Beyond i.i.d. in the Resource Theory of Asymmetry: An Information-Spectrum Approach for Quantum Fisher Information

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Energetic coherence is indispensable for various operations, including precise measurement of time and acceleration of quantum manipulations. Since energetic coherence is fragile, it is essential to understand the limits in distillation and dilution to restore damage. The resource theory of asymmetry (RTA) provides a rigorous framework to investigate energetic coherence as a resource to break time-translation symmetry. Recently, in the independent and identically distributed (i.i.d.) regime where identical copies of a state are converted into identical copies of another state, it was shown that the convertibility of energetic coherence is governed by a standard measure of energetic coherence, called the quantum Fisher information (QFI). This fact means that QFI in the theory of energetic coherence takes the place of entropy in thermodynamics and entanglement entropy in entanglement theory. However, distillation and dilution in realistic situations take place in regimes beyond i.i.d., where quantum states often have complex correlations. Unlike entanglement theory, the conversion theory of energetic coherence in pure states in the non-i.i.d. regime has been an open problem. In this Letter, we solve this problem by introducing a new technique: an information-spectrum method for QFI. Two fundamental quantities, coherence cost and distillable coherence, are shown to be equal to the spectral QFI rates for arbitrary sequences of pure states. As a consequence, we find that both entanglement theory and RTA in the non-i.i.d. regime are understood in the information-spectrum method, while they are based on different quantities, i.e., entropy and QFI, respectively.

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Introduction.—In quantum mechanics, different states can be superposed. Of vital importance is the superposition between different energy eigenstates, called energetic coherence. Energetic coherence is mandatory for creating accurate clocks [1–6], accelerating quantum operations [7], and measuring physical quantities noncommutative with conserved quantities [8–12]. Recently, it was shown that energetic coherence also plays important roles for gate implementation in quantum computation [13–17], quantum measurements [12,17], quantum error correction [16–21], and black hole physics [16,17].

Energetic coherence and other fundamental properties of quantum systems are better understood by treating them as resources for quantum tasks. Quantum resource theories (QRTs) provide a versatile framework for analyzing seemingly unrelated resources with different origins, including entanglement [22], athermality [23,24], and energetic coherence [5,25–27]. Unexpected similarities arise in different branches of QRTs [28], leading to a unified understanding of the underlying laws. Since valuable resources are often fragile, it is fundamental to develop a theoretical understanding of the distillation and dilution of resources to restore their damage. Here, distillation is the operation of extracting as much resource as possible from a given state, and dilution is the opposite (Fig. 1).

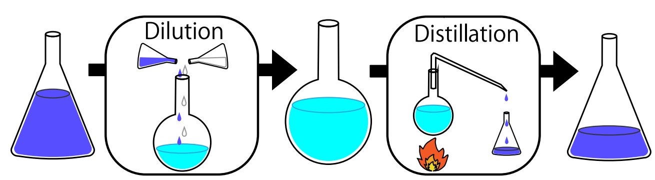

FIG. 1. Schematic of dilution and distillation. In dilution, a given state or a given sequence of states (depicted as light blue liquid) is generated by consuming as little resource (depicted as dark blue liquid) as possible. In distillation, as much resource as possible is extracted from a given state or a given sequence of states.

Revealing the limits of distillation and dilution of energetic coherence is of great importance when assembling multiple inaccurate clocks into an accurate clock [5]. These limits have been studied [5,27] in the resource theory of asymmetry (RTA), a branch of QRTs that analyzes symmetries and conservation laws [12,14–16,25–27,29–32]. In the independent and identically distributed (i.i.d.) regime where identical copies of a pure state are converted to identical copies of another pure state, the conversion rate is shown to be given by the ratio of their quantum Fisher information (QFI), a quantifier of energetic coherence [27]. In other words, QFI is in the position of entropy in the second law of thermodynamics. The same thermodynamic structure is known to exist in the i.i.d. regime for entanglement entropy in entanglement theory [33].

Toward practical applications, it is essential to extend the conversion theory in the i.i.d. regime to the non-i.i.d. setting because a realistic resource often has complex correlations while an i.i.d. resource state has no correlation. In entanglement theory [34,35] and quantum thermodynamics [36–38], conversion theories in the non-i.i.d. regime have been established. However, the counterpart in RTA remains elusive.

The obstacle to analyzing the non-i.i.d. regime in RTA is the limitation of the traditional information-spectrum method. This method gives a universal way of dealing with entropy-related problems for general states with arbitrary correlations in classical and quantum information theory, e.g., source coding, channel coding, and hypothesis testing [39–43]. Furthermore, entanglement theory and quantum thermodynamics in the non-i.i.d. regime [34–38,44] are established with the information-spectrum method since they are based on entropy. However, a central measure in converting energetic coherence is QFI, which is quite different from entropy. Therefore, the non-i.i.d. theory in RTA has been out of the scope of the information-spectrum method.

In this Letter, we establish the conversion theory of energetic coherence in non-i.i.d. pure states by constructing an information-spectrum approach for QFI. The key ingredients we introduce here are the following: the spectral supremum (sup-) and infimum (inf-)QFI rates, the maximum (max-) and minimum (min-)QFI, and asymmetric majorization. All of them clarify the correspondence in the conversion theories of entanglement and energetic coherence in the non-i.i.d. regime, which are characterized by entropies and QFIs, respectively. First, we prove a general formula for the coherence cost and the distillable coherence, i.e., the optimal conversion rates of a sequence of arbitrary pure states from and to a reference state. Concretely, they are shown to be equal to the spectral sup- and inf-QFI rates, respectively. This result corresponds to the general formula in entanglement theory [34,35], asserting that the entanglement cost and the distillable entanglement are equal to the spectral sup- and inf-entropy rates. Second, these spectral QFI rates are constructed as asymptotic rates of the smooth max- and min-QFIs. Their construction is parallel to that of spectral entropy rates, given as the asymptotic rates of the smooth max- and min-entropies with the smoothing technique [45,46]. Third, the asymmetric majorization relation between energy distribution is shown to provide a necessary and sufficient condition for the exact convertibility among pure states in RTA. This result is the counterpart in RTA to Nielsen's theorem [47], which characterizes the pure-state convertibility in entanglement theory by the majorization relation of the Schmidt coefficients.

Our findings highlight a clear correspondence in non-i.i.d. resource conversion in entanglement theory and RTA. See Figs. 2 and 3. Although they treat quite different resources, i.e., entanglement and energetic coherence, both are understood within a unified framework of the information-spectrum method for each resource.

Resource theory of asymmetry.—This Letter aims to construct a general theory of manipulating energetic coherence. To this end, we begin by identifying states with and without energetic coherence. Consider a quantum system S and its Hamiltonian H. Energetic coherence means superposition between eigenstates of H with different eigenvalues. Thus, a state ρ has energetic coherence if and only if the time evolution $e^{-\mathrm{i}Ht}$ changes it. Conversely, a state without energetic coherence is symmetric under time evolution, i.e., $e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}=\rho$ for any $t\in\mathbb{R}$. From these facts, we call a state without energetic coherence *symmetric* and a state with energetic coherence *asymmetric*. By definition, a state ρ is symmetric if and only if $[\rho, H]=0$.

We next consider transformations of states to manipulate energetic coherence. A basic element is an operation which does not create energetic coherence in the sense that it transforms a symmetric state to a symmetric state. This condition is satisfied if the operation is described by a completely positive trace-preserving map $\mathcal{E}$ satisfying [59]

$$\mathcal{E}(e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}) = e^{-\mathrm{i}Ht}\mathcal{E}(\rho)e^{\mathrm{i}Ht}, \quad \forall\ \rho, \quad \forall\ t\in\mathbb{R}. \tag{1}$$

A channel $\mathcal{E}$ satisfying Eq. (1) is called covariant (under time evolution $e^{-\mathrm{i}Ht}$).

Based on these ideas, RTA is constructed as a resource theory of energetic coherence. The framework of a resource theory is determined by defining "free states" that can be freely prepared and "free operations" that can be freely performed. In RTA, symmetric states are free states, and covariant operations are free operations. With these definitions, energetic coherence in asymmetric states becomes a resource. This structure in RTA is the same as in entanglement theory, where entanglement becomes a resource by defining separable states and local operations and classical communication (LOCC) as free states and free operations.

By adopting the above resource-theoretic perspective, coherence is quantified by resource measures, which monotonically decrease under covariant operations. A well-known and important one is the symmetric logarithmic derivative Fisher information [60,61] with respect to $\{e^{-iHt}\rho e^{iHt}\}_t$, given by

$$\mathcal{F}(\rho, H) := 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle i|H|j\rangle|^2, \tag{2}$$

where $\rho = \sum_i \lambda_i |i\rangle\langle i|$ is the eigenvalue decomposition. See, e.g., [32,62,63] for details and its generalization. Hereafter, we call this quantity quantum Fisher information, simply written as $\mathcal{F}(\rho)$. For a pure state, QFI equals four times the variance of H [64].

Following the standard argument [27], we hereafter analyze a system with a Hamiltonian

$$H = \sum_{n=0}^{\infty} n|n\rangle\langle n|, \tag{3}$$

where $\{|n\rangle\}$ denotes an orthogonal basis. With the method in Ref. [27], pure-state conversion theory in this system can be extended to a more general setup in RTA with arbitrary Hamiltonians [48].

An essential characteristic of a pure state $\psi = |\psi\rangle\langle\psi|$ in the manipulation of energetic coherence is its energy distribution $p_\psi = \{p_\psi(n)\}_{n=0}^{\infty}$, where $p_\psi(n) := |\langle n|\psi\rangle|^2$. This is because any pure state $|\psi\rangle$ can be mapped to $\sum_{n=0}^{\infty} \sqrt{p_\psi(n)}|n\rangle$ by an energy-conserving unitary operation, which is covariant and invertible. In fact, necessary and sufficient conditions for the exact convertibility between pure states have been obtained in terms of the energy distributions [25,30,67].

From a practical viewpoint, the exact conversion is typically impossible and too restrictive. Therefore, it is common to explore the convertibility with vanishing error in the asymptotic regime. We adopt the trace distance $D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$ as a quantifier of error, where $\|A\|_1 := \mathrm{Tr}(\sqrt{A^\dagger A})$. For $\epsilon \in (0, 1]$, we denote $\rho \approx_\epsilon \sigma$ if and only if two states ρ and σ satisfy $D(\rho, \sigma) \leq \epsilon$. For two sequences of states $\hat{\rho} = \{\rho_m\}_m$ and $\hat{\sigma} = \{\sigma_m\}_m$, we denote $\hat{\rho} \overset{\mathrm{cov}}{\succ}_\epsilon \hat{\sigma}$ if and only if there exists a sequence of covariant channels $\{\mathcal{E}_m\}_m$ such that $\mathcal{E}_m(\rho_m) \approx_\epsilon \sigma_m$ for all sufficiently large m. If $\hat{\rho} \overset{\mathrm{cov}}{\succ}_\epsilon \hat{\sigma}$ holds for all $\epsilon \in (0, 1]$, we say $\hat{\rho}$ is asymptotically convertible to $\hat{\sigma}$ and denote $\hat{\rho} \overset{\mathrm{cov}}{\succ} \hat{\sigma}$. For simplicity, we only analyze systems with Hamiltonian given by Eq. (3). Our main theorem on the pure-state conversion (Theorem 1) for this setup can be extended to a more general setup with arbitrary Hamiltonians. Of course, this includes the i.i.d. case, where a Hamiltonian is given by a sum of copies of a free Hamiltonian of a subsystem. See the Supplemental Material [48] for a general formula.

In the analysis of asymptotic convertibility, we adopt a coherence bit, i.e., a qubit with Hamiltonian $|1\rangle\langle 1|$ in a state $|\phi_{\mathrm{coh}}\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$ as a reference. There are two fundamental resource measures: the coherence cost and the distillable coherence. They are defined as the optimal rates for converting a sequence of states from and to coherence bits, i.e.,

$$C_{\mathrm{cost}}(\hat{\rho}) := \inf\left\{R | \widehat{\phi_{\mathrm{coh}}}(R) \overset{\mathrm{cov}}{\succ} \hat{\rho}\right\}, \tag{4}$$

$$C_{\mathrm{dist}}(\hat{\rho}) := \sup\left\{R | \hat{\rho} \overset{\mathrm{cov}}{\succ} \widehat{\phi_{\mathrm{coh}}}(R)\right\}, \tag{5}$$

where $\widehat{\phi_{\mathrm{coh}}}(R) := \{\phi_{\mathrm{coh}}^{\otimes[Rm]}\}_m$ for $R > 0$ and $\phi_{\mathrm{coh}} := |\phi_{\mathrm{coh}}\rangle\langle\phi_{\mathrm{coh}}|$. Note that the infimum of the empty set is formally defined as $+\infty$.

Finally, we introduce several notations for later convenience. For $a = \{a(n)\}_{n\in\mathbb{Z}}$, we denote $a \geq 0$ if and only if $a(n) \geq 0$ for all $n \in \mathbb{Z}$. A product sequence $a * b$ is defined by $a * b(n) := \sum_{k\in\mathbb{Z}} a(k)b(n-k)$. Similarly, we define $[a * b]_s^t(n) := \sum_{k=s}^{t} a(k)b(n-k)$.

For a given sequence $q = \{q(n)\}_{n\in\mathbb{Z}}$, another sequence $\tilde{q} = \{\tilde{q}(n)\}_{n\in\mathbb{Z}}$ satisfying

$$\delta_{0,n} = \tilde{q} * q(n) \tag{6}$$

plays a central role in our analysis. Here, $\delta_{m,n}$ is the Kronecker delta. If there exists a finite $n_\star := \min\{n | q(n) > 0\}$, such a sequence is constructed as

$$\tilde{q}(n) := \begin{cases} 0 & (n < -n_\star) \\ \frac{1}{q(n_\star)} & (n = -n_\star) \\ -\frac{1}{q(n_\star)}[\tilde{q} * q]_{-n_\star}^{n-1}(n_\star + n) & (n > -n_\star) \end{cases}. \tag{7}$$

Note that $\tilde{q}(n)$ is defined recursively for $n > -n_\star$. If q is an energy distribution, $n_\star \geq 0$ exists. In this case, $\tilde{q}$ satisfies $\sum_n \tilde{q}(n) = 1$. However, it is not a probability distribution in general since it can contain negative elements. Such a sequence $\tilde{q}$ is utilized to define central quantifiers of our analysis, the max- and min-QFI, just below.

We also introduce a generalized Poisson distribution $\mathrm{P}_\lambda = \{\mathrm{P}_\lambda(n)\}_{n\in\mathbb{Z}}$ for $\lambda \in \mathbb{R}$, where $\mathrm{P}_\lambda(n) := e^{-\lambda}\lambda^n/n!$ for $n \geq 0$ and $\mathrm{P}_\lambda(n) := 0$ for $n < 0$. For $\lambda \geq 0$, P_λ is the ordinary Poission distribution. Although P_λ with negative λ is not a probability distribution, this notation is useful since $\tilde{\mathrm{P}}_\lambda = \mathrm{P}_{-\lambda}$ [48].

Main results.—Now, let us construct an information-spectrum theory for QFI and show our main results. We first introduce key quantities. For a pure state ψ, we define the max-QFI $\mathcal{F}_{\max}$ and the min-QFI $\mathcal{F}_{\min}$ by

$$\mathcal{F}_{\max}(\psi) := \inf\{4\lambda | \mathrm{P}_\lambda * \widetilde{p_\psi} \geq 0\}, \tag{8}$$

$$\mathcal{F}_{\min}(\psi) \coloneqq \sup\{4\lambda | p_\psi * \mathrm{P}_{-\lambda} \geq 0\}. \tag{9}$$

They quantify the amounts of coherence in ψ transformable from and to a state whose energy distribution follows a Poisson distribution [48]. For a general state ρ, we define the max-QFI by $\mathcal{F}_{\max}(\rho) \coloneqq \inf_{\Phi_\rho} \mathcal{F}_{\max}(\Phi_\rho)$, where the infimum is taken over the set of all purifications Φ_ρ of ρ and the Hamiltonians of the auxiliary system with integer eigenvalues. This notation is consistent with that for pure states [48].

The max- and min-QFI have similar properties to the max- and min-entropies in entanglement theory [48]. For example, they provide the upper and lower bounds for QFI:

$$\mathcal{F}_{\max}(\psi) \geq \mathcal{F}(\psi) \geq \mathcal{F}_{\min}(\psi). \tag{10}$$

For a general sequence of pure states $\hat{\psi} = \{\psi_m\}$, the spectral sup- and inf-QFI rates are defined as

$$\overline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to+0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\max}(\psi_m), \tag{11}$$

$$\underline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to+0} \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\min}(\psi_m), \tag{12}$$

where $\mathcal{F}^{\epsilon}_{\max}(\psi) \coloneqq \inf_{\rho\in B^\epsilon(\psi)} \mathcal{F}_{\max}(\rho)$ and $\mathcal{F}^{\epsilon}_{\min}(\psi) \coloneqq \sup_{\phi\in B^\epsilon_{\mathrm{pure}}(\psi)} \mathcal{F}_{\min}(\phi)$ are smooth max- and min-QFI. Here, we defined $B^\epsilon(\psi) \coloneqq \{\rho\colon \text{states} | D(\rho,\psi) \leq \epsilon\}$ and $B^\epsilon_{\mathrm{pure}}(\psi) \coloneqq \{\phi\colon \text{pure states} | D(\phi,\psi) \leq \epsilon\}$. Note that $\limsup_{m\to\infty}(1/m)\mathcal{F}^{\epsilon}_{\max}(\psi_m)$ and $\liminf_{m\to\infty}(1/m)\mathcal{F}^{\epsilon}_{\min}(\psi_m)$ monotonically increases and decreases as ϵ becomes smaller, and hence limit values $\overline{\mathcal{F}}(\psi)$ and $\underline{\mathcal{F}}(\psi)$ exist.

The main theorem of this Letter is the following:

Theorem 1.—For a general sequence of pure states $\hat{\psi} = \{\psi_m\}$, the coherence cost and the distillable coherence are equal to the spectral sup- and inf-QFI rates, respectively. That is

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}), \qquad C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi}). \tag{13}$$

As a corollary of Theorem 1, we immediately get [48]

$$\hat{\psi} \overset{\mathrm{cov}}{\succ} \hat{\phi} \Rightarrow \overline{\mathcal{F}}(\hat{\psi}) \geq \overline{\mathcal{F}}(\hat{\phi}), \qquad \underline{\mathcal{F}}(\hat{\psi}) \geq \underline{\mathcal{F}}(\hat{\phi}), \tag{14}$$

$$\underline{\mathcal{F}}(\hat{\psi}) > \overline{\mathcal{F}}(\hat{\phi}) \Rightarrow \hat{\psi} \overset{\mathrm{cov}}{\succ} \hat{\phi}. \tag{15}$$

Replacing $\overline{\mathcal{F}}$, $\underline{\mathcal{F}}$, and $\overset{\mathrm{cov}}{\succ}$ by $\overline{S}$, $\underline{S}$, and $\overset{\mathrm{LOCC}}{\succ}$, the same relations as Eqs. (14) and (15) hold in entanglement theory. Here, $\overline{S}$ and $\underline{S}$ denote the spectral sup- and inf-entropy rates, while $\hat{\psi} \overset{\mathrm{LOCC}}{\succ} \hat{\phi}$ means that $\hat{\psi}$ is asymptotically convertible to $\hat{\phi}$ by LOCC [48].

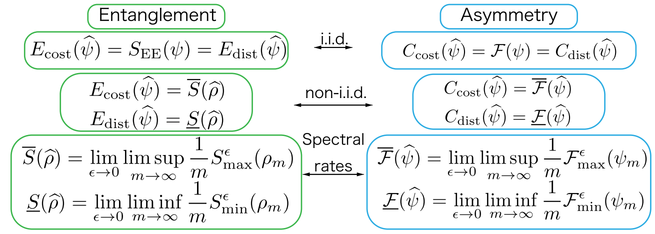

FIG. 2. Comparison of entanglement theory and RTA on the asymptotic convertibility [48]. The quantities E_{cost} and E_{dist} denote the entanglement cost and the distillable entanglement. For sequences of general bipartite pure states $\hat{\psi} = \{\psi_{AB,m}\}_m$, we define $\hat{\rho} = \{\rho_m\}_m$, where $\rho_m \coloneqq \mathrm{Tr}_B(\psi_{AB,m})$. The quantities $S_{\mathrm{EE}}(\psi)$, $S^{\epsilon}_{\max}(\rho)$, and $S^{\epsilon}_{\min}(\rho)$ denote entanglement entropy, the smooth max- and min-entropies.

Theorem 1 for a system with Hamiltonian in Eq. (3) can be extended to an arbitrary sequence of systems with any Hamiltonians in pure states having a finite period [48]. In particular, the spectral QFI rates $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$ are equal to QFI $\mathcal{F}$ in the i.i.d. setting [48], which reproduces the result in earlier i.i.d. studies [25,27]. We remark that $\overline{S}$ and $\underline{S}$ are equal to entanglement entropy in the i.i.d. regime in entanglement theory [48].

These results show that the spectral sup- and inf-QFI rates, $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$, in RTA play the same roles as the spectral sup- and inf-entropy rates, $\overline{S}$ and $\underline{S}$, in entanglement theory [48]. See Fig. 2. In other words, RTA in the non-i.i.d. regime has the same structure on convertibility as Lieb-Yngvason's nonequilibrium theory [68], based on QFI-related quantities rather than entropies.

Theorem 1 for the coherence cost is directly extended to general states, including mixed states. That is, defining $\overline{\mathcal{F}}(\hat{\rho}) \coloneqq \lim_{\epsilon\to+0}\limsup_{m\to\infty}(1/m)\mathcal{F}^{\epsilon}_{\max}(\rho_m)$, where $\mathcal{F}^{\epsilon}_{\max}(\rho) \coloneqq \inf_{\sigma\in B^\epsilon(\rho)} \mathcal{F}_{\max}(\sigma)$, the following holds [48]:

Theorem 2.—For a general sequence of states $\hat{\rho} = \{\rho_m\}$, it holds $C_{\mathrm{cost}}(\hat{\rho}) = \overline{\mathcal{F}}(\hat{\rho})$.

One-shot convertibility between pure states.—We here define a notion of asymmetric majorization, which we abbreviate a-majorization, as follows:

Definition 3.—For probability distributions $p = \{p(n)\}_{n\in\mathbb{Z}}$ and $q = \{q(n)\}_{n\in\mathbb{Z}}$, we say that p a-majorizes q if and only if $p * \tilde{q} \geq 0$ hold. In this case, we denote $p \succ_{\mathrm{a}} q$.

For comparison, we review the definition of majorization. A probability distribution $p = \{p(i)\}_{i=1}^d$ majorizes another probability distribution $q = \{q(i)\}_{i=1}^d$ if and only if $\sum_{l=1}^k p^{\downarrow}(l) \geq \sum_{l=1}^k q^{\downarrow}(l)$ for all $k = 1, \ldots, d$, where $\downarrow$ indicates that the distributions are rearranged in decreasing order so that $p^{\downarrow}(i) \geq p^{\downarrow}(j)$ and $q^{\downarrow}(i) \geq q^{\downarrow}(j)$ for $i > j$.

The a-majorization has properties similar to the ordinary majorization [48]. Among them, a significant one is the following:

Theorem 4.—A pure state ψ is convertible to a pure state ϕ by a covariant operation if and only if $p_\psi \succ_{\mathrm{a}} p_\phi$.

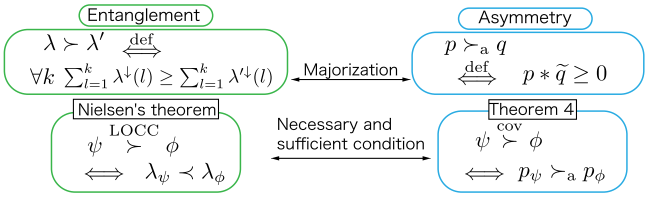

FIG. 3. Comparison of entanglement theory and RTA on the one-shot convertibility. If a pure state ψ is exactly convertible to another pure state ϕ by LOCC (respectively, covariant operations), we denote $\psi \overset{\mathrm{LOCC}}{\succ} \phi$ (respectively, $\psi \overset{\mathrm{cov}}{\succ} \phi$).

This is the counterpart in RTA to Nielsen's theorem in entanglement theory [47]: A bipartite pure state ψ is convertible to a bipartite pure state ϕ by LOCC if and only if $\lambda_\psi \prec \lambda_\phi$, where λ_ψ and λ_ϕ are the probability distributions given by the Schmidt coefficients of ψ and ϕ, respectively. This correspondence is the motivation for introducing the terminology of a-majorization. See Fig. 3.

We remark that other necessary and sufficient conditions on one-shot convertibility in RTA were proven in earlier studies [25,30,67]. Our contribution here is to provide the one-shot convertibility condition in terms of a-majorization to make it useful for our purpose to analyze the asymptotic convertibility in the non-i.i.d. regime. In particular, this reformulation makes the correspondence between RTA and entanglement theory clearer.

Proof of Theorem 1.—For a Poisson distribution P_λ, we denote $\chi_\lambda \coloneqq \sum_{n,n'=0}^{\infty} \sqrt{\mathrm{P}_\lambda(n)\mathrm{P}_\lambda(n')}|n\rangle\langle n'|$ and $\hat{\chi}_\lambda \coloneqq \{\chi_{\lambda m}\}_m$. This sequence is interconvertible with $\widehat{\phi_{\mathrm{coh}}}(R)$ by covariant operations, i.e., $\widehat{\phi_{\mathrm{coh}}}(R) \overset{\mathrm{cov}}{\succ} \hat{\chi}_{R/4}$ and $\hat{\chi}_{R/4} \overset{\mathrm{cov}}{\succ} \widehat{\phi_{\mathrm{coh}}}(R)$ [48].

The followings are key lemmas [48]:

Lemma 5.—Let $\mathcal{E}$ be a covariant channel. A state $\mathcal{E}(\chi_\lambda)$ has a purification Ψ such that $p_\Psi = \mathrm{P}_\lambda$, where the Hamiltonian of the ancilla added to purify $\mathcal{E}(\chi_\lambda)$ has integer eigenvalues.

Lemma 6.—Let ψ and ϕ be pure states. Assume that a covariant channel satisfies $\mathcal{E}(\psi) \approx_\epsilon \phi$. Then there exists a pure state ψ' such that $\psi' \in B_{\mathrm{pure}}^{2\epsilon^{1/4}}(\psi)$ and $p_{\psi'} \succ_{\mathrm{a}} p_\phi$.

To show $C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi})$, we introduce $C_{\mathrm{cost}}^{\epsilon}(\hat{\rho}) \coloneqq \inf\left\{R \middle| \widehat{\phi_{\mathrm{coh}}}(R) \overset{\mathrm{cov}}{\succ}_{\epsilon} \hat{\rho}\right\}$. Defining $4\lambda^{\epsilon/2} \coloneqq C_{\mathrm{cost}}^{\epsilon/2}(\hat{\psi})$, for any $\delta > 0$, there exists δ' such that $\delta > \delta' \geq 0$ and $\widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} + \delta')] \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \hat{\psi}$. Since $\hat{\chi}_{(\lambda^{\epsilon/2}+\delta')} \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} + \delta')]$, we have $\hat{\chi}_{(\lambda^{\epsilon/2}+\delta)} \overset{\mathrm{cov}}{\succ}_{\epsilon} \hat{\psi}$, where we have used the fact that $\mathrm{P}_\lambda \succ_{\mathrm{a}} \mathrm{P}_{\lambda'}$ holds for any $\lambda \geq \lambda'$ [48]. From Lemma 5, for all sufficiently large m, there exists a state $\rho_m \in B^{\epsilon}(\psi_m)$ whose purification Φ_m satisfies $\mathrm{P}_{(\lambda^{\epsilon/2}+\delta)m} = p_{\Phi_m}$. Therefore, $4(\lambda^{\epsilon/2} + \delta)m \geq \mathcal{F}_{\max}^{\epsilon}(\psi_m)$ for sufficiently large m, which implies

$$\forall\ \delta > 0, \qquad C_{\mathrm{cost}}^{\epsilon/2}(\hat{\psi}) + 4\delta \geq \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon}(\psi_m). \quad (16)$$

As $\epsilon \to +0$, we get $C_{\mathrm{cost}}(\hat{\psi}) \geq \overline{\mathcal{F}}(\hat{\psi})$.

To show the opposite inequality, we define $4\lambda_m^{\epsilon/2} \coloneqq \mathcal{F}_{\max}^{\epsilon/2}(\psi_m)$. For any $\delta > 0$, there exist δ'_m, satisfying $\delta > \delta'_m \geq 0$, such that there exist a state $\rho_m \in B^{\epsilon/2}(\psi_m)$ and its purification Φ_m satisfying $\mathrm{P}_{(\lambda_m^{\epsilon/2}+\delta'_m)} \succ_{\mathrm{a}} p_{\Phi_m}$. Note that for all sufficiently large m, $m(\lambda^{\epsilon/2} + \delta) \geq \lambda_m^{\epsilon/2}$ holds for $\lambda^{\epsilon/2} \coloneqq \limsup_{m\to\infty}(1/m)\lambda_m^{\epsilon/2}$. Therefore, we get $\mathrm{P}_{(\lambda^{\epsilon/2}+2\delta)m} \succ_{\mathrm{a}} p_{\Phi_m}$, where we have used $m\delta > \delta'_m$. Since $\widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} + 2\delta)] \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \hat{\chi}_{(\lambda^{\epsilon/2}+2\delta)}$, we have $\widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} + 2\delta)] \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \{\Phi_m\}_m$. Since the partial trace is a covariant operation, we have $\widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} + 2\delta)] \overset{\mathrm{cov}}{\succ}_{\epsilon} \hat{\psi}$. Therefore,

$$\forall\ \delta > 0, \qquad C_{\mathrm{cost}}^{\epsilon}(\hat{\psi}) \leq \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon/2}(\psi_m) + 8\delta. \quad (17)$$

As $\epsilon \to +0$, we get $C_{\mathrm{cost}}(\hat{\psi}) \leq \overline{\mathcal{F}}(\hat{\psi})$. Therefore, $C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi})$.

To show $C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi})$, we introduce $C_{\mathrm{dist}}^{\epsilon}(\hat{\psi}) \coloneqq \sup\{R | \hat{\psi} \overset{\mathrm{cov}}{\succ}_{\epsilon} \widehat{\phi_{\mathrm{coh}}}(R)\}$. Defining $4\lambda^{\epsilon/2} \coloneqq C_{\mathrm{dist}}^{\epsilon/2}(\hat{\psi})$, for any $\delta > 0$, there exists δ' such that $\delta > \delta' \geq 0$ and $\hat{\psi} \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} - \delta')]$. Since $\widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} - \delta')] \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \hat{\chi}_{\lambda^{\epsilon/2}-\delta'}$, we get $\hat{\psi} \overset{\mathrm{cov}}{\succ}_{\epsilon} \hat{\chi}_{\lambda^{\epsilon/2}-\delta'}$. From Lemma 6, for all sufficiently large m, there exist pure states $\psi'_m \in B_{\mathrm{pure}}^{2\epsilon^{1/4}}(\psi_m)$ such that $\psi'_m \succ_{\mathrm{a}} \mathrm{P}_{(\lambda^{\epsilon/2}-\delta)m}$, where we used $\delta > \delta'$. Therefore,

$$\forall\ \delta > 0, \qquad C_{\mathrm{dist}}^{\epsilon/2}(\hat{\psi}) - 4\delta \leq \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}_{\min}^{2\epsilon^{1/4}}(\hat{\psi}). \quad (18)$$

As $\epsilon \to +0$, we get $C_{\mathrm{dist}}(\hat{\psi}) \leq \underline{\mathcal{F}}(\hat{\psi})$.

To show the opposite inequality, we define $4\lambda_m^{\epsilon/2} \coloneqq \mathcal{F}_{\min}(\psi_m)$. For any $\delta > 0$, there exists δ'_m, satisfying $\delta > \delta'_m \geq 0$, such that there exists a pure state $\psi'_m \in B_{\mathrm{pure}}^{\epsilon/2}(\psi_m)$ satisfying $p_{\psi'_m} \succ_{\mathrm{a}} \mathrm{P}_{\lambda_m^{\epsilon/2}-\delta'_m}$. For all sufficiently large m, $\lambda_m^{\epsilon/2} \geq m(\lambda^{\epsilon/2} - \delta)$, where $\lambda^{\epsilon/2} \coloneqq \liminf_{m\to\infty}(1/m)\lambda_m^{\epsilon/2}$. Since $m\delta > \delta'_m$, we have $p_{\psi'_m} \succ_{\mathrm{a}} \mathrm{P}_{(\lambda^{\epsilon/2}-2\delta)m}$. By using $\hat{\chi}_{\lambda^{\epsilon/2}-2\delta} \overset{\mathrm{cov}}{\succ}_{\epsilon/2} \widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} - 2\delta)]$ and $\psi'_m \in B^{\epsilon/2}(\psi_m)$, we get $\hat{\psi} \overset{\mathrm{cov}}{\succ}_{\epsilon} \widehat{\phi_{\mathrm{coh}}}[4(\lambda^{\epsilon/2} - 2\delta)]$, which implies

$$\forall\ \delta > 0, \qquad C_{\mathrm{dist}}^{\epsilon}(\hat{\psi}) \geq \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}_{\min}^{\epsilon/2}(\psi_m) - 8\delta. \quad (19)$$

As $\epsilon \to +0$, we get $C_{\mathrm{dist}}(\hat{\psi}) \geq \underline{\mathcal{F}}(\hat{\psi})$. Therefore, $C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi})$.

Conclusion and discussions.—In this Letter, we established the pure-state conversion theory in RTA in the asymptotic non-i.i.d. regime. Unlike entanglement theory, the traditional information-spectrum method for entropy cannot be applied to RTA since its standard measure, QFI, is quite different from entropy. To overcome this issue, we constructed an information-spectrum approach for QFI by carefully analyzing the correspondence between RTA and entanglement theory. It opens the possibility of exploring a unified understanding of asymptotic conversion theory in each branch of quantum resource theories by extending the information-spectrum method for its resource measure. Such an extension may trigger research that has been out of the scope of the information-spectrum method. We speculate that the information-spectrum approach for QFI can be helpful in research areas where QFI plays an essential role, such as in non-equilibrium thermodynamics [69] and general resource theories [70].

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