Evolution Operator Can Always Be Separated into the Product of Holonomy and Dynamic Operators

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The geometric phase is a fundamental quantity characterizing the holonomic feature of quantum systems. It is well known that the evolution operator of a quantum system undergoing a cyclic evolution can be simply written as the product of holonomic and dynamical components for the three special cases concerning the Berry phase, adiabatic non-Abelian geometric phase, and nonadiabatic Abelian geometric phase. However, for the most general case concerning the nonadiabatic non-Abelian geometric phase, how to separate the evolution operator into holonomic and dynamical components is a long-standing open problem. In this Letter, we solve this open problem. We show that the evolution operator of a quantum system can always be separated into the product of holonomy and dynamic operators. Based on it, we further derive a matrix representation of this separation formula for cyclic evolution, and give a necessary and sufficient condition for a general evolution being purely holonomic. Our finding is not only of theoretical interest itself, but also of vital importance for the application of quantum holonomy. It unifies the representations of all four types of evolution concerning the adiabatic/nonadiabatic Abelian/non-Abelian geometric phase, and provides a general approach to realizing purely holonomic evolution.

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Introduction.—Nature is replete with phenomena where a quantity fails to return to its original value although the driving parameters undergo a cyclic evolution. Holonomy is used to characterize the geometrical essence of such phenomenon, in which the value difference remains even if the local rate of change is always zero [1]. This kind of holonomic effect plays crucial roles in various fields of physics [2], including, for example: in mechanics the change of the swing plane of a Foucault pendulum after one rotation of the Earth, in optics the change in the direction of linear polarization of light along a coiled optical fiber, and in general relativity the change of reference frames around a closed loop in spacetime.

In the quantum regime, despite of earlier studies in specific systems [3–7], the Berry phase found in 1984 [8] is often considered as the seminal theory on the quantum holonomic phenomenon. This finding represents a special case of quantum holonomy, the adiabatic Abelian geometric phase. The generalization of the Berry phase to adiabatic non-Abelian geometric phase [9], nonadiabatic Abelian geometric phase [10], and nonadiabatic non-Abelian geometric phase [11] were soon established. The holonomic nature of geometric phases is of broad importance in various research fields, such as in condensed matter physics [12], in quantum chemistry [13], in quantum field theory [14,15], in quantum gravity [16,17], and in quantum information [18]. Moreover, this kind holonomic nature has also been experimentally observed and manipulated in a variety of physical platforms, such as in superconductors [19,20], in nitrogen-vacancy centers [21–23], in trapped ions [24], in molecular ensembles [25,26], and in photonic systems [27,28].

For a quantum system undergoing a cyclic evolution, the state difference, i.e., the evolution operator transforming the initial state to the final state, is not purely holonomic, in general, but it can be simply written as the product of the holonomic and dynamical components for the three special cases concerning the Berry phase, adiabatic non-Abelian geometric phase, and nonadiabatic Abelian geometric phase. However, the situation is different for the most general case concerning the nonadiabatic non-Abelian geometric phase. In all the previous works on this issue, the holonomic component is blended with the dynamical component, and the evolution operator cannot be separated into the product of them except for some special cases such that they commute with each other.

How to separate the evolution operator of a general quantum system into the product of holonomic and dynamical components has been a long-standing open problem ever since the discovery of nonadiabatic quantum holonomy. The difficulty comes from the noncommutativity of the holonomic and dynamical components, both of which are related to the time-ordered integral. Solving this problem is not only of theoretical interest itself, but also of vital importance for the application of quantum holonomy. For example, holonomy-based quantum computation and quantum control relay on the separation of holonomic and dynamical components, which ensures the possibility of

eliminating the dynamical component from cyclic evolution and hence taking full advantage of the holonomy against control errors [29–34].

In this Letter, we solve this open problem. We first show that the evolution operator of a quantum system can always be separated into the product of holonomy and dynamic operators. Based on it, we further derive a matrix representation of this separation formula for cyclic evolution, and give a necessary and sufficient condition for a general evolution being purely holonomic. Our finding unifies the representations of all four types of evolution concerning the adiabatic/nonadiabatic Abelian/non-Abelian geometric phase, and provides a general approach to realizing purely holonomic evolution.

Preliminaries.—We start by recalling the notion of quantum holonomy. Since Abelian geometric phase can be taken as a special case of non-Abelian geometric phase, our discussion focuses on non-Abelian geometric phase, to which we refer as quantum holonomy or holonomy for simplicity.

Quantum holonomy was first studied in adiabatic evolution as a generalization of the Berry phase to degenerate eigenstates [9]. It arises in the quantum system governed by a slowly changing Hamiltonian $H(\boldsymbol{s})$ with a degenerate eigenvalue $E(\boldsymbol{s})$ of order ℓ, where $\boldsymbol{s}=\boldsymbol{s}(t)$ is a set of time-dependent parameters with $\boldsymbol{s}(T)=\boldsymbol{s}(0)$. Without ambiguity, we will use $P(\boldsymbol{s})$ to denote both the subspace spanned by the ℓ degenerate eigenstates and the corresponding rank-ℓ orthogonal projector. Then, $P(\boldsymbol{s})=|\phi_i(\boldsymbol{s})\rangle\langle\phi_i(\boldsymbol{s})| \coloneqq \sum_{i=1}^{\ell}|\phi_i(\boldsymbol{s})\rangle\langle\phi_i(\boldsymbol{s})|$, where $\{|\phi_i(\boldsymbol{s})\rangle\}_{i=1}^{\ell}$ is an arbitrary orthonormal basis of the ℓ-degenerate eigenspace. Here and in the following, the Einstein summation convention is employed, i.e., repeated indices are implicitly summed over from 1 to ℓ. For a quantum system evolving adiabatically, any state initially in the subspace $P[\boldsymbol{s}(0)]$ will be in the subspace $P[\boldsymbol{s}(t)]$ at time t, and go back to the initial subspace at $t=T$ as $P[\boldsymbol{s}(T)]=P[\boldsymbol{s}(0)]$.

Let $|\psi_j(t)\rangle$ be the state of the quantum system at time t, which is initially in $|\psi_j(0)\rangle=|\phi_j(0)\rangle$. Then, $|\psi_j(t)\rangle$ admits the expression $|\psi_j(t)\rangle=U_{ij}(t)|\phi_i(\boldsymbol{s}(t))\rangle$, where $U(t)=[U_{ij}(t)]_{i,j=1}^{\ell}$ is the transformation matrix between the basis and the states, and $U(T)$ gives the evolution operator after a cyclic evolution. By substituting $|\psi_j(t)\rangle$ into the Schrödinger equation, one can obtain that $U(t)=\mathrm{e}^{-\mathrm{i}\int_0^t E(\boldsymbol{s}(\tau))d\tau}\Gamma[\boldsymbol{s}(t)]$ with $\Gamma(\boldsymbol{s})=\mathcal{P}\exp[\int \boldsymbol{A}(\boldsymbol{\varsigma})\cdot d\boldsymbol{\varsigma}]$, where $\mathcal{P}$ denotes the path ordering along the curve $\{\boldsymbol{s}(\tau):0\le\tau\le t\}$, and $\boldsymbol{A}(\boldsymbol{s})=[\boldsymbol{A}_{ij}(\boldsymbol{s})]_{i,j=1}^{\ell}$ with $\boldsymbol{A}_{ij}(\boldsymbol{s})=-\langle\phi_i(\boldsymbol{s})|\boldsymbol{\nabla}_{\boldsymbol{s}}|\phi_j(\boldsymbol{s})\rangle$. Thus, after a cyclic evolution with period T, the evolution operator acting on the initial subspace reads $U(T)=\mathrm{e}^{-\mathrm{i}\int_0^T E(\boldsymbol{s}(\tau))d\tau}\Gamma(T)$, where $\Gamma(T)=\mathcal{P}\exp[\oint \boldsymbol{A}(\boldsymbol{s})\cdot d\boldsymbol{s}]$ is the quantum holonomy in adiabatic evolution. $\Gamma(T)$ is gauge invariant in the sense that it depends only on the subspace $P(\boldsymbol{s})$ but not on the choice of the basis $\{|\phi_i(\boldsymbol{s})\rangle\}_{i=1}^{\ell}$, as long as $|\phi_i(T)\rangle=|\phi_i(0)\rangle=|\psi_i(0)\rangle$.

Quantum holonomy can also be generalized to the nonadiabatic case [11]. Consider a d-dimensional quantum system governed by Hamiltonian $H(t)$. If there exists a set of the orthonormal states $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ satisfying the Schrödinger equation $\mathrm{i}|\dot{\psi}_i(t)\rangle=H|\psi_i(t)\rangle$ and the cyclic evolution condition $|\psi_i(T)\rangle\langle\psi_i(T)|=|\psi_i(0)\rangle\langle\psi_i(0)|$, i.e., $\sum_{i=1}^{\ell}|\psi_i(T)\rangle\langle\psi_i(T)|=\sum_{i=1}^{\ell}|\psi_i(0)\rangle\langle\psi_i(0)|$, with T being the evolution period, then one can define the quantum holonomy for the subspace $P(t)$ spanned by $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ similarly to the adiabatic case,

$$\Gamma(T)=\mathcal{P}\exp\left[\int_0^T A(t)dt\right], \tag{1}$$

where $A(t)=[A_{ij}(t)]_{i,j=1}^{\ell}$ with

$$A_{ij}(t)=\langle\dot{\phi}_i(t)|\phi_j(t)\rangle \tag{2}$$

is an anti-Hermitian matrix. Here, $\{|\phi_i(t)\rangle\}_{i=1}^{\ell}$ satisfying $|\phi_i(T)\rangle=|\phi_i(0)\rangle=|\psi_i(0)\rangle$ is an arbitrary basis of the subspace $P(t)$, and the dot $\dot{(\cdot)}$ denotes the derivative with respect to t. Again, $\Gamma(T)$ does not depend on any special choice of the basis, being gauge invariant.

A critical problem existing in all the previous works on the nonadiabatic quantum holonomy is that the evolution operator acting on the subspace cannot be separated into the product of holonomic and dynamical components. To see this clearly, we recall the processing procedure in the literature (see, e.g., [11]). The state $|\psi_j(t)\rangle$ is expressed as

$$|\psi_j(t)\rangle=U_{ij}(t)|\phi_i(t)\rangle. \tag{3}$$

Still, $U(t)=[U_{ij}(t)]_{i,j=1}^{\ell}$ represents the transformation matrix between the basis and the states, and $U(T)$ gives the evolution operator after a cyclic evolution. Substituting $|\psi_j(t)\rangle$ into the Schrödinger equation gives the differential equation,

$$\frac{d}{dt}U(t)=[A(t)+K(t)]U(t), \tag{4}$$

where $K(t)=[K_{ij}(t)]_{i,j=1}^{\ell}$ with $K_{ij}(t)=-\mathrm{i}\langle\phi_i(t)|H(t)|\phi_j(t)\rangle$. Equation (4) implies that $U(T)=\mathcal{T}\exp[\int_0^T[A(t)+K(t)]dt]$ with $\mathcal{T}$ denoting the time ordering. Clearly, although $U(T)$ can be written as the product of $\mathcal{T}\exp[\int_0^T A(t)dt]$ and $\mathcal{T}\exp[\int_0^T K(t)dt]$ in the special case that $[A(t_1),K(t_2)]=0$ for any t_1 and t_2, this separation is invalid in general. Therefore, one cannot simply separate the holonomic component $\Gamma(T)$ from the dynamical component by starting from the known differential equation (4). To realize the separation, we need to construct a new differential equation.

Main results.—With the above preliminaries, we can now present our results. We first construct a new differential equation of the evolution operator, based on which we derive a universal formula for separating the evolution operator into the product of holonomy and dynamic operators.

The evolution operator acting on the subspace spanned by $\{|\psi_i(0)\rangle\}_{i=1}^{\ell}$ plays the role that it transforms an arbitrary state in the initial subspace $P(0)$ to a corresponding state in the subspace $P(t)$ at time t. For example, if the quantum system is initially in the superposition $c_i|\psi_i(0)\rangle$, it will be in the state $c_i|\psi_i(t)\rangle$ at time t. Thus, the evolution operator acting on the subspace can be written as

$$\hat{U}(t) = |\psi_j(t)\rangle\langle\psi_j(0)|. \tag{5}$$

Here, we use the hat $(\hat{})$ to emphasize that $\hat{U}(t)$ is an operator, which should not be simply viewed as an $\ell \times \ell$ matrix. The evolution operator $\hat{U}(t)$ is different from the transformation matrix $U(t)$ defined in Eq. (3), in general, but coincides with $U(t)$ at time T.

Based on the expression (5), we can construct the following differential equation satisfied by the evolution operator [35]

$$\frac{d}{dt}\hat{U}(t) = \dot{P}(t)\hat{U}(t) + \hat{U}(t)\hat{F}(t), \tag{6}$$

where

$$\hat{F}(t) = F_{ij}(t)\big|\psi_i(0)\rangle\langle\psi_j(0)\big|, \tag{7}$$

with

$$F_{ij}(t) = -\langle\dot{\psi}_i(t)\big|\psi_j(t)\rangle = -\mathrm{i}\langle\psi_i(t)\big|H(t)|\psi_j(t)\rangle. \tag{8}$$

It is interesting to note that the form of Eq. (6) is different from Eq. (4) in the sense that $\dot{P}(t)$ and $\hat{F}(t)$ appear on different sides of $\hat{U}(t)$, while their counterparts $A(t)$ and $K(t)$ appear on the same side of $U(t)$. It is exactly this subtle difference that makes the separation possible. Mathematically, if $\dot{X}(t) = L(t)X(t)$ and $\dot{Y}(t) = Y(t)R(t)$, then $Z(t) = X(t)Y(t)$ satisfies that $\dot{Z}(t) = L(t)Z(t) + Z(t)R(t)$. Note also that $P(0)$ is the identity operator on the initial subspace, and so is $\hat{U}(0)$. Thus, Eq. (6) implies the following separation formula in operator form.

Theorem 1.—Let $P(t)$ be a subspace spanned by ℓ orthonormal states $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ of a quantum system with Hamiltonian $H(t)$, then the evolution operator $\hat{U}(t)$ acting on the subspace can always be separated into the product of the holonomy operator $\hat{\Gamma}(t)$ and dynamic operator $\hat{D}(t)$,

$$\hat{U}(t) = \hat{\Gamma}(t)\hat{D}(t), \tag{9}$$

with

$$\hat{\Gamma}(t) = \mathcal{P}\exp\left[\int_0^t \dot{P}(\tau)d\tau\right]P(0), \tag{10}$$

$$\hat{D}(t) = P(0)\bar{\mathcal{T}}\exp\left[\int_0^t \hat{F}(\tau)d\tau\right], \tag{11}$$

where $P(t) = |\psi_i(t)\rangle\langle\psi_i(t)|$, $\hat{F}(t) = F_{ij}(t)|\psi_i(0)\rangle\langle\psi_j(0)|$, $F_{ij}(t) = -\mathrm{i}\langle\psi_i(t)|H(t)|\psi_j(t)\rangle$, and $\mathcal{P}$ and $\bar{\mathcal{T}}$ are the path ordering and reverse time ordering, respectively.

Alternatively, one can express Eqs. (10),(11) in differential form

$$\frac{d}{dt}\hat{\Gamma}(t) = \dot{P}(t)\hat{\Gamma}(t), \qquad \frac{d}{dt}\hat{D}(t) = \hat{D}(t)\hat{F}(t), \tag{12}$$

with the initial conditions that $\hat{\Gamma}(0) = \hat{D}(0) = P(0)$. In Eq. (10), the time ordering $\mathcal{T}$ has been replaced by the path ordering $\mathcal{P}$, as $\hat{\Gamma}(t)$ is independent of the evolution details such as the evolution rate but only depends on the path of the ℓ-dimensional subspaces $\{P(\tau):0 \le \tau \le t\}$. Hence, $\hat{\Gamma}(t)$ is a holonomic component, and thus we call it the holonomy operator for the evolution. The other component $\hat{D}(t)$ depends on the dynamical details of the evolution, hence we call it the dynamic operator for the evolution. Besides, $\hat{F}(t)$ defined by Eq. (7) is an anti-Hermitian operator, i.e., $\hat{F}^{\dagger}(t) = -\hat{F}(t)$, and therefore the Hermitian conjugate of the dynamic operator can be written in the time ordering form as

$$\hat{D}^{\dagger}(t) = \mathcal{T}\exp\left[-\int_0^t \hat{F}(\tau)d\tau\right]P(0). \tag{13}$$

So far, we have proved that the evolution operator can always be separated into the product of holonomy and dynamic operators as in Theorem 1. In the following, we will apply the theorem to the cyclic evolution, and give the matrix representation of the separation formula.

To this end, we need to express operators $\hat{U}(T)$, $\hat{D}(T)$, and $\hat{\Gamma}(T)$ with their corresponding matrices. By using Eq. (3), the evolution operator $\hat{U}(t)$ can be written as $\hat{U}(t) = U_{ij}(t)|\phi_i(t)\rangle\langle\phi_j(0)|$, which coincides with the transformation matrix $U(t)$ at time T, i.e.,

$$\hat{U}(T) = U_{ij}(T)\big|\phi_i(0)\rangle\langle\phi_j(0)\big|. \tag{14}$$

Here, $\{|\phi_i(t)\rangle\}_{i=1}^{\ell}$ satisfying $|\phi_i(T)\rangle = |\phi_i(0)\rangle = |\psi_i(0)\rangle$ is still used to denote a basis of the subspace $P(t)$.

In analogy to Eq. (14), we can derive the matrix representation of the dynamic operator $\hat{D}(T)$ from its definition. Equations (7),(11) directly imply that

$$\hat{D}(t) = D_{ij}(t)\big|\phi_i(0)\rangle\langle\phi_j(0)\big| \tag{15}$$

with $D(t)=\bar{\mathcal{T}}\exp[\int_0^t F(\tau)d\tau]$, where $F(t)=[F_{ij}(t)]_{i,j=1}^{\ell}$ is an anti-Hermitian matrix defined by Eq. (8). Equation (15) gives $\hat{D}(T)=D_{ij}(T)|\phi_i(0)\rangle\langle\phi_j(0)|$ at the time T.

The remaining task is to show that $\hat{\Gamma}(T)$ coincides with the holonomy matrix $\Gamma(T)$ defined in Eq. (1) for cyclic evolution. For this, we first use $\hat{C}(t)$ to denote $\Gamma_{ij}(t)|\phi_i(t)\rangle\langle\phi_j(0)|$, where $\Gamma(t)=[\Gamma_{ij}(t)]_{i,j=1}^{\ell}=\mathcal{T}\exp[\int_0^t A(\tau)d\tau]$ with $A(t)=[\langle\dot{\phi}_i(t)|\phi_j(t)\rangle]_{i,j=1}^{\ell}$, and will then prove $\hat{\Gamma}(t)=\hat{C}(t)$ by demonstrating that they satisfy the same differential equation with the same initial condition. From the definitions of $\hat{C}(t)$ and $\Gamma(t)$, we have

$$\frac{d}{dt}\hat{C}(t)=\dot{\Gamma}_{ij}(t)\big|\phi_i(t)\big\rangle\langle\phi_j(0)\big|+\Gamma_{ij}(t)\big|\dot{\phi}_i(t)\big\rangle\langle\phi_j(0)\big| \quad (16)$$

with $\dot{\Gamma}_{ij}(t)=A_{ik}(t)\Gamma_{kj}(t)=\langle\dot{\phi}_i(t)|\phi_k(t)\rangle\Gamma_{kj}(t)$. Simple calculations show that the first term on the right-hand side of Eq. (16) reduces to $|\phi_i(t)\rangle\langle\dot{\phi}_i(t)|\hat{C}(t)$ and the second term reduces to $|\dot{\phi}_i(t)\rangle\langle\phi_i(t)|\hat{C}(t)$. Therefore, $\hat{C}(t)$ satisfies the differential equation $(d/dt)\hat{C}(t)=\dot{P}(t)\hat{C}(t)$ with the initial condition that $\hat{C}(0)=P(0)$, which is the same to $\hat{\Gamma}(t)$. Thus, we prove that $\hat{\Gamma}(t)=\hat{C}(t)$, i.e.,

$$\hat{\Gamma}(t)=\Gamma_{ij}(t)\big|\phi_i(t)\big\rangle\langle\phi_j(0)\big|, \quad (17)$$

which gives $\hat{\Gamma}(T)=\Gamma_{ij}(T)|\phi_i(0)\rangle\langle\phi_j(0)|$ at the time T.

With Eqs. (9), (14), (15), (17), we obtain the following separation formula for cyclic evolution in matrix form.

Theorem 2.—If the subspace $P(t)$ spanned by ℓ orthonormal states $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ evolves cyclically with period T, i.e., $P(T)=P(0)$, then the evolution operator acting on the subspace at time T has the matrix representation,

$$U(T)=\Gamma(T)D(T), \quad (18)$$

where $\Gamma(T)=\mathcal{P}\exp[\int_0^T A(t)dt]$ and $D(T)=\bar{\mathcal{T}}\exp[\int_0^T F(t)dt]$ are the holonomic and dynamical components, respectively.

Before proceeding further, we would like to add a few remarks. First, our finding unifies the representations of all four types of evolution concerning the adiabatic/nonadiabatic Abelian/non-Abelian geometric phase. In the adiabatic Abelian/non-Abelian case, $F_{ij}(t)=-\mathrm{i}E(t)\delta_{ij}$, and thus Eq. (18) reduces to $U(T)=\mathrm{e}^{-\mathrm{i}\int_0^T E(t)dt}\Gamma(T)$, which is just the well-known results of Berry [8] (when $\ell=1$) and Wilczek and Zee [9] (when $\ell\neq 1$). In the nonadiabatic Abelian case, i.e., $P(t)=|\psi(t)\rangle\langle\psi(t)|$, Eq. (18) reduces to the celebrated formula of Aharonov and Anandan [10], $\gamma(T)=\arg\langle\psi(0)|\psi(T)\rangle+\int_{t=0}^{T}\langle\psi(t)|H(t)|\psi(t)\rangle dt$. Second, Eqs. (9) and (18) provide separation formulas in the operator and matrix forms, respectively, but Eq. (9) is more general than Eq. (18). The operator form (9) holds for any time t, or equivalently, it also holds for noncyclic evolution, while the matrix form (18) holds only for cyclic evolution. Third, a fundamental difference between $\hat{\Gamma}(t)$ and $\Gamma(t)$ is that the gauge invariance holds at any time t for the former but only at time T for the latter. Moreover, the gauge-invariant quantity $\hat{\Gamma}(t)$ gives the operator form of the so-called parallel transport, i.e., it satisfies that $\hat{\Gamma}^{\dagger}(t)(d/dt)\hat{\Gamma}(t)=0$ [36].

Purely holonomic evolution.—A crucial issue for the application of quantum holonomy is to determine when a quantum evolution is purely holonomic. Explicitly, we call a cyclic evolution purely holonomic if $U(T)$ is equal to $\Gamma(T)$ up to a global phase, i.e., $U(T)=\mathrm{e}^{\mathrm{i}\alpha}\Gamma(T)$ for some real number α. We note that this is different from Abelian geometric phases, in the applications of which two or more paths are considered and thus the phases, or rather, the difference of the phases matters. From Theorem 2, one can directly obtain the following necessary and sufficient condition for purely holonomic evolution.

Corollary 3.—If the subspace $P(t)$ spanned by ℓ orthonormal states $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ evolves cyclically with period T, i.e., $P(T)=P(0)$, then the evolution is purely holonomic if and only if

$$D^{\dagger}(T)=\mathcal{T}\exp\left[-\int_0^T F(t)dt\right]=\mathrm{e}^{\mathrm{i}\alpha}\mathbb{1}_{\ell}, \quad (19)$$

where α is some real number, $\mathbb{1}_{\ell}$ is the $\ell\times\ell$ identity matrix, and $F(t)=[F_{ij}(t)]_{i,j=1}^{\ell}$ with $F_{ij}(t)=-\mathrm{i}\langle\psi_i(t)|H(t)|\psi_j(t)\rangle$.

Corollary 3 shows that for a quantum system, the evolution operator acting on the subspace $P(0)$ is purely holonomic if and only if the Hamiltonian governing the quantum satisfies Eq. (19). This provides a general approach for realizing purely holonomic evolution. Specially, this opens a new avenue for holonomic quantum computation and holonomic quantum control. Note that in the previous schemes, either the systems must be in adiabatic evolution [32] or satisfy the parallel transport condition [33,34], which are just corresponding to the special cases of Corollary 3 with $F(t)=-iE(t)\mathbb{1}_{\ell}$ and $F(t)=0$, respectively. Yet, our result shows that these requirements are unnecessary for purely holonomic evolution. This largely extends the applicability of holonomic quantum computation. In the Supplemental Material [37], we take a one-parameter Hamiltonian, which is widely used in holonomic quantum computation, as a concrete example to illustrate this point.

Besides, the use of Corollary 3 can be flexible. For a cyclic evolution $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ $(0\le t\le T_1)$ with the dynamical component $D(T_1)\propto\mathbb{1}_{\ell}$, it is possible to construct an adjacent cyclic evolution $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ $(T_1\le t\le T_1+T_2)$ such that $D^{\dagger}(T_1+T_2;T_1)\propto D(T_1)$, where $D^{\dagger}(T_1+T_2;T_1)=\mathcal{T}\exp[-\int_{T_1}^{T_1+T_2}F(t)dt]$. In this way,

the dynamical components cancel out and only the holonomic components remain. This strategy works because we can regard $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ $(0 \le t \le T_1 + T_2)$ as an overall cyclic evolution, whose dynamical component satisfies $D^\dagger(T_1+T_2) = D^\dagger(T_1+T_2;T_1)D^\dagger(T_1)$.

Conclusions.—We have shown that the evolution operator of a quantum system can always be separated into the product of holonomy and dynamic operators, of which the operator expression and the matrix representation are stated as Theorem 1 and Theorem 2, respectively. From the fundamental perspective, our finding solves a long-standing open problem in the study of quantum holonomy, and unifies the representations of all four types of evolution concerning the adiabatic/nonadiabatic Abelian/non-Abelian geometric phase. From the practical perspective, our separation formula provides a general approach for realizing purely holonomic evolution, which can find widespread applications in quantum information and quantum control. For example, our approach can largely extend the applicability of holonomic quantum computation.

For future research, it would be very interesting to apply our approach for implementing holonomic quantum computation and holonomic quantum control in actual quantum systems, both theoretically and experimentally. Furthermore, as quantum holonomy is a fundamental geometric quantity in quantum physics, our result may also lead to deeper understandings of the geometric phenomena in various fields. For example, as our approach holds for any quantum evolution, it opens a more flexible avenue for simulating the non-Abelian gauge field. This may shed light on the investigation of the factional quantum Hall effect [40], lattice gauge theory [41], topological field theory [14,15], and loop quantum gravity [16,17], in various quantum simulation platforms [42,43].

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