Identification of Quantum Change Points for Hamiltonians

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Detecting sudden changes in the environment is crucial in many statistical applications. We mainly focus on identifying sudden changes in weak signals transmitted by electromagnetic or gravitational waves. Assuming that the Hamiltonians representing the signals before and after the change are known, we aim to find a discrimination strategy that can detect the change point with the best possible accuracy. This problem has potential applications in accurately detecting the precise timing of events such as stellar explosions, foreign object intrusions, specific chemical bonds, and phase transitions. We formulate this problem as a quantum process discrimination problem by discretizing the time evolution of a quantum system as a sequence of unitary channels. However, due to the complexity of the dynamics, solving such a multiple process discrimination problem is typically challenging. We demonstrate that the maximum success probability for the Hamiltonian change point problem with any finite number of candidate change points can be determined and has a simple analytical form.

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Identifying transition points is a crucial issue in many fields, including quality control [1], bioinformatics [2], network traffic monitoring [3], and climatology [4]. In various situations, the environment may undergo a sudden change, and determining the exact moment of this transformation can provide valuable insights. This challenge, known as the change point problem, has attracted significant research interest. Considerable efforts have been made to develop effective and efficient detection strategies for this problem [5–11].

Our main focus lies in detecting sudden changes in electromagnetic or gravitational waves. When the signal indicating the change point is weak, it is important to perform quantum mechanical identification for high-sensitivity detection. Working on the quantum version of the change point problem is expected to be in demand in a wide range of fields such as quantum sensing [12], quantum imaging [13], and quantum biology [14]. In this Letter, we aim to accurately identify change points by assuming that the Hamiltonians representing weak signals before and after the change are known. Solving this problem could potentially aid in detecting the precise timing of events such as stellar explosions, foreign object intrusions, specific chemical bonds, and phase transitions. Figure 1 shows a schematic of the problem of detecting change points in Hamiltonians. The problem of detecting transitions between quantum states has recently been addressed, establishing upper and lower bounds for the maximum success probability [15]. Detecting the change point in Hamiltonians poses a considerably greater challenge compared to that in quantum states because it necessitates optimizing not only output measurements but also input states, driving Hamiltonians, etc. To obtain optimal discrimination for Hamiltonians, it is necessary to consider various types of distinguishing strategies, including those that combine entangled input states with ancillary systems and adaptive strategies.

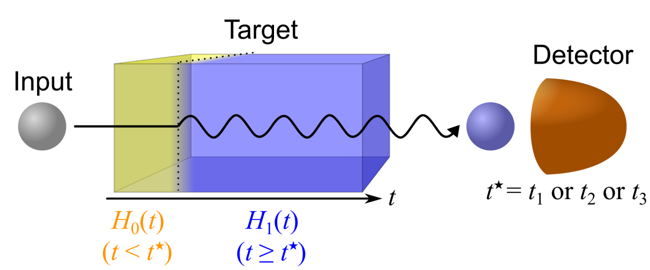

FIG. 1. Problem of discriminating change points in Hamiltonians. The Hamiltonian of a quantum system suddenly changes from $H_0(t)$ to $H_1(t)$ at a specific time $t = t^\star$. The exact values of $H_0(t)$ and $H_1(t)$, which may be time dependent, are known but the point of the transition occurrence (i.e., $t^\star$) is unknown. Assuming that candidate change points are given, we wish to identify $t^\star$ as accurately as possible by optimizing the state input to the system, the measurement for the output, and so on. Let us assume that the number of candidate change points is finite and that each candidate has an equal chance of becoming a change point.

When discretizing the time evolution of a quantum system as a sequence of unitary channels, the Hamiltonian change point problem can be seen as the problem of identifying quantum processes. The problem of distinguishing between quantum channels [16–27] or more general quantum processes (also known as quantum memory channels or quantum strategies) [28–34] has recently received significant attention. Analytical solutions have been found for distinguishing between two simple quantum processes or processes with significant symmetry, such as those covariant with respect to unitary operators. However, to achieve our goals, we need to differentiate between multiple processes that do not exhibit high symmetry, making it difficult to obtain an analytical solution. Our task is to identify multiple time-dependent Hamiltonians, but due to the complexity of the dynamics, solving such a problem is highly challenging and little is known about optimal solutions. The formulation of the quantum process discrimination problem as a semidefinite programming problem is known [31], which often proves helpful in obtaining analytical or numerical solutions. Nevertheless, the conclusions of Ref. [31] cannot be explicitly applied when the Hamiltonians vary continuously. Also, as the dimension of the system and the number of candidate change points increase, the computational complexity increases exponentially, which limits the feasibility of obtaining a solution to small-scale problems.

We demonstrate that optimal performance can be analytically obtained and expressed in a simple form for the Hamiltonian change point problem, regardless of the number of candidate change points. This result may appear surprising, considering that analytical optimal solutions to quantum process discrimination problems have only been found for very simple cases. We should emphasize that even in the change point problem for quantum states, which intuitively seems to be easier to solve, an analytical solution is only found in the limiting case where the number of candidate change points is infinite [15]. To begin with, we tackled the task of discriminating the change point when a sudden change occurs from a unitary channel to the next. We provide a specialized method for solving this problem. In this method, after formulating the problem as a semidefinite programming problem, we devise a discrimination strategy aimed at maximizing the success probability. We then show the optimality of this strategy by examining the dual problem. We also demonstrate that adaptive strategies and ancillary systems are not required for optimal discrimination. This is in contrast to general channel discrimination problems, which require the use of adaptive strategies in conjunction with ancillary systems [19,35]. Then, this result is applied to analytically obtain an optimal solution to the Hamiltonian change point problem.

Identification of change points for unitary channels.— We first formalize the task of recognizing transition points within unitary channels. We assume a unitary channel, where the first n uses correspond to $\mathcal{U}_0$, and the remaining uses correspond to $\mathcal{U}_1$. The channels $\mathcal{U}_0$ and $\mathcal{U}_1$ are known and n can be any integer between 0 to N. The objective is to precisely determine the value of n. Let $\mathcal{U}_{n<k}$ be $\mathcal{U}_1$ if $n < k$, and $\mathcal{U}_0$ otherwise. Let $\mathcal{E}_n$ be the process consisting of a sequence of N channels $(\mathcal{U}_{n<1}, \ldots, \mathcal{U}_{n<N})$. This problem can be formulated as differentiating between the $N+1$ processes $\mathcal{E}_0, \ldots, \mathcal{E}_N$. For example, in the case of $N = 2$, there is a need to distinguish between the three possible sequences: $\mathcal{E}_0 = (\mathcal{U}_1, \mathcal{U}_1)$, $\mathcal{E}_1 = (\mathcal{U}_0, \mathcal{U}_1)$, and $\mathcal{E}_2 = (\mathcal{U}_0, \mathcal{U}_0)$. Let V_k and W_k, with equal dimensions, denote the input and output systems, respectively, for the channel $\mathcal{U}_{n<k}$.

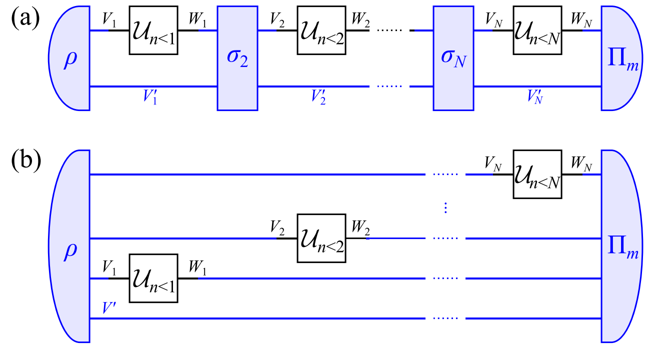

FIG. 2. (a) The most general protocol of change point discrimination for unitary channels. Each process $\mathcal{E}_n$ consists of a sequence of N channels $(\mathcal{U}_{n<1}, \ldots, \mathcal{U}_{n<N})$. Any discrimination strategy is expressed as a collection of a state ρ, channels $\sigma_2, \ldots, \sigma_N$, and a measurement $\{\Pi_m\}_{m=0}^N$. (b) The most general nonadaptive protocol, which consists of a state ρ and a measurement $\{\Pi_m\}_{m=0}^N$.

The most general discrimination strategy, presented in Fig. 2(a), involves ancillary systems $V_1', \ldots, V_N'$. We begin by preparing a bipartite system with initial conditions of $V_1 \otimes V_1'$. The first segment V_1 is sent through the channel $\mathcal{U}_{n<1}$, followed by a channel σ_2. Subsequently, V_2 is sent through the channel $\mathcal{U}_{n<2}$, followed by a channel σ_3, until N steps have been completed. The system $W_N \otimes V_N'$ is then subjected to a quantum measurement, $\Pi \coloneqq \{\Pi_m\}_{m=0}^N$. A collection $(\rho, \sigma_2, \ldots, \sigma_N, \Pi)$ can be used to define any quantum discrimination strategy allowed by quantum mechanics, including an entanglement-assisted and/or adaptive one. This collection of objects is known as a quantum tester [29]. We want to find a discrimination strategy that maximizes the success probability. The problem of obtaining the maximum success probability, denoted by P, is an optimization problem over quantum testers, which is formulated as a semidefinite programming problem [31]. We assume $P < 1$, which means that $\mathcal{U}_0$ and $\mathcal{U}_1$ are not perfectly distinguishable with a single evaluation.

Figure 2(b) shows the most general nonadaptive protocol, which can be regarded as a special case of the protocol shown in Fig. 2(a). An initial state ρ is prepared for the multipartite system $V_N \otimes \cdots \otimes V_1 \otimes V'$ of the protocol, where V' is an ancillary system. Subsystems $V_1, \ldots, V_N$ are then exposed to their respective channels $\mathcal{U}_{n<1}, \ldots, \mathcal{U}_{n<N}$

and the system $W_N \otimes \cdots \otimes W_1 \otimes V'$ is subjected to a quantum measurement $\Pi \coloneqq \{\Pi_m\}_{m=0}^N$. Any nonadaptive discrimination strategy can be described using a collection (ρ, Π). The determination of the optimal performance by using only a nonadaptive strategy is simple; however, the resulting performance could be inferior to those obtained using adaptive strategies.

Considering the optimal performance of nonadaptive strategies, let Λ_n represent the unitary channel composed of N unitary channels $\mathcal{U}_{n<N}, \ldots, \mathcal{U}_{n<1}$ connected in parallel, i.e.,

$$\Lambda_n \coloneqq \mathcal{U}_{n<N} \otimes \cdots \otimes \mathcal{U}_{n<1} = \mathcal{U}_1^{\otimes(N-n)} \otimes \mathcal{U}_0^{\otimes n}.$$

This problem can be expressed as the following optimization problem:

$$\text{maximize} \frac{1}{N+1} \sum_{n=0}^{N} \mathrm{Tr}[\Pi_n \, (\Lambda_n \otimes \mathbb{1}_{V'})(\rho)],$$

where the maximization is taken over all possible input states ρ of the system $V_N \otimes \cdots \otimes V_1 \otimes V'$ and over all possible measurements Π of the system $W_N \otimes \cdots \otimes W_1 \otimes V'$. We denote the optimal value of this problem, i.e., the maximum success probability, by P_{na}. $P_{\mathrm{na}} \le P$ clearly holds. For each $b \in \{0,1\}$, the channel $\mathcal{U}_b$ is associated with a unitary matrix U_b such that $\mathcal{U}_b(\rho) = U_b \rho U_b^\dagger$.

We first consider the simplest case $N = 1$; then, the problem is reduced to distinguishing two unitary channels $\mathcal{U}_0$ and $\mathcal{U}_1$ in a single trial, and thus $P = P_{\mathrm{na}}$ holds. P_{na} was obtained in Refs. [36,37]; we here briefly review their results. The maximum success probability for distinguishing two output states $\mathcal{U}_0|\psi\rangle$ and $\mathcal{U}_1|\psi\rangle$ from a pure input state $|\psi\rangle$ is given by

$$\frac{1}{2}\left(1 + \sqrt{1 - |\langle\psi|U_0^\dagger U_1|\psi\rangle|^2}\right). \tag{1}$$

The optimal input state $|\psi\rangle$ minimizes the absolute value of the inner product of $U_0|\psi\rangle$ and $U_1|\psi\rangle$. Let Γ be the polygon in the complex plane whose vertices are the eigenvalues of the unitary matrix $U_0^\dagger U_1$; then, the distance between the polygon Γ and the origin is equal to the minimum value of $|\langle\psi|U_0^\dagger U_1|\psi\rangle|$ [38]. If λ_0 and λ_1 are the eigenvalues representing the points at both ends of the polygon Γ closest to the origin, and $|\lambda_0\rangle$ and $|\lambda_1\rangle$ are the corresponding normalized eigenvectors, then $|\langle\psi|U_0^\dagger U_1|\psi\rangle|$ attains its minimum value $|\lambda_0 + \lambda_1|/2$ at $|\psi\rangle = |\pm\rangle \coloneqq (|\lambda_0\rangle \pm |\lambda_1\rangle)/\sqrt{2}$. Thus, from Eq. (1), we obtain

$$P = \frac{\gamma+1}{2}, \qquad \gamma \coloneqq \frac{1}{2}|\lambda_1 - \lambda_0|. \tag{2}$$

Next, we examine the case $N = 2$, which presents us with the challenge of distinguishing between three processes. Analogous to the $N = 1$ scenario, we consider a discrimination strategy that employs a pure state $|\psi\rangle$ of the system $V_2 \otimes V_1$ as the input state. The outputs from the three processes then become $U_1 \otimes U_1|\psi\rangle$, $U_1 \otimes U_0|\psi\rangle$, and $U_0 \otimes U_0|\psi\rangle$. Consequently, our task is to optimize $|\psi\rangle$ to maximize the distinguishability of these three states. However, the relationship between these three states and the maximum success probability is complex, rendering it unclear how to optimize $|\psi\rangle$. This stands in contrast to the case $N = 1$, where the optimization of the input state $|\psi\rangle$ simply involves minimizing the inner product of $U_0|\psi\rangle$ and $U_1|\psi\rangle$. The utilization of an ancillary system may enhance the success probability, and it would not be surprising if P were strictly larger than P_{na}. These considerations are equally applicable to cases of $N > 2$.

Considering the above discussion, obtaining an analytical expression of P for $N \ge 2$ is challenging. However, we discovered that a nonadaptive strategy can achieve optimal discrimination for any N. The following theorem provides a simple expression for the exact value of P as a function of N and γ.

Theorem 1.—In the problem of change point discrimination for unitary channels, we have

$$P = P_{\mathrm{na}} = \frac{N\gamma + 1}{N+1}. \tag{3}$$

Proof.—We present a summary of the proof; for more details, please refer to Sec. III of Supplemental Material (SM) [39]. Consider a nonadaptive discrimination strategy in which pure state $|\psi\rangle$ of the system $V_N \otimes \cdots \otimes V_1$ is input into the channel Λ_n. Assume that the measurement in an orthonormal system $\{|\pi_m\rangle\}_{m=0}^N$ identifies the pure output state from the channel Λ_n, which is represented by $[U_1^{\otimes(N-n)} \otimes U_0^{\otimes n}]|\psi\rangle$; then, the conditional probability, denoted by $p_{m|n}$, that the measurement result is m, given that the change point n is represented as

$$p_{m|n} = |\langle\pi_m|[U_1^{\otimes(N-n)} \otimes U_0^{\otimes n}]|\psi\rangle|^2.$$

Let us choose

$$|\psi\rangle = \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_N \in \{-1,1\}} a_{s_1,\ldots,s_N} |\phi_{s_N}\rangle \cdots |\phi_{s_1}\rangle \tag{4}$$

with $|\phi_1\rangle \coloneqq |+\rangle$ and $|\phi_{-1}\rangle \coloneqq |-\rangle$, where $a_{s_1,\ldots,s_N}$ is $\prod_{k=1}^{N-1}[(s_k s_{k+1}\sqrt{\gamma}+1)/\sqrt{2(\gamma+1)}]$ if the sequence $s_1, \ldots, s_N$ has an even number of elements equal to -1, and 0 otherwise. It can be seen that there exists an orthonormal system $\{|\pi_m\rangle\}_{m=0}^N$ satisfying

$$p_{m|n} = \begin{cases} \sum_{k=-\infty}^{0} \mathcal{L}(k-n;\zeta), & m = 0, \\ \mathcal{L}(m-n;\zeta), & 0 < m < N, \\ \sum_{k=N}^{\infty} \mathcal{L}(k-n;\zeta), & m = N, \end{cases}$$

where $\zeta \coloneqq (1-\gamma)/(1+\gamma)$ and $\mathcal{L}(n;\zeta)$ is the probability mass function of the discrete Laplace distribution, given by

$$\mathcal{L}(n;\zeta) \coloneqq \frac{1-\zeta}{1+\zeta}\zeta^{|n|}, \qquad 0<\zeta<1.$$

This nonadaptive strategy provides the success probability of

$$\frac{1}{N+1}\sum_{n=0}^{N} p_{n|n} = \frac{N\gamma+1}{N+1} \eqqcolon q,$$

which is clearly not greater than P_{na}. In addition, we can demonstrate that the optimal value, denoted by D, of the Lagrange dual of the change point problem is upper bounded by q. As the weak duality inequality $P \le D$ holds, we have $q \le P_{\mathrm{na}} \le P \le D \le q$, and thus all these inequalities are equalities. ■

In practice, it may be challenging to implement the entangled states described by Eq. (4). Alternatively, consider using a separable state, $|+\rangle^{\otimes N}$, as an input. In this situation, the problem is to identify the state $\{(U_1|+\rangle)^{\otimes(N-n)}(U_0|+\rangle)^{\otimes n}\}_n$, and thus reduces to a quantum change point problem for pure states, which has been studied in Ref. [15]. However, the use of a separable state results in performance degradation, as shown in Fig. 3.

Identification of change points for Hamiltonians.—The above-mentioned discussion can be expanded to address the problem of identifying Hamiltonian change points. Consider a situation where a Hamiltonian $H_0(t)$ acting on a quantum system changes to $H_1(t)$ at a particular time $t^\star$. Assume that the change point $t^\star$ is known to be one of the possible candidates $t_0,\ldots,t_N$ with equal prior probabilities. To frame this problem as a process discrimination problem,

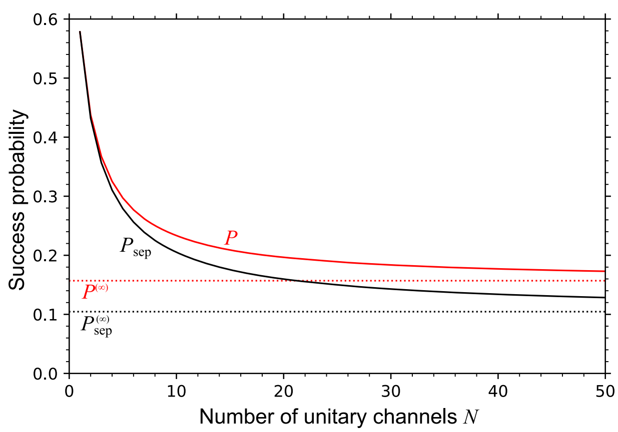

FIG. 3. Probability of successful identification of change points for $\lambda_0 = 1$ and $\lambda_1 = \exp(\mathbf{i}\pi/10)$. P_{sep} is the maximum success probability when the separable state $|+\rangle^{\otimes N}$ is used as an input, which is obtained by numerically solving a semidefinite programming problem. $P^{(\infty)} = \gamma$ and $P_{\mathrm{sep}}^{(\infty)}$ are limits as $N \to \infty$. The analytical solution of $P_{\mathrm{sep}}^{(\infty)}$ is given in Ref. [15].

it is imperative to consider time in a discrete manner. Also, it is necessary to contemplate any quantum discrimination strategy permitted by quantum mechanics, which may involve applying channels at time points other than $t_0,\ldots,t_N$. Consequently, we explore the concept of discretizing time into sufficiently short intervals. We arbitrarily choose a natural number R and time instants $\tau_0 < \tau_1 < \ldots < \tau_{NR}$ such that $\tau_{kR} = t_k$ holds for each $k \in \{0,\ldots,N\}$. For each $b \in \{0,1\}$, $1 \le k \le N$, and $1 \le r \le R$, let $\mathcal{U}_b^{(k,r)}$ be the unitary channel representing the time evolution with the Hamiltonian $H_b(t)$ between the time interval $\tau_{(k-1)R+r-1} \le t \le \tau_{(k-1)R+r}$, i.e.,

$$\mathcal{U}_b^{(k,r)}(\rho) \coloneqq U_b^{(k,r)}\rho U_b^{(k,r)\dagger},$$
$$U_b^{(k,r)} \coloneqq \mathcal{T}\left[\exp\left[-\mathbf{i}\int_{\tau_{(k-1)R+r-1}}^{\tau_{(k-1)R+r}} H_b(t)dt\right]\right],$$

where $\mathcal{T}$ is the time-ordered operator. Also, let $\mathcal{E}_n$ be the sequence expressed by

$$\mathcal{E}_n \coloneqq \Big[\mathcal{U}_{n<1}^{(1,1)},\mathcal{U}_{n<1}^{(1,2)},\ldots,\mathcal{U}_{n<1}^{(1,R)},\ \mathcal{U}_{n<2}^{(2,1)},\mathcal{U}_{n<2}^{(2,2)},\ldots,\mathcal{U}_{n<2}^{(2,R)},\ldots,\ \mathcal{U}_{n<N}^{(N,1)},\mathcal{U}_{n<N}^{(N,2)},\ldots,\mathcal{U}_{n<N}^{(N,R)}\Big].$$

The problem is then reduced to distinguishing quantum processes $\mathcal{E}_0,\ldots,\mathcal{E}_N$ similar to the problem of unitary channels. However, the main difference in this case is that the Hamiltonians can be time dependent and we can use arbitrarily short time intervals. As a result, this problem is challenging to solve analytically.

Let $\mu_{\max}(t)$ and $\mu_{\min}(t)$, respectively, be the maximum and minimum eigenvalues of $H_1(t) - H_0(t)$. Also, let

$$\gamma_k \coloneqq \Delta\left[\int_{t_{k-1}}^{t_k}[\mu_{\max}(t)-\mu_{\min}(t)]dt\right], \qquad 1\le k\le N, \tag{5}$$

where $\Delta(\theta) \coloneqq \sin[\min(\theta,\pi)/2]$.

For $N = 1$, the problem is to identify two processes $\mathcal{E}_0$ and $\mathcal{E}_1$. In essence, this problem involves distinguishing whether the Hamiltonian applied to the system between the time interval $t_0 \le t \le t_1$ is $H_0(t)$ or $H_1(t)$. In this problem, the maximum success probability is known as $(\gamma_1+1)/2$ [36,40], which is obtained as limit $R \to \infty$. In addition, some experiments have been conducted using this result [41]. We find that by extending the proof of Theorem 1, an analytical expression of the ultimate performance for each N is obtained, as stated in the following theorem (the proof is given in Sec. III C of SM [39]).

Theorem 2.—The maximum success probability in the change point problem for two Hamiltonians $H_0(t)$ and

$H_1(t)$ with any integer $N \geq 1$ is given by

$$\frac{1}{N+1}\left(\sum_{k=1}^{N}\gamma_k + 1\right). \tag{6}$$

Let P_k represent the maximum success probability of distinguishing whether the Hamiltonian applied to the system between the time interval $t_{k-1} \leq t \leq t_k$ is $H_0(t)$ or $H_1(t)$. Drawing from the discussion for the case $N = 1$, we deduce $P_k = (\gamma_k + 1)/2$. Therefore, the maximum success probability of Eq. (6) can be expressed as a linear function of the sum of these probabilities, $\sum_{k=1}^{N} P_k$. Moreover, from Eq. (5), we ascertain that for each k, P_k is monotonically nondecreasing with respect to $\gamma'_k \coloneqq \int_{t_{k-1}}^{t_k} [\mu_{\max}(t) - \mu_{\min}(t)]dt$. Therefore, the maximum success probability generally increases as each γ'_k increases. In the limit of large N, it follows from Eq. (6) that the maximum success probability tends to the average of $\gamma_1, \gamma_2, \ldots, \gamma_N$ (i.e., $\sum_{k=1}^{N} \gamma_k/N$).

In Sec. VI of SM [39], we provide a tangible example of a Hamiltonian change point problem, in which H_0 and H_1 oscillate at disparate frequencies. Given that frequency often encodes information about energy, position, and structure, detecting changes in frequency could be of paramount importance for several applications.

Conclusions.—In this Letter, we investigated the difficulty of identifying a precise moment when the Hamiltonian suddenly changes, and we presented an analytical expression of the maximum success probability. We first discussed the quantum change point problem for unitary channels, as a simpler problem. The objective of this task is to accurately identify the exact moment when a unitary channel changes to another. We demonstrated that the maximum success probability can be expressed in a simple analytical form by using only the number of possible change points and a parameter reflecting the ease of recognizing the channels before and after the change, assuming identical prior probabilities. The proposed method was then applied to derive the optimal performance for the problem of discriminating change points for Hamiltonians.

This Letter lays the foundation for future research on related topics, including the estimation of a continuous-valued change point and the detection of multiple change points. In addition, it can facilitate research on the change point problem for channels in open systems (i.e., nonunitary channels) and the optimization with other criteria such as unambiguous or Neyman-Pearson. We anticipate that our results will provide a solid starting point for addressing these challenges.

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[39] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.210804, which includes Refs. [40–46], for additional information about the problem formulation, the maximum success probability, and an example of Hamiltonian change point identification.

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