hetterich, P. Sala, and F. Pollmann, Dynamics of strongly interacting systems: From Fock-space fragmentation to many-body localization, Phys. Rev. B **100**, 214313 (2019).

[39] S. Roy and A. Lazarides, Strong ergodicity breaking due to local constraints in a quantum system, Phys. Rev. Res. **2**, 023159 (2020).

[40] L. Herviou, J. H. Bardarson, and N. Regnault, Many-body localization in a fragmented Hilbert space, Phys. Rev. B **103**, 134207 (2021).

[41] N. Shibata, N. Yoshioka, and H. Katsura, Onsager's scars in disordered spin chains, Phys. Rev. Lett. **124**, 180604 (2020).

[42] C. M. Langlett and S. Xu, Hilbert space fragmentation and exact scars of generalized Fredkin spin chains, Phys. Rev. B **103**, L220304 (2021).

[43] S. Scherg, T. Kohlert, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Observing non-ergodicity due to kinetic constraints in tilted Fermi-Hubbard chains, Nat. Commun. **12**, 4490 (2021).

[44] J. Feldmeier and M. Knap, Critically slow operator dynamics in constrained many-body systems, Phys. Rev. Lett. **127**, 235301 (2021).

[45] A. Caldeira and A. Leggett, Path integral approach to quantum Brownian motion, Physica (Amsterdam) **121A**, 587 (1983).

[46] A. Buchleitner, K. Hornberger, and B. Kümmerer, Quantum Markov processes, in *Coherent Evolution in Noisy Environments* (Springer, New York, 2011).

[47] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, Towards the fast scrambling conjecture, J. High Energy Phys. 04 (2013) 022.

[48] T. Zhou and X. Chen, Operator dynamics in a Brownian quantum circuit, Phys. Rev. E **99**, 052212 (2019).

[49] S. Xu and B. Swingle, Locality, quantum fluctuations, and scrambling, Phys. Rev. X **9**, 031048 (2019).

[50] X. Chen and T. Zhou, Quantum chaos dynamics in long-range power law interaction systems, Phys. Rev. B **100**, 064305 (2019).

[51] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. **10**, 285 (1975).

[52] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rep. Math. Phys. **3**, 275 (1972).

[53] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.220403 for technical details and broader context.

[54] It measures the spreading of an operator O located at $\boldsymbol{x}$ at time t by measuring its overlap with an operator at $\boldsymbol{y}$.

[55] Since $O_{\boldsymbol{x}}$ is local, the overlap $\langle\langle \boldsymbol{k}, \nu \| O \rangle\rangle = \mathrm{tr}(O X_{k,\nu})$ should be near constant for small k, where $X_{k,\nu}$ is the operator corresponding to the Choi state $\|\boldsymbol{k}, \nu\rangle\rangle$.

[56] R. P. Feynman, Atomic theory of the λ transition in helium, Phys. Rev. **91**, 1291 (1953).

[57] R. P. Feynman, Atomic theory of the two-fluid model of liquid helium, Phys. Rev. **94**, 262 (1954).

[58] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Magneto-roton theory of collective excitations in the fractional quantum Hall effect, Phys. Rev. B **33**, 2481 (1986).

[59] X. Deng, I. Khaymovich, and A. L. Burin, Superdiffusion in random two dimensional system with ubiquitous long-range hopping, arXiv:2205.14715.

[60] R. M. Nandkishore and M. Hermele, Fractons, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).

[61] M. Pretko, X. Chen, and Y. You, Fracton phases of matter, Int. J. Mod. Phys. A **35**, 2030003 (2020).

[62] C. Chamon, Quantum glassiness in strongly correlated clean systems: An example of topological overprotection, Phys. Rev. Lett. **94**, 040402 (2005).

[63] J. Haah, Local stabilizer codes in three dimensions without string logical operators, Phys. Rev. A **83**, 042330 (2011).

[64] B. Yoshida, Exotic topological order in fractal spin liquids, Phys. Rev. B **88**, 125122 (2013).

[65] S. Vijay, J. Haah, and L. Fu, A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations, Phys. Rev. B **92**, 235136 (2015).

[66] M. Pretko and L. Radzihovsky, Fracton-elasticity duality, Phys. Rev. Lett. **120**, 195301 (2018).

[67] M. Pretko, Subdimensional particle structure of higher rank U(1) spin liquids, Phys. Rev. B **95**, 115139 (2017).

[68] M. Pretko, The fracton gauge principle, Phys. Rev. B **98**, 115134 (2018).

[69] S. Moudgalya and O. I. Motrunich, Hilbert space fragmentation and commutant algebras, Phys. Rev. X **12**, 011050 (2022).

[70] J. Iaconis, A. Lucas, and R. Nandkishore, Multipole conservation laws and subdiffusion in any dimension, Phys. Rev. E **103**, 022142 (2021).

[71] H. Singh, B. A. Ware, R. Vasseur, and A. J. Friedman, Subdiffusion and many-body quantum chaos with kinetic constraints, Phys. Rev. Lett. **127**, 230602 (2021).

[72] X. Feng and B. Skinner, Hilbert space fragmentation produces an effective attraction between fractons, Phys. Rev. Res. **4**, 013053 (2022).

[73] P. Zechmann, E. Altman, M. Knap, and J. Feldmeier, Fractonic Luttinger liquids and supersolids in a constrained Bose-Hubbard model, Phys. Rev. B **107**, 195131 (2023).

[74] D. Horn, Finite matrix models with continuous local gauge invariance, Phys. Lett. **100B**, 149 (1981).

[75] P. Orland and D. Rohrlich, Lattice gauge magnets: Local isospin from spin, Nucl. Phys. **B338**, 647 (1990).

[76] S. Chandrasekharan and U.-J. Wiese, Quantum link models: A discrete approach to gauge theories, Nucl. Phys. **B492**, 455 (1997).

[77] E. Fradkin and S. Kivelson, Short range resonating valence bond theories and superconductivity, Mod. Phys. Lett. B **04**, 225 (1990).

[78] R. Moessner, S. L. Sondhi, and E. Fradkin, Short-ranged resonating valence bond physics, quantum dimer models, and Ising gauge theories, Phys. Rev. B **65**, 024504 (2001).

[79] E. Lake, H.-Y. Lee, J. H. Han, and T. Senthil, Dipole condensates in tilted Bose-Hubbard chains, Phys. Rev. B **107**, 195132 (2023).