Full Counting Statistics of Charge in Chaotic Many-Body Quantum Systems

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We investigate the full counting statistics of charge transport in U(1)-symmetric random unitary circuits. We consider an initial mixed state prepared with a chemical potential imbalance between the left and right halves of the system and study the fluctuations of the charge transferred across the central bond in typical circuits. Using an effective replica statistical mechanics model and a mapping onto an emergent classical stochastic process valid at large on-site Hilbert space dimension, we show that charge transfer fluctuations approach those of the symmetric exclusion process at long times, with subleading $t^{-1/2}$ quantum corrections. We discuss our results in the context of fluctuating hydrodynamics and macroscopic fluctuation theory of classical nonequilibrium systems and check our predictions against direct matrix-product state calculations.

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Introduction.—The longtime dynamics of generic many-body quantum systems is expected to be effectively classical. Starting from a pure initial state, the local properties of chaotic systems quickly thermalize: the expectation value of local operators can be described by an effective Gibbs ensemble with spatially varying Lagrange multipliers such as temperature. The resulting evolution from local to global equilibrium is then described by the classical equations of hydrodynamics. However, the advent of quantum simulator platforms such as cold atoms [1–14], trapped ions [15–17], or superconducting arrays [18–21] has made it possible to measure not only local expectation values but also their full quantum statistics. Whether there exists an emergent classical description of such fluctuations in generic, chaotic many-body quantum systems is an open question.

Consider a one-dimensional quantum system with a conserved charge that is prepared with a domain-wall chemical potential imbalance across the central bond $\mu_L = -\mu_R = \mu$. By measuring the charge in the right half of the system at times 0 and t, experiments reveal "quantum snapshots" of the charge transfer Q across the central bond (from the left to the right). By repeating the experiment, one has access to the full distribution of measurement outcomes $P_t(Q)$. While the average of that distribution is described by hydrodynamics—which in the case of a single conserved charge simply reduces to a diffusion equation—higher cumulants describe current fluctuations and the full counting statistics (FCS) of charge transport [22–29].

Computing the FCS in many-body quantum systems is a formidable task, and exact or mean field results have only been achieved in a few cases, notably in noninteracting fermion models [30–37], integrable systems [38–49], and in quantum dots or few qubit models [50–57]. While there is currently no exact result pertaining to chaotic many-body quantum systems, charge current fluctuations are expected to be subject to the large deviation principle [58–60]: all cumulants of charge transfer should scale in the same way with time, as $\sqrt{t}$ for a diffusive system in one dimension. In the context of classical stochastic models with a conserved charge, the emergence of the large deviation principle is understood within a general formalism known as macroscopic fluctuation theory (MFT) [61]. MFT is a toolbox for solving the noisy diffusion equation obtained from promoting the hydrodynamic equation to a nonlinear *fluctuating hydrodynamic* theory by adding a noise term to the current, with strength determined by a fluctuation-dissipation theorem. MFT has been very successful in describing stochastic classical systems, and has recently been used to compute the FCS of a paradigmatic integrable Markov chain, the (simple) symmetric exclusion process (SEP) [62,63].

Quantum systems have intrinsic quantum fluctuations, and it is natural to wonder whether they can be captured by an emergent classical description such as MFT. In this Letter, we investigate the FCS in an ensemble of diffusive chaotic models—random unitary circuits with a conserved U(1) charge [64,65]. Quantum systems with a conserved charge are endowed with current fluctuations and counting statistics. While the quantum many-body dynamics of individual circuit realizations is generally inaccessible, by ensemble averaging, we will study the dynamics of typical circuit realizations. At the level of mean transport, this is known to yield a classical stochastic description [64–67]. In this work

we show that a classical stochastic process in fact describes the entire (late time) FCS and quantify the subleading corrections.

In order to capture typical current fluctuations within a single circuit realization, circuit averaging must be performed at the level of cumulants, which are polynomial in the system's density matrix. By doing so, we map the problem of computing cumulants onto that of expectation values in replica statistical mechanics (SM) models. By simulating the SM time evolution using matrix-product states, and separately, by introducing an effective stochastic model of coupled SEP chains, we show that the quantum corrections to the higher order cumulants are subleading. This leads to a latetime FCS consistent with a simple fluctuating hydrodynamics for the coarse-grained charge density $\rho(x,\tau)$ [62] with rescaled space-time coordinates $x = j/\ell$ and $\tau = t/\ell^2$,

$$\partial_\tau \rho = -\partial_x j, \qquad j = -D(\rho)\partial_x \rho + \sqrt{\frac{2\sigma(\rho)}{\ell}}\xi, \tag{1}$$

where $\xi(x,\tau)$ is a Gaussian white noise with zero mean and unit variance, and ℓ is the size of the hydrodynamic cells over which ρ is coarse grained. The only microscopic input in this equation are the diffusion constant $D(\rho) = 1$ and the conductivity $\sigma(\rho) = D(\rho)\chi_s(\rho)$, with $\chi_s(\rho) = \rho(1-\rho)$, which characterize both random quantum circuits and SEP. The noise term in Eq. (1) is set by the fluctuation-dissipation theorem to preserve equilibrium charge fluctuations, making this equation a natural candidate for a fluctuating hydrodynamic theory of random quantum circuits. We confirm this result by computing the FCS in individual quantum circuits using matrix-product state techniques [68] (Fig. 1) as an independent check to our effective stochastic theory. Our results establish the emergence of "classicality" at long times in quantum systems, even at the level of fluctuations.

Model and measurement scheme.—We work with a one-dimensional chain, in which each site is composed of a charged qubit with basis states $|q = 0, 1\rangle$, and a neutral qudit of dimension d, yielding a single-site Hilbert space $\mathcal{H}_{\text{loc}} \equiv \mathbb{C}^2 \otimes \mathbb{C}^d$. The system evolves via the application of layers of random nearest-neighbor unitary gates in a brick-wall pattern (see Fig. 1). The unitary gates conserve the total charge on the two sites, but are otherwise Haar random [64,65]. This is a generic model of hard-core bosons, or equivalently spinless fermions (via a Jordan-Wigner transformation); since hard-core and non-hard-core bosons have the same coarse-grained dynamics at high temperatures [70,71], we expect our results to also apply to generic bosons.

Unitary evolution and projective measurement ensures that the system's charge dynamics is endowed with current fluctuations. We will investigate the charge transfer Q across the central bond in a time window $[0, t]$ by following the two-time projective measurement protocol [72–77] in Fig. 1,

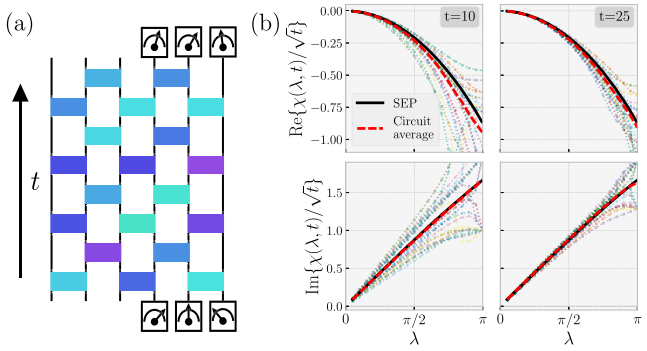


FIG. 1. (a) A two-time measurement protocol for charge transfer across the central bond in a random unitary circuit with a U(1) conserved charge. The charge in the right half of the system is measured at times 0 and t. (b) The cumulant generating function $\chi(\lambda, t)$ with a step initial state ($\mu = \infty$) at times $t = 10$ and $t = 25$ for different circuit realizations (multicolored) from TEBD simulations, the circuit averaged CGF with 35 samples (red dashed line), and the latetime analytical prediction for the SEP CGF [69] (black solid line). The two FCS snapshots show self-averaging of the FCS; circuit-to-circuit fluctuations in the rescaled CGF $\chi/\sqrt{t}$ decay as $\mathcal{O}(1/t)$ [68].

i.e., measuring the operator $\hat{Q}_R$ for the charge in the right half of the system at times 0 and t. The FCS for this measurement setup is characterized by the cumulant generating function (CGF) $\chi(\lambda) \equiv \log\langle e^{i\lambda Q}\rangle_t$, where the average $\langle f(Q)\rangle_t = \sum_Q P_t(Q) f(Q)$ is over repetitions of the measurement protocol and $P_t(Q)$ is the probability to measure a charge transfer Q. As shown in [50], writing $P_t(Q)$ in terms of Born probabilities enables us to write the average over measurements as a quantum expectation value [68],

$$\langle e^{i\lambda Q}\rangle_t = \langle \mathcal{T} e^{i\lambda \Delta\hat{Q}_R}\rangle' \equiv \mathrm{Tr}\left[\mathcal{T} e^{i\lambda\Delta\hat{Q}_R}\hat{\rho}'\right], \tag{2}$$

where $\Delta\hat{Q}_R \equiv \hat{Q}_R(t) - \hat{Q}_R(0)$ and $\hat{Q}_R(t) \equiv U(t)\hat{Q}_R U(t)^\dagger$ is the Heisenberg evolved charge operator. The noncommutativity of quantum dynamics requires the use of the time-ordering $\mathcal{T}$ [24,78,79]. The density matrix $\hat{\rho}'$ is related to the initial state $\hat{\rho}$ by the quantum channel $\hat{\rho}' = \sum_q P_q \hat{\rho} P_q$, where P_q are projectors onto the charge sector $Q_R = q$. For initial states with a chemical potential imbalance, $\hat{\rho} \propto \exp[\mu\hat{Q}_L - \mu\hat{Q}_R]$, we simply have $\hat{\rho}' = \hat{\rho}$.

The circuit averaged charge dynamics is known to map onto that of a discrete-time symmetric simple exclusion process [64,65,80] with a brick-wall geometry, i.e., $\overline{P_t(Q)} = P_{t,\text{SEP}}(Q)$, where $\overline{O}$ refers to the averaging O over circuits—all of the quantum fluctuations are lost in the circuit averaged moments of charge transfer. To capture the FCS in typical quantum circuits, we focus on self-averaging quantities, in particular, the cumulants of charge transfer. The cumulants are related to the generating function by $C_m(t) \equiv (-i\partial_\lambda)^m \chi(\lambda)|_{\lambda=0}$. To compute the nth cumulant, we introduce an often-used n-replica

statistical mechanics model [64,81–86], expressing each cumulant as a statistical expectation value.

Mapping to a statistical mechanics model.—By circuit averaging, we reduce the size of the state space needed to describe the replicated model. The Haar average of a replicated gate, $\bar{\mathcal{U}} \equiv \overline{U^{\otimes n} \otimes U^{*\otimes n}}$, projects onto a smaller space of states characterized by only the local charge degrees of freedom and a permutation degree of freedom $\sigma \in S_n$ that defines a pairing between the n replicas at each site (specifically, between the n conjugated and unconjugated replicas).

The circuit average of the replicated circuit is equivalent to a statistical mechanics model [64,66,81–85,87–91] with the permutation degrees of freedom living on the vertices and the charge configurations on the edges. The partition function for this statistical mechanics model is given by a sum over the charge configurations and permutations (compatible with the charges) with statistical weights associated with each edge [64,66,67].

In the SM model, $d \to \infty$ locks together neighboring permutations, and together with the initial and final boundary conditions $\sigma_0 = \sigma_t = \mathbb{1}$, the n-replica model decouples into n independent discrete-time SEP chains. Letting d be large but finite allows different permutations to appear during the dynamics; domain walls between domains of different permutations σ and τ have an energy cost of $\mathcal{O}[|\sigma\tau^{-1}| \log(d)]$ per unit length of domain wall [82] ($|\sigma|$ is the transposition distance of σ from $\mathbb{1}$). This is the basis of a large-d expansion that is the focus of the next section.

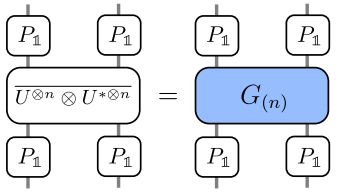
We use the time-evolving block decimation (TEBD) algorithm [92–94] to apply the $n = 2$ SM transfer matrix, and compute exactly the charge transfer variance $\bar{C}_2$, which is given as an SM expectation value. Denoting the n-replica expectation value by $\langle\cdot\rangle_{n\text{-rep}}$, and using superscripts to indicate in which replica an observable acts, the variance is given by

$$\overline{C_2}(t) = \langle \mathcal{T} \Delta\hat{Q}_R^{(1)2} - \Delta\hat{Q}_R^{(1)} \Delta\hat{Q}_R^{(2)} \rangle_{2\text{-rep}}. \tag{3}$$

Using maximum bond dimension $\chi = 1500$, we compute $\bar{C}_2$ for different initial chemical potential imbalances μ and for local Hilbert space dimensions $q \equiv 2d = 3, 4, 6, 8$ [95]. The results for $\mu = 0.1$ are shown in Fig. 2(a) and results for $\mu = 2$ and ∞ can be found in the Supplemental Material [68]. By subtracting the variance for $q = \infty$ (i.e., the SEP variance), we isolate the quantum contributions to $\bar{C}_2$, which we call ΔC_2, and find that these decay as $t^{-1/2}$ for all q [inset of Fig. 2(a)]. The n-replica SM model requires a local state space of dimension $2^n n!$, putting higher cumulants beyond reach with TEBD. In order to access the higher cumulants, and to find a theoretical explanation for the approach to SEP at $n = 2$, we develop an effective stochastic model for the charge dynamics in the replicated SM models.

Effective stochastic model.—At large d, the lowest energy contributions to the SM free energy come from dilute configurations of small domains of single transpositions in an "all-identity" background. The smallest of these domains—or *bubbles*—have the lowest possible energy cost of $4 \log(d)$. All configurations of these bubbles can be counted in the brick-wall circuit picture by inserting a projector $P_{\mathbb{1}}$ onto the identity permutation subspace in between every replicated gate $\bar{\mathcal{U}}$.

Upon doing this, we can replace $\bar{\mathcal{U}}$ with a gate $G_{(n)}$ that explores only the $\sigma = \mathbb{1}$ subspace but has a modified charge dynamics [68]:



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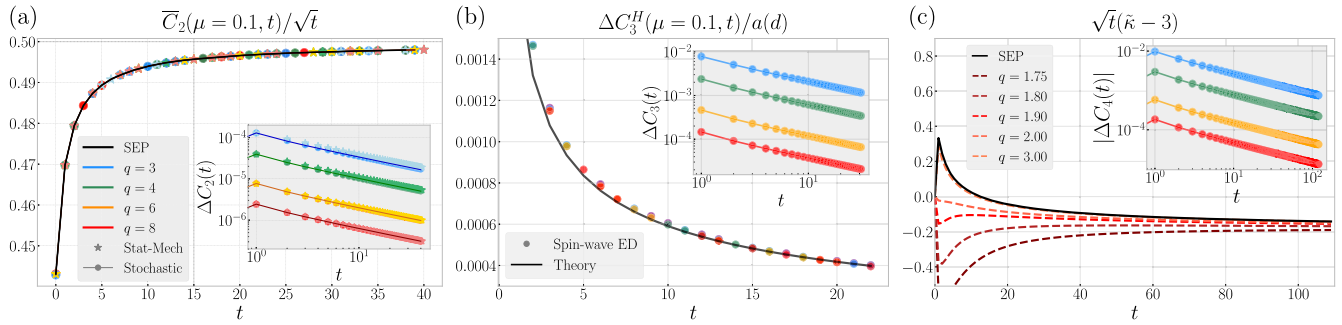
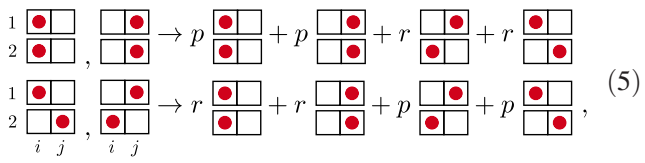


FIG. 2. Circuit averaged charge transfer cumulants $\bar{C}_n$ for U(1) charge-conserving random unitary circuits at different local Hilbert space dimension $q = 3, 4, 6, 8$ and in a discrete-time symmetric simple exclusion process, computed using TEBD applied to the SM transfer matrix: (a) the variance at chemical potential imbalance $\mu = 0.1$ (main) and the difference from SEP ΔC_2 (inset) with data from a replica statistical mechanics model and an effective stochastic process; (b) the third cumulant (rescaled by the interchain coupling $a(d)$) for a softened stochastic model with Hamiltonian H_3 [see Eq. (7)] (main) and the approach to SEP of the circuit averaged third cumulant (inset); (c) a proxy for the excess kurtosis showing a $t^{-1/2}$ approach to a Gaussian $\kappa = 3$ (main) and the approach to SEP of the circuit averaged fourth cumulant at equilibrium (inset).

The result is an effective Markov process described by an n-chain ladder with hard-core random walkers on each chain and a hopping rate that is conditional on the local occupancy of the other chains. More concretely, the model is that of n discrete-time SEP chains with pairwise local interactions between chains—when two chains have the same (different) charge configuration at a pair of neighboring sites, the interaction biases transitions in favor of states in which both chains have the same (different) configurations. The transfer matrix is given by a product of even and odd layers of two-site operators, $T = T_E T_O$ with $T_{E/O} = \prod_{j \in \text{even/odd}} T_{j,j+1}$. Representing a charge with a red dot and focusing on $n = 2$ replicas (labeled 1,2), the modified transitions on a pair of sites (i, j) are given by



(5)

where the transition probabilities are $p = [(1 + a)/4]$ and $r = [(1 - a)/4]$ with $a(d) = [4d^4 - 1]^{-1}$. All other transitions are as given for decoupled SEP chains (charges hopping with probability $1/2$). The derivation of the Markov process is described in detail in the Supplemental Material [68].

This effective model inherits an n-fold SU(2) invariance (one for each chain) from the SM model, allowing for arbitrary rotations of the charge basis $|Q = 0, 1)$ in each chain (see Supplemental Material for details [68]). Choosing a rotated basis $[\{|\uparrow) \propto |0) + |1), |\downarrow) \propto |0) - |1)\}]$, the nth cumulant can be written in terms of matrix elements of the n-chain transfer matrix T_n, with the initial and final states having at most n magnons (overturned spins). This reduces the problem of calculating C_n to the diagonalization of an $L^n \times L^n$ matrix.

Results.—By applying the Markov process transfer matrix exactly, we calculate the second and third cumulants at different biases and the fourth cumulant in equilibirum. We find that in all cases, the effective evolution approaches SEP as $\Delta C_n \equiv \bar{C}_n - C_n^{\text{SEP}} \sim a(d) t^{-1/2}$ (see the insets in Fig. 2 and also Ref. [68]). The variance data show excellent agreement between the SM model and the effective model.

In chaotic models at equilibrium (no bias, $\mu = 0$), we expect that the distribution $P_t(Q)$ will approach a Gaussian at late times. However, even longtime deviations from Gaussianity are *universal* and are captured by an effective classical stochastic model—SEP in the case of random circuits. For example, using standard SEP results [69], we find that at half filling, the average equilibrium excess kurtosis decays in a universal way as

$$\kappa - 3 = \frac{(4 - 3\sqrt{2})\sqrt{\pi}}{2\sqrt{t}} + \cdots, \tag{6}$$

independently of the value of q. Circuit averaging quantities with the evolution unitary in the denominator, such as kurtosis, requires a replica trick. To avoid this, we calculate the proxy $\tilde{\kappa} \equiv (\overline{\mu_4}/\overline{\sigma^4})$ that averages the numerator and denominator separately (μ_4 is the fourth central moment and σ is the standard deviation) and find the same universal approach to a Gaussian, $\kappa = 3$, for different q [Fig. 2(c)]. We have accentuated the variations between models by using unphysical local Hilbert space dimensions q.

Effective Hamiltonian.—To understand the approach to SEP at long times, we can map the effective n-chain Markov processes to an effective ferromagnetic Hamiltonian. We do this by softening the transfer matrix, $T_n \to e^{-H_n}$. The effective Hamiltonian is given by

$$H_n \equiv \sum_j \sum_{\alpha=1}^{n} P_{j,j+1}^{(\alpha)} - a(d) \sum_j \sum_{\alpha<\beta} P_{j,j+1}^{(\alpha)} P_{j,j+1}^{(\beta)}, \tag{7}$$

where the superscripts indicate in which chain an operator acts and where the second term contains a sum over distinct pairs of chains. We have dropped subleading $\mathcal{O}(1/d^8)$ terms. In terms of Heisenberg spin interactions, the projector P is given by $P_{j,j+1} = \frac{1}{4} - \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$. The imaginary time dynamics is then dominated at late times by the low energy physics of (7). We study the low energy spectrum for $n = 2$ using standard spin-wave methods [68] and find that, at late times, the quantum contribution to the charge transfer variance is

$$\Delta C_2^H \approx \frac{a \tanh(\mu/2)^2}{16\sqrt{\pi t}}, \tag{8}$$

where the superscript H indicates that this prediction is for the continuous time stochastic model with imaginary time Hamiltonian dynamics [68]. We also consider the third cumulant in the softened stochastic model, finding the familiar $t^{-1/2}$ decay of quantum fluctuations [Fig. 2(b)] from numerics and theoretical predictions in the linear response regime ($\mu \ll 1$ [68]). This general scaling can be generalized to higher cumulants using a simple renormalization group (RG) argument based on power counting: because of the imaginary time evolution, the longtime dynamics is controlled by the low energy properties of Eq. (7). Using standard spin-coherent state path integral techniques, it is straightforward to show that the perturbation coupling the replicas with strength $a(d)$ has scaling dimension $\Delta = 4$, and is thus irrelevant in the RG sense. At long times, we thus expect the different replicas (SEP chains) to be effectively decoupled so that $\langle O \rangle_{n\text{-chain}} = \langle O \rangle_{\text{SEP}} (1 + \mathcal{O}(t^{-1}))$.

The asymptotic decoupling between replicas also establishes that circuit-to-circuit fluctuations are suppressed at long times. To see this, consider an n-copy quantity A (this could be mean charge transfer for $n = 1$ or charge transfer

variance for $n = 2$), the circuit average of A is given by $\bar{A} = \langle X \rangle_{n\text{-chain}}$ for some operator X on n replicas, whereas the circuit-to-circuit fluctuations are controlled by $\overline{(A - \bar{A})^2} = \langle X \otimes X \rangle_{2n\text{-chain}} - \langle X \rangle^2_{n\text{-chain}}$. Using the asymptotic decoupling of the SEP chains, we have the aforementioned suppression of circuit-to-circuit fluctuations, $\mathrm{Var}(A)/\bar{A}^2 \sim t^{-1}$. Therefore, the FCS of individual quantum circuits approaches the SEP predictions as

$$\chi(\lambda)/\sqrt{t} = \overline{\chi(\lambda)}/\sqrt{t} + \mathcal{O}(1/t), \tag{9}$$

with $\bar{\chi}/\sqrt{t} \to \chi_{\mathrm{SEP}}/\sqrt{t}$ as $t \to \infty$. To verify this prediction, we have computed the FCS of individual random quantum circuits for a domain-wall initial state ($\mu = \infty$) using standard counting field techniques [68] (Fig. 1). We find that the rescaled CGF $\chi(\lambda)/\sqrt{t}$ is indeed self-averaging with $\mathcal{O}(1/t)$ fluctuations, and does approach the SEP predictions at long times [68].

Discussion.—Our main result is that charge transfer fluctuations in random charge-conserving quantum circuits is controlled by an effective SEP stochastic model at long times: $C_n = C_n^{\mathrm{SEP}} + \mathcal{O}(t^{-1/2})$. The full cumulant generating function of individual random circuits $\chi(\lambda) \approx \overline{\chi(\lambda)}$ must then take the same form as that of SEP at late times, $\overline{\chi(\lambda)} \equiv \overline{\log\langle e^{i\lambda Q}\rangle} \approx \chi_{\mathrm{SEP}}(\lambda)$. The symmetric exclusion process generating function is known analytically [69] from integrability, and is given by

$$\chi(\lambda) \approx \sqrt{t} F(\omega), \qquad F(\omega) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/2}} \omega^n, \tag{10}$$

where $\omega = \rho_L(e^{i\lambda} - 1) + \rho_R(e^{-i\lambda} - 1) + \rho_L\rho_R(e^{i\lambda} - 1)(e^{-i\lambda} - 1)$ and $\rho_{L/R} = [e^{\mu_{L/R}}/(1 + e^{\mu_{L/R}})]$ is the initially local charge density in the left (L) and right (R) halves of the system [96]. The same FCS was recently shown to emerge from MFT [62] from solving Eq. (1) directly. Our results thus establish that the current fluctuations of *individual realizations* of random quantum circuits are described by the simple fluctuating hydrodynamic equation (1). To fully establish the validity of MFT to many-body quantum systems, it would be interesting to consider ensembles of circuits with more general diffusion constants $D(\rho)$: there as well we expect a similar mapping onto effective classical stochastic models to the one we have found here, with irrelevant interreplica couplings as in (7). We leave the study of such generalizations to future work.

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