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# Wigner-Araki-Yanase Theorem for Continuous and Unbounded Conserved Observables

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The Wigner-Araki-Yanase (WAY) theorem states that additive conservation laws imply the commutativity of exactly implementable projective measurements and the conserved observables of the system. Known proofs of this theorem are only restricted to bounded or discrete-spectrum conserved observables of the system and are not applicable to unbounded and continuous observables like a momentum operator. In this Letter, we present the WAY theorem for possibly unbounded and continuous conserved observables under the Yanase condition, which requires that the probe positive operator-valued measure should commute with the conserved observable of the probe system. As a result of this WAY theorem, we show that exact implementations of the projective measurement of the position under momentum conservation and of the quadrature amplitude using linear optical instruments and photon counters are impossible. We also consider implementations of unitary channels under conservation laws and find that the conserved observable $L_S$ of the system commutes with the implemented unitary $U_S$ if $L_S$ is semibounded, while $U_S^\dagger L_S U_S$ can shift up to possibly nonzero constant factor if the spectrum of $L_S$ is upper and lower unbounded. We give simple examples of the latter case, where $L_S$ is a momentum operator.

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*Introduction.*—One of the fundamental questions of the quantum measurement theory is whether there exist limitations on the implementable measurements imposed by the physical laws. One of the fundamental findings of quantum measurement theory [1] is the fact that the physical conservation laws restrict our ability to implement measurements. By considering specific examples of spin measurements, Wigner [2] found that additive conservation law prohibits the projective and repeatable measurements of an observable that does not commute with the conserved one. He also discussed that an approximate measurement is possible if a probe state has a large coherence in the conserved quantity. Later, Araki and Yanase [3,4] generalized Wigner's result to arbitrary repeatable projective measurements and bounded conserved observables. The former no-go result is now called the Wigner-Araki-Yanase (WAY) theorem.

From these pioneering works by Wigner, Araki, and Yanase, many results have been published that sophisticate the WAY theorem and extend it to various directions. The first and exciting direction is to extend the WAY theorem to a quantitative form. Since the original WAY theorem was a qualitative theorem, many researchers, including Yanase and Ozawa, extended it to provide necessary conditions for an approximate implementation of desired measurements [4–7]. By imposing the Yanase condition, which requires that the probe observable of the measurement model should commute with the conserved observable, it became clear that the size of the measurement device [4], the variance [5], and quantum fluctuations [6,7] of the conserved quantities must be inversely proportional to the error in implementing the desired measurement. The second direction is extending the WAY theorem to general quantum information processings beyond quantum measurements. This extension was first made as a restriction on the implementation of controlled-NOT gates [8], extended to various limited unitary gates [9–11]; then it was shown that, for an arbitrary unitary gate [12,13], the same restriction is given as in measurements. This direction has been further deepened in recent years, and now extended versions of the WAY theorem are given for various objects, such as error-correcting codes [14,15], thermodynamic processes [15], and the toy model of black holes [14,15].

Most of the existing WAY-type results are, however, restricted to bounded conserved observables and not applicable to physically important examples in which unbounded conserved observables are common. This problem is crucial, since if the WAY theorem is correct for unbounded operators, the position measurement without error is impossible under the momentum conservation law (see Fig. 1).

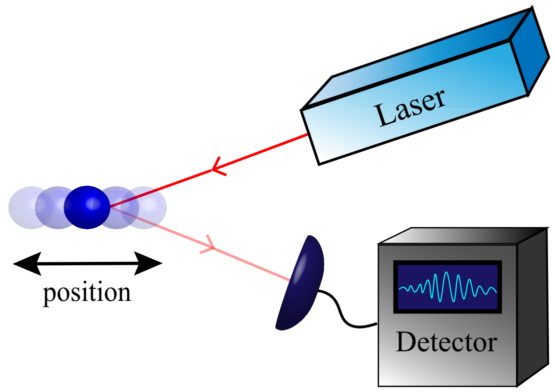


FIG. 1. It is well known that position and momentum cannot be measured without error *simultaneously*. However, in natural settings like the above schematic, position measurements will be performed under the momentum conservation law. If the WAY theorem is correct for unbounded operators, the position itself cannot be measured without error under such natural settings.

Because of its importance, there has been active research on extending the WAY theorem to unbounded systems. However, despite previous important progress [16–21], this problem remains unsolved. The extensions proved in [16,17] require some technical additional conditions, which do not hold for the position measurement under the momentum conservation. There are also detailed accounts of the position measurements under the momentum conservation [18,20]. Particularly in Ref. [20], a trade-off relation is obtained for the accuracy of the position measurement and the momentum coherence of the probe under the momentum conservation. However, in the derivation of the trade-off, as pointed out in [21], issues related to domains of unbounded operators are ignored. Moreover, even if the trade-off relation was valid, it would not imply the impossibility of the exact implementation of the position measurement under momentum conservation because we can prepare a probe state with a divergent momentum coherence and the trade-off relation gives the trivial inequality $0 \geq 0$ in this case.

Here we give a positive answer to this question: we present the WAY theorem for general unbounded and continuous conserved observables under the Yanase condition, which is a basic condition introduced in Refs. [4,5] and used in [20]. We also consider the unitary channel implementation and show that a similar theorem holds in that case. For the unitary channel implementation without error under the conservation law, we show that the implemented unitary $U_S$ and the conserved observable $L_S$ must commute, except for a very limited scenario in which $L_S$ is upper and lower unbounded and the change of $L_S$ by $U_S$ is a constant shift: $U_S^\dagger L_S U_S = L_S + \gamma \mathbb{1}$ [22].

*Notation and definitions.*—In this Letter, the Hilbert space of a quantum system $S$ is denoted by $\mathcal{H}_S$, which may be finite or infinite dimensional. The unit operator and sets of bounded and trace-class operators on a Hilbert space $\mathcal{H}_S$ are, respectively, denoted by $\mathbb{1}_S$ $\mathbf{B}(\mathcal{H}_S)$, and $\mathbf{T}(\mathcal{H}_S)$. A nonnegative operator $\rho_S \in \mathbf{T}(\mathcal{H}_S)$ with a unit trace is called a density operator, which corresponds to a quantum state. The set of density operators on $\mathcal{H}_S$ is denoted by $\mathbf{D}(\mathcal{H}_S)$.

For a linear map $\Lambda\colon \mathbf{T}(\mathcal{H}_A) \to \mathbf{T}(\mathcal{H}_B)$ that is bounded with respect to the trace norms on $\mathbf{T}(\mathcal{H}_A)$ and $\mathbf{T}(\mathcal{H}_B)$, the adjoint map $\Lambda^\dagger\colon \mathbf{B}(\mathcal{H}_B) \to \mathbf{B}(\mathcal{H}_A)$ is well defined by $\mathrm{Tr}[\Lambda(\rho_A)b] = \mathrm{Tr}[\rho_A \Lambda^\dagger(b)][\rho_A \in \mathbf{T}(\mathcal{H}_A); b \in \mathbf{B}(\mathcal{H}_B)]$, where $\mathrm{Tr}[\cdot]$ denotes the trace. A linear map $\Lambda\colon \mathbf{T}(\mathcal{H}_A) \to \mathbf{T}(\mathcal{H}_B)$ is called a *quantum channel* if $\Lambda$ is trace preserving and $\Lambda^\dagger$ is completely positive (CP) [23,24]. The map $\Lambda^\dagger$ represents the channel in the Heisenberg picture.

A triple $(\Omega, \Sigma, \mathsf{E}_S)$ is called a *positive operator-valued measure* (POVM) [25,26] on $\mathcal{H}_S$ if $\Sigma$ is a $\sigma$-algebra on the set $\Omega$ and $\mathsf{E}_S\colon \Sigma \to \mathbf{B}(\mathcal{H}_S)$ satisfies (i) $\mathsf{E}_S(X) \geq 0 (X \in \Sigma)$, (ii) $\mathsf{E}_S(\emptyset) = 0$, $\mathsf{E}_S(\Omega) = \mathbb{1}_S$, and (iii) $\mathsf{E}_S(\cup_k X_k) = \sum_k \mathsf{E}(X_k)$ in the weak operator topology [27,28] for any disjoint sequence $(X_k) \subseteq \Sigma$. A POVM $(\Omega, \Sigma, \mathsf{E}_S)$ is called a projection-valued measure (PVM) if each $\mathsf{E}_S(X)(X \in \Sigma)$ is a projection. A POVM $(\Omega, \Sigma, \mathsf{E}_S)$ on $\mathcal{H}_S$ describes the outcome statistics of a general measurement process so that the outcome probability measure when the state is prepared in $\rho_S \in \mathbf{D}(\mathcal{H}_S)$ is given by $\Sigma \ni X \mapsto \mathrm{Tr}[\rho_S \mathsf{E}_S(X)]$.

We now consider the implementation of a quantum channel $\Psi\colon \mathbf{T}(\mathcal{H}_S) \to \mathbf{T}(\mathcal{H}_{S'})$ by a system-environment model. Here a tuple $(\mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U)$ is called a *system-environment model* if $P$, $S'$, and $P'$ are quantum systems, $\rho_P \in \mathbf{D}(\mathcal{H}_P)$ is a density operator on $P$ called the probe state, and $U\colon \mathcal{H}_S \otimes \mathcal{H}_P \to \mathcal{H}_{S'} \otimes \mathcal{H}_{P'}$ is a unitary operator. The system-environment model $(\mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U)$ is said to *implement* $\Psi$ if

$$\Psi(\rho_S) = \mathrm{Tr}_{P'}[U(\rho_S \otimes \rho_P)U^\dagger] \qquad [\rho_S \in \mathbf{D}(\mathcal{H}_S)], \quad (1)$$

where $\mathrm{Tr}_A[\cdot]$ denotes the partial trace over a system $A$ and the dagger denotes the adjoint. The condition (1) says that the channel $\Psi$ is realized if we first prepare the system $S$ in an arbitrary state $\rho_S$ and $P$ in the fixed probe state $\rho_P$, then they interact according to the unitary $U$, and finally discard the output probe system $P'$ (see Fig. 2).

The implementation of a measurement by a measurement model is defined in a similar way as follows. A tuple $\mathbb{M} = (\mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U, (\Omega, \Sigma, \mathsf{F}_{P'}))$ is called a "'measurement model" if $(\mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U)$ is a system-environment model and $(\Omega, \Sigma, \mathsf{F}_{P'})$ is a POVM on $\mathcal{H}_{P'}$.

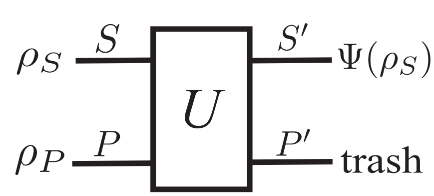


FIG. 2. A system-environment model that implements a channel $\Psi$.

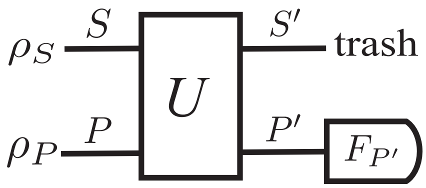


FIG. 3. A measurement model that implements a POVM $\mathsf{E}_S$.

The measurement model $\mathbb{M}$ is said to implement a POVM $(\Omega, \Sigma, \mathsf{E}_S)$ on $\mathcal{H}_S$ if

$$\mathsf{E}_S(X) = \mathrm{Tr}_P[(\mathbb{1}_S \otimes \rho_P) U^\dagger (\mathbb{1}_{S'} \otimes \mathsf{F}_{P'}(X)) U] \tag{2}$$

for all $X \in \Sigma$ (see Fig. 3).

Let $L_A$ be a possibly unbounded self-adjoint operator [27,28] on a Hilbert space $\mathcal{H}_A$ and let $\mathrm{dom}(L_A) \subseteq \mathcal{H}_A$ denote the domain of $L_A$. A bounded operator $a \in \mathbf{B}(\mathcal{H}_A)$ is said to *commute* with $L_A$ if $a$ commutes with the spectral measure [27,28] of $L_A$. If $U_A \in \mathbf{B}(\mathcal{H}_A)$ is unitary, then $U_A$ commutes with $L_A$ if and only if $L_A = U_A^\dagger L_A U_A$, where the domain of the self-adjoint operator $U_A^\dagger L_A U_A$ is $U_A^\dagger \mathrm{dom}(L_A)$.

The *spectrum* $\sigma(L_A)$ of a self-adjoint operator $L_A$ on $\mathcal{H}_A$ is the set of $\lambda \in \mathbb{C}$ such that the operator $L_A - \lambda \mathbb{1}_A$ has no bounded inverse. The spectrum $\sigma(L_A)$ is a closed subset of the reals $\mathbb{R}$ and, if $\mathcal{H}_A$ is finite dimensional, coincides with the set of the eigenvalues of $L_A$. A self-adjoint operator $L_A$ is said to be semibounded (respectively, unbounded) if $\sigma(L_A)$ is an upper or lower bounded (respectively, unbounded) subset of $\mathbb{R}$. For example, the quantum harmonic oscillator Hamiltonian is unbounded but still semibounded.

*Main results.*—We now state the main results of this Letter.

*Theorem 1: WAY theorem for projective measurements.*—Let $[\mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U, (\Omega, \Sigma, \mathsf{F}_{P'})]$ be a measurement model that implements a PVM $(\Omega, \Sigma, \mathsf{E}_S)$ on $\mathcal{H}_S$. Suppose that there are (possibly unbounded) self-adjoint operators $L_S$, $L_P$, $L_{S'}$, and $L_{P'}$ that act, respectively, on $\mathcal{H}_S$, $\mathcal{H}_P$, $\mathcal{H}_{S'}$, and $\mathcal{H}_{P'}$ and satisfy the conservation law

$$U^\dagger L_{S'P'} U = L_{SP}, \tag{3}$$

where $L_{SP} \coloneqq L_S \otimes \mathbb{1}_P + \mathbb{1}_S \otimes L_P$ and $L_{S'P'} \coloneqq L_{S'} \otimes \mathbb{1}_{P'} + \mathbb{1}_{S'} \otimes L_{P'}$. We also assume the Yanase condition that $\mathsf{F}_{P'}(X)$ commutes with $L_{P'}$ for every $X \in \Sigma$. Then $\mathsf{E}_S(X)$ commutes with $L_S$ for every $X \in \Sigma$.

*Theorem 2: WAY theorem for unitary channels.*—Let $U_{S\to S'}\colon \mathcal{H}_S \to \mathcal{H}_{S'}$ be a unitary operator between Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_{S'}$, let $\mathcal{U}_{S\to S'}\colon \mathbf{T}(\mathcal{H}_S) \to \mathbf{T}(\mathcal{H}_{S'})$ be the unitary channel defined by $\mathcal{U}_{S\to S'}(\rho_S) \coloneqq U_{S\to S'} \rho_S U_{S\to S'}^\dagger$ $[\rho_S \in \mathbf{T}(\mathcal{H}_S)]$, and let $(\mathcal{H}_S, \mathcal{H}_P, \mathcal{H}_{S'}, \mathcal{H}_{P'}, \rho_P, U)$ be a system-environment model that implements $\mathcal{U}_{S\to S'}$. Suppose that there are (possibly unbounded) self-adjoint operators $L_S$, $L_P$, $L_{S'}$, and $L_{P'}$ that act, respectively, on $\mathcal{H}_S$, $\mathcal{H}_P$, $\mathcal{H}_{S'}$, and $\mathcal{H}_{P'}$ and satisfy the conservation law (3). Then there exists a real number $\gamma \in \mathbb{R}$ such that

$$U_{S\to S'}^\dagger L_{S'} U_{S\to S'} = L_S + \gamma \mathbb{1}_S. \tag{4}$$

Moreover, if $\mathcal{H}_S = \mathcal{H}_{S'}$ and $L_S = L_{S'}$ hold and $L_S$ is semibounded, then $U_S \coloneqq U_{S\to S'}$ commutes with $L_S$.

The latter part of Theorem 2 can be immediately proved from the former part as follows. Assume $\mathcal{H}_S = \mathcal{H}_{S'}$ and $L_S = L_{S'}$. Then Eq. (4) implies

$$U_S^\dagger L_S U_S = L_S + \gamma \mathbb{1}_S. \tag{5}$$

Suppose that $\sigma(L_S)$ is lower bounded and let $\lambda_{\min}(L_S) \in \mathbb{R}$ denote the finite infimum of the spectrum $\sigma(L_S)$. Since the spectra of $L_S$ and $U_S^\dagger L_S U_S$ coincide, Eq. (5) implies $\lambda_{\min}(L_S) = \lambda_{\min}(L_S) + \gamma$ and therefore $\gamma = 0$; hence $U_S$ commutes with $L_S$. The claim is similarly proved by considering the supremum of $\sigma(L_S)$ when $L_S$ is upper bounded.

Theorems 1 and 2 can be proved by using the notion of the multiplicative domains of unital CP maps [24,29]. This notion is recently used in [30] to derive WAY-type trade-off relations for bounded observables. In the proof, arguments on the topological group $\mathbb{R}$ and its unitary representations are also essential that derive statements valid for all $t \in \mathbb{R}$ from those valid only for restricted $t$. We also remark that Theorems 1 and 2 can be generalized to general continuous symmetries described by connected topological groups [31,32]. All the details of the proofs, including the generalization to continuous symmetries, are given in the Supplemental Material [33].

*Applications of the WAY theorem for projective measurements.*—Now we see two applications of Theorem 1, which show that some kinds of measurements are not implementable.

The first one is the position measurement under the momentum conservation [18,20]. Since the position and momentum operators of a one-dimensional quantum particle are noncommutative in the sense that their spectral measures do not commute, it immediately follows from Theorem 1 that no measurement model satisfying the momentum conservation and the Yanase condition can implement the projective position measurement of the particle. This gives a positive answer to the open question in [20].

The next one is the projective measurement of a quadrature amplitude of a single-mode optical field by using beam splitters, phase shifters, and photon counters. We consider fixed-frequency optical fields and denote by $\hat{a}_A$ the annihilation operator acting on the Hilbert space $\mathcal{H}_A$ of a mode $A$. In this situation, the accurate implementation of the projective measurement of the quadrature amplitude operator $\hat{q}_S = (\hat{a}_S + \hat{a}_S^\dagger)/2$ is important in continuous-variable (CV) quantum technologies like CV quantum

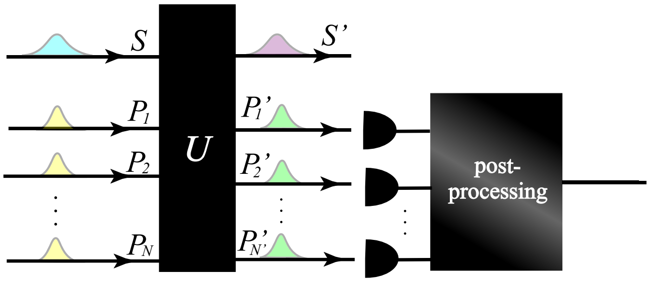


FIG. 4. Measurement model with passive optical operations and photon-counting measurements.

key distribution [46] or CV quantum teleportation [47]. However, since the quadrature amplitude operator $\hat{q}_S = (\hat{a}_S + \hat{a}_S^\dagger)/2$ does not commute with the number operator $L_S = \hat{n}_S = \hat{a}_S^\dagger \hat{a}_S$, Theorem 1 implies that the errorless projective measurement $\mathsf{E}_S$ of $\hat{q}_S$ is not implementable by any measurement model satisfying the conservation law of the total photon number and the Yanase condition.

To see the detail of the above, let us introduce a measurement model of the passive optical operations (see Fig. 4) and how Theorem 1 works on this model. A two-mode passive optical unitary $V\colon \mathcal{H}_{A_{\rm in}} \otimes \mathcal{H}_{B_{\rm in}} \to \mathcal{H}_{A_{\rm out}} \otimes \mathcal{H}_{B_{\rm out}}$ is a unitary such that $V^\dagger \hat{a}_{A_{\rm out},B_{\rm out}} V$ is a linear combination of $\hat{a}_{A_{\rm in}}$ and $\hat{a}_{B_{\rm in}}$ and energy (photon number) conservation

$$V^\dagger(\hat{n}_{A_{\rm out}} + \hat{n}_{B_{\rm out}})V = \hat{n}_{A_{\rm in}} + \hat{n}_{B_{\rm in}} \tag{6}$$

holds. Here we abbreviated the identities and tensors. The Hilbert spaces $\mathcal{H}_P = \mathcal{H}_{P_1} \otimes \ldots \otimes \mathcal{H}_{P_N}$, $\mathcal{H}_{S'} = \mathcal{H}_{S'_1} \otimes \ldots \otimes \mathcal{H}_{S'_M}$, $\mathcal{H}_{P'} = \mathcal{H}_{P'_1} \otimes \ldots \otimes \mathcal{H}_{P'_{N-M+1}}$ are finite tensor products of single-mode Hilbert spaces and the total unitary $U\colon \mathcal{H}_S \otimes \mathcal{H}_P \to \mathcal{H}_{S'} \otimes \mathcal{H}_{P'}$ is finite compositions of passive optical unitaries satisfying Eq. (6). We assume that the probe POVM $[\Omega, \Sigma, \mathsf{F}_{P'}(\cdot)]$ on $\mathcal{H}_{P'}$ commutes with the outcome photon number operators $\hat{n}_{P'_1}, \ldots, \hat{n}_{P'_{N-M+1}}$ so that the Yanase condition holds. For example, if the probe measurement $\mathsf{F}_{P'}$ is realized by postprocessing the outcomes of photon-counting measurements on the modes $P'_1, \ldots, P'_{N-M+1}$, this condition holds.

Because of Theorem 1, the above model cannot implement the projective measurement $\mathsf{E}_S$ of $\hat{q}_S$. To see that, let us put the conserved observables in Theorem 1 as $L_S \coloneqq \hat{n}_S$, $L_P \coloneqq \sum_k \hat{n}_{P_k}$, $L_{S'} \coloneqq \sum_k \hat{n}_{S'_k}$, $L_{P'} \coloneqq \sum_k \hat{n}_{P'_k}$. Then the conservation law (3) holds, which is in this case the total photon number conservation $U^\dagger(\hat{n}_S + \hat{n}_P)U = \hat{n}_{S'} + \hat{n}_{P'}$. Moreover, since $\hat{q}_S$ does not commute with $\hat{n}_S = L_S$, the projective measurement $\mathsf{E}_S$ of $\hat{q}_S$ also does not commute with $L_S$. Therefore, Theorem 1 prohibits the implementation of $\mathsf{E}_S$. We remark that we do not require any condition on the probe state $\rho_P$.

We can still realize *approximate* measurement of $\hat{q}_S$ by the balanced homodyne detection [48,49], in which the signal optical field is mixed with a strong local oscillator (LO) field by a half beam splitter and the properly normalized difference of the photocounts of the output fields is recorded. The measurement model of the homodyne detection apparently satisfies the above assumptions and hence does not implement the projective measurement of $\hat{q}_S$.

On the other hand, it can be shown [48] that if we prepare the probe LO state as a coherent state $|\beta_{\rm LO}\rangle_P = e^{-|\beta_{\rm LO}|^2/2}\sum_{n=0}^\infty (\beta_{\rm LO}^n/\sqrt{n!})|n\rangle_P$ $(\beta_{\rm LO} \in \mathbb{R})$, where $|n\rangle_P$ denotes the photon number eigenstate of the probe LO field, then, for every initial state $\rho_S$, the probability distribution of the homodyne measurement converges in distribution to that of the projective measurement of $\hat{q}_S$ in the strong LO limit $\beta_{\rm LO} \to \infty$. This is in accordance with the "positive part" of the original WAY arguments, since strong LO means a large spread of $|\beta_{\rm LO}\rangle_P$ in the photon number basis. We should still be careful about the statewise nature of the convergence that results from the unboundedness of the conserved observable $\hat{n}_S$. For example, if we prepare the input state as a coherent state $|\alpha_S\rangle_S$ and $|\alpha_S|$ is comparable with the LO amplitude $\beta_{\rm LO}$, the outcome distribution is far from that of the projective measurement of $\hat{q}_S$.

*Examples of implementations of unitary channels.*—We now give two examples of implementations of a unitary channel in which the constant term $\gamma \mathbb{1}_S$ in Eq. (4) is nonzero.

In the models, the final systems $S'$ and $P'$ are, respectively, the same as the initial systems $S$ and $P$. We take one-dimensional quantum particles as the system and probe systems so that the Hilbert spaces $\mathcal{H}_S = \mathcal{H}_{S'}$ and $\mathcal{H}_P = \mathcal{H}_{P'}$ are both the space $L^2(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$. Let $\hat{x}_\alpha$ and $\hat{p}_\alpha$ $(\alpha = S, P)$ denote, respectively, the position and momentum operators of the system $\alpha$, which satisfy the Weyl relation

$$e^{it\hat{x}_\alpha} e^{is\hat{p}_\alpha} = e^{-ist} e^{is\hat{p}_\alpha} e^{it\hat{x}_\alpha} \qquad (s, t \in \mathbb{R}; \alpha = S, P), \tag{7}$$

where $\hbar$ is set to 1. We fix an arbitrary real number $\gamma \neq 0$ and give two implementation models of the unitary channel $\mathcal{U}_S(\rho_S) = U_S \rho_S U_S^\dagger$ with $U_S \coloneqq e^{i\gamma\hat{x}_S}$. We put $L_S = L_{S'} = \hat{p}_S$ and $L_P = L_{P'} = \hat{p}_P$. Then from Eq. (7) we can see that $U_S^\dagger L_S U_S = L_S + \gamma \mathbb{1}_S$ holds.

In the first example, we take the following total unitary:

$$U_{SP}^{(1)} \coloneqq e^{i\gamma\hat{x}_S} \otimes e^{-i\gamma\hat{x}_P}. \tag{8}$$

Then from (7) this unitary satisfies the momentum conservation law

$$U_{SP}^{(1)\dagger}(\hat{p}_S + \hat{p}_P)U_{SP}^{(1)} = \hat{p}_S + \hat{p}_P, \tag{9}$$

where we omitted the tensors and units. Moreover, for an arbitrary probe state $\rho_P \in \mathbf{D}(\mathcal{H}_P)$ we have

$$\mathrm{Tr}_P[U_{SP}^{(1)}(\rho_S \otimes \rho_P)U_{SP}^{(1)\dagger}] = U_S \rho_S U_S^\dagger \quad (10)$$

$[\rho_S \in \mathbf{D}(\mathcal{H}_S)]$. This shows that the system-environment model $(\mathcal{H}_S, \mathcal{H}_P, \mathcal{H}_S, \mathcal{H}_P, \rho_P, U_{SP}^{(1)})$ satisfies all the assumptions of Theorem 2 together with (4) with nonzero $\gamma$.

There is another example of an implementation of the unitary channel $\mathcal{U}_S$ in which the total unitary is not in product form. For simplicity, we assume $\gamma > 0$ and define the total unitary

$$U_{SP}^{(2)} \coloneqq e^{i\gamma \hat{x}_S} \otimes e^{-i\gamma \hat{x}_P} 1_X(\hat{p}_P) + \mathbb{1}_S \otimes 1_{\mathbb{R}\setminus X}(\hat{p}_P), \quad (11)$$

which is not in product form. Here,

$$1_A(\lambda) \coloneqq \begin{cases} 1 & (\lambda \in A); \\ 0 & (\lambda \notin A) \end{cases} \quad (12)$$

is the indicator function of a subset $A \subseteq \mathbb{R}$, and $X \subseteq \mathbb{R}$ is a measurable set such that $X + \gamma \coloneqq \{x + \gamma : x \in X\} = X$, and neither $X$ nor $\mathbb{R}\setminus X$ is a null set. For definiteness, we take as $X = \cup_{n:\ \mathrm{integer}} [\gamma n - \gamma/3, \gamma n + \gamma/3]$. Then, since $e^{i\gamma \hat{x}_P} 1_X(\hat{p}_P) e^{-i\gamma \hat{x}_P} = 1_X(\hat{p}_P - \gamma) = 1_{X+\gamma}(\hat{p}_P) = 1_X(\hat{p}_P)$, that is, $e^{i\gamma \hat{x}_P}$ and $1_X(\hat{p}_P)$ commute, the operator $U_{SP}^{(2)}$ in Eq. (11) is unitary. Moreover, from

$$\begin{aligned} &U_{SP}^{(2)\dagger}(e^{it\hat{p}_S} \otimes e^{it\hat{p}_P})U_{SP}^{(2)} \\ &= e^{-i\gamma \hat{x}_S} e^{it\hat{p}_S} e^{i\gamma \hat{x}_S} \otimes e^{i\gamma \hat{x}_P} e^{it\hat{p}_P} e^{-i\gamma \hat{x}_P} 1_X(\hat{p}_P) \\ &\quad + e^{it\hat{p}_S} \otimes e^{it\hat{p}_P} 1_{\mathbb{R}\setminus X}(\hat{p}_P) \quad (13) \\ &= e^{it\hat{p}_S} \otimes e^{it\hat{p}_P} \qquad (t \in \mathbb{R}), \quad (14) \end{aligned}$$

the momentum conservation

$$U_{SP}^{(2)\dagger}(\hat{p}_S + \hat{p}_P)U_{SP}^{(2)} = \hat{p}_S + \hat{p}_P \quad (15)$$

holds. If we take a state $\rho_P \in \mathbf{D}(\mathcal{H}_P)$ supported by the projection $1_X(\hat{p}_P)$, then we have $U_{SP}^{(2)}(\rho_S \otimes \rho_P)U_{SP}^{(2)\dagger} = U_{SP}^{(1)}(\rho_S \otimes \rho_P)U_{SP}^{(1)\dagger} [\rho_S \in \mathbf{D}(\mathcal{H}_S)]$ and therefore from Eq. (10) we can see that the system-environment model $(\mathcal{H}_P, \mathcal{H}_S, \mathcal{H}_P, \rho_P, U_{SP}^{(2)})$ implements $\mathcal{U}_S$.

*Conclusion.*—We investigated measurement implementations under conservation laws of unbounded observables and established the WAY theorem for projective measurements under the Yanase condition. Applications of this WAY theorem revealed that the projective measurements of the position and the quadrature amplitude are incompatible with the conservation of the momentum and the photon number, respectively. It is still open whether the original WAY theorem [3] (or Theorem 8.1 of [50]) for *repeatable* measurement models can be generalized to unbounded conserved observables.

We also considered implementation of unitary channels under conservation laws and found that the implemented unitary commutes with the conserved observable if it is semibounded, while the conserved observable can shift up to a constant factor if the conserved observable is upper and lower unbounded. The former case in finite dimensions can be immediately derived from the more general trade-off relation [13], while the latter case is essentially infinite dimensional and cannot be expected from the finite-dimensional existing works.

Our work has several possible directions of future extensions. One such possibility is the generalization to the state-dependent scenario (e.g., energy-constrained states), while our results are restricted to the state-independent case. Another possible extension is to consider approximate implementations. This Letter concerns only the extreme case of *exact* implementations of projective measurements or unitary channels. On the other hand, as mentioned in the Introduction, results on approximate implementations of measurements and unitary gates have been actively studied in recent years. With few exceptions, these have not been extended to infinite-dimensional systems. (See the brief review in Supplemental Material [33].) It is an interesting future direction to extend these results to unbounded observables.

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