A No-Go Result for Implementing Chiral Symmetries by Locality-Preserving Unitaries in a Three-Dimensional Hamiltonian Lattice Model of Fermions

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We argue that the chiral $\mathrm{U}(1)_A$ symmetry of a Weyl fermion cannot be implemented by a shallow depth quantum circuit operation in a fermionic lattice Hamiltonian model with finite dimensional onsite Hilbert spaces. We also extend this result to discrete $\mathbb{Z}_{2N}$ subgroups of $\mathrm{U}(1)_A$, in which case we show that for N_f Weyl fermions of the same helicity, this group action cannot be implemented with shallow depth circuits when N_f is not an integer multiple of $2N$.

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Introduction.—Understanding how chiral symmetry of Weyl fermions can be implemented on the lattice is an important problem to multiple disciplines of physics. In the field of condensed matter theory, the well-known 't Hooft anomaly of the chiral $\mathrm{U}(1)_A$ symmetry is an archetypal example of the boundary of a symmetry protected topological (SPT) phase, which should preclude the possibility of realizing the $\mathrm{U}(1)_A$ as an exact on-site symmetry on the lattice. For the high energy theory community, realizing chiral fermions on the lattice is of both fundamental and practical importance, as the Standard Model and various versions of grand unified theories are chiral gauge theories. A celebrated no-go theorem by Nielsen and Ninomiya [1] explicitly concluded that the $\mathrm{U}(1)_A$ symmetry cannot be engineered in a local lattice model in three spatial dimensions for free fermions. Over the years this problem has been approached in several ways, for example by introducing an extra dimension [2], so that the chiral fermions and the $\mathrm{U}(1)_A$ symmetry are realized at the boundary of a higher dimensional system, consistent with the general spirit of realizing the anomalous chiral symmetries at the boundary of a bulk SPT phase. It was also realized that the axial rotation symmetry could be an emergent symmetry in the infrared, which is implemented as a nonlocal transformation on a four-dimensional Euclidean spacetime lattice and action [3–5].

In this Letter, we will approach this problem from the standpoint of the fermionic analog of the many-qubit model. That is, we will assume a many-body Hilbert space that is a graded tensor product of finite dimensional on-site fermionic Hilbert spaces, and investigate the action of chiral symmetry from a quantum information theoretic local unitary perspective. This perspective has proven useful in elucidating aspects of anomalies in other contexts 9–6]]. Specifically, we ask the following question. Suppose that a local Hamiltonian H acting in such a Hilbert space realizes in its low energy limit a single Weyl fermion [10]. Is it then possible to find an operator Q which is a sum of quasilocal terms such that $\exp(2\pi i Q) = 1$, $[Q, H] = 0$, and Q coincides with the usual U(1) particle number associated to the Weyl fermion at low energies? That is, can the chiral U(1) symmetry be implemented by a possibly non-on-site, but still quasilocal operator acting on this Hilbert space?

We argue that the answer to this question is no, for any number N_f of Weyl fermions of the same helicity. In particular, the $\mathrm{U}(1)_A$ axial symmetry of a Dirac fermion cannot be realized by a non-on-site operator of this form [11]. We note that in Ref. [12] a non-on-site free fermion operator Q implementing axial $\mathrm{U}(1)_A$ symmetry in such a setting was written down; however, this Q fails to satisfy $\exp(2\pi i Q) = 1$, i.e. it does not generate a compact U(1) group. There is in fact a simple topological argument which shows that any such free-fermion Q must have a vanishing eigenvalue at some point in the Brillouin zone, implying that in the thermodynamic limit excitations with arbitrarily small nonzero charges exist, a violation of charge quantization. In this work, we show that, more generally, an obstruction exists even at the level of interacting but still quasilocal Q. We also show that the conclusion still holds if only a $\mathbb{Z}_{2N} \subset \mathrm{U}(1)_A$ subgroup is preserved, assuming that N_f is not an integer multiple of N.

The significance of our result arises from the existence of a large class of 't Hooft anomalies for which the typically discrete symmetries *can* be realized by shallow depth circuits, but not on-site. This includes in particular the boundaries of all of the in-cohomology SPT phases [6]. Another example can be obtained by considering k Dirac fermions in $1+1$ dimensions, and restricting the symmetry group to $\mathbb{Z}_2 \times \mathbb{Z}_2^f$, where $\mathbb{Z}_2$ is the fermion parity of the left movers and $\mathbb{Z}_2^f$ the overall fermion parity. It is known that with interactions the anomaly in this case depends only on k mod 8 [13,14], and that for even k the symmetries can be realized by shallow depth circuits [15,16], but for odd k they cannot [17]. Thus, ‘t Hooft anomalies may be classified by severity according to whether or not the corresponding symmetries admit shallow depth circuit representations, where increasing severity corresponds to being further away from the cohomology classification. Now, for a continuous group like $U(1)_A$, the natural generalization of “acting by shallow depth circuits” is that the Hermitian generator of the $U(1)_A$ be a quasilocal operator. Thus, modulo this assumption, our result demonstrates that the $U(1)_A$ cannot act by shallow depth circuits and hence suffers a severe form of ‘t-Hooft anomaly.

Our result can be viewed as a partial interacting generalization of the Nielsen and Ninomiya theorem [1]. Our argument relies on certain physical assumptions, the most important one being that it is impossible to have a chiral edge for a $(2+1)d$ commuting projector Hamiltonian. The key part of the argument for the U(1) case is that a U(1) vortex line in the Majorana mass term carries $1d$ chiral modes, and a sufficiently local Q would allow one to build a shallow depth circuit that inserts such a vortex at the boundary of a $2d$ membrane. This circuit effectively builds a $2d$ chiral phase on this membrane from a trivial state, in particular giving it a commuting projector Hamiltonian, which is a contradiction.

Assumptions and statement of the result.—We assume the Hilbert space is a graded tensor product, over the sites of some lattice, of two-dimensional fermionic Hilbert spaces with one even and one odd dimensional subspace. We assume the existence of a fixed microscopic length scale l (setting the lattice constant to 1), that controls the locality of various operators as described below. We assume the existence of a quasilocal Hermitian $Q = \sum_j Q_j$ with $\exp(2\pi i Q) = 1$ and Q_j acting near lattice site j. We will not formalize the notion of quasilocal beyond the following. First, Q must satisfy the locality conditions of the Lieb-Robinson theorem [18,19], so that the unitary operators $\exp(i\tau Q)$, $0 \le \tau < 2\pi$ have a common Lieb-Robinson length [18] bounded by l. We will further consider operators of the form $\exp(i\tau \sum_j f_j Q_j)$, where $0 \le f_j \le 1$ is a slowly varying function on the lattice. We assume that these operators also have Lieb-Robinson length bounded by l. Finally, we require that for any local operator A_k supported on sites within distance d of k, the operator norm of the difference between $\exp(i\tau \sum_j f_j Q_j) A_k \exp(-i\tau \sum_j f_j Q_j)$ and $\exp(i\tau Q) A_k \exp(-i\tau Q)$ is bounded by $Cd|\nabla f| \cdot \|A_k\|$ where C is some constant, and $|\nabla f|$ the maximum gradient of f, in lattice units. This condition formalizes the idea that a slowly spatially varying U(1) rotation acting on a local operator can be approximated by a constant U(1) rotation acting on that local operator, with the quality of the approximation controlled by the gradient of the spatial variation.

We will assume the existence of a local Hamiltonian H, possibly interacting, with the locality of the terms bounded by l, whose effective low energy theory is described by a Weyl fermion

$$H_{\mathrm{Weyl}} = \hbar v \sum_{|\vec{k}|<\Lambda} \psi^\dagger(\vec{k})^T \left(\sum_{i=1}^{3} k_i \sigma_i \right) \psi(\vec{k}), \tag{1}$$

valid below a cutoff scale of $\hbar v / l$. This means that the field operator

$$\psi(\vec{r}) = N \sum_{|\vec{k}|<\Lambda} e^{i\vec{k}\cdot\vec{r}} \psi(\vec{k})$$

is quasilocal in the sense of being well approximated by an operator acting on a region of diameter l around $\vec{r}$. Furthermore, we will assume that the only low energy excitations are those of H_{Weyl}, even on spatial manifolds of nontrivial topology. This rules out the possibility of H allowing additional gapped topological quantum field theory excitations, and means in particular that when H is gapped out with a Majorana mass term below, the result is an invertible state.

Finally, we will assume that $[Q, H] = 0$ and at low energies Q generates the chiral U(1) particle number symmetry of H_{Weyl}. We will show that, taken together, these assumptions lead to a contradiction.

No quasilocal Q generating chiral $U(1)$.—We first compactify along one of the spatial directions, which we call z, and impose periodic boundary conditions along that direction, with length $l_z \gg l$. We will parametrize the z direction with a coordinate $z \in [0, l_z]$, with 0 and l_z identified.

We now construct a unitary operator V which smoothly interpolates between doing a π U(1) rotation on half of the space (in the z coordinate) and doing nothing on the other half, as illustrated in Fig. 1. Note that this is a π rotation on the fermions, so a 2π rotation of the order parameter $\Delta(z)$. To define V, we first pick w such that $l \ll w \ll l_z$, and let $f(z)$ be an indicator function for $[0, l_z/2]$ smoothed out on

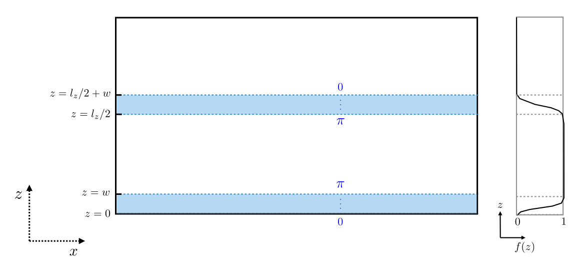

FIG. 1. Dimensionally reduced system, extended in the x direction and y direction (into the page). The operator V interpolates smoothly between a trivial operator for $l_z/2 + w < z < l_z$ and a π U(1) rotation for $w < z < l_z$ [acting as fermion parity $(-1)^F$]. The function $f(z)$ used to define Q_f in Eq. (2) shown on the right.

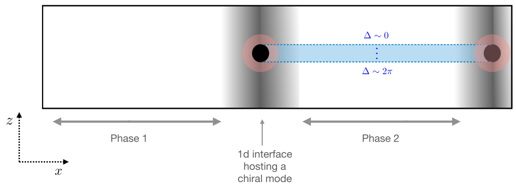

FIG. 2. A $1d$ spatial interface along the y direction (into the page) between two different $2d$ gapped phases in the x, y plane, at $x > 0$ and $x < 0$ respectively. A 2π vortex in the order parameter Δ extends along this interface, as can be seen by the fact that the phase of Δ winds by 2π when crossing the blue region.

the scale w; its defining properties are that it is equal to 1 for $w < z < l_z/2$, it is equal to 0 for z outside of $[0, l_z/2 + w]$, and it interpolates smoothly between those values in the intervening regions, with a maximum gradient of order $w^{-1} \ll l^{-1}$. We then define

$$Q_f = \sum_i f(z_i) Q_i \tag{2}$$

where for each i, z_i is a point in the support of Q_i, and define $V = \exp(\mathrm{i}\pi Q_f)$.

V can be viewed as a finite time evolution of a bounded local pseudo-Hamiltonian and has Lieb-Robinson length bounded by l. Hence, after possibly increasing w by an amount of order l, we can see that $\tilde{V}$ acts as the fermion parity operation $(-1)^F$ on all operators supported on the region $w < z < l_z/2$, and as the identity on all operators supported on the region $l_z/2 + w < z < l_z$.

We now perturb the Hamiltonian in our dimensionally reduced geometry as follows. We first define

$$H_\Delta = H + \int d\vec{r} \Delta \psi_1(\vec{r}) \psi_2(\vec{r}) + \text{H.c.}$$

We choose $|\Delta|$ sufficiently small so that the effect of the perturbation can be analyzed within the low energy free fermion theory, where it is just a Majorana mass term that gaps out the Weyl fermion Hamiltonian of Eq. (1). Note that this Majorana mass term induces a correlation length $\xi \sim \Delta^{-1}$, which we can assume to satisfy $l \ll \xi \ll w$. As mentioned above, this gapped state must be invertible. Using the fact that the conjectured classification of $(3+1)d$ invertible fermionic states is trivial, or by stacking with the conjugate invertible state, we can assume that this state is trivial, i.e. connected by a shallow depth circuit to a product state. This means, in particular, that it has another parent Hamiltonian H'_Δ which is of local commuting projector form (obtained by conjugating the trivial commuting projector for the product state by this circuit).

Now, for $x < 0$ we simply define $H_{x<0} = H_\Delta$. We also define $H'_{x<0} = H'_\Delta$ to be the commuting projector parent Hamiltonian with the same ground state. For $x > 0$, we define $H_{x>0}$ as follows. For z away from the interval $[l_z/2, l_z/2 + w]$, we take the local terms in $H_{x>0}$ to be identical to those of H_Δ. For z in the interval $[l_z/2 - \epsilon, l_z/2 + w + \epsilon]$, we take the local terms to be those of $V H_\Delta V^{-1}$, where ϵ is some length scale with $l \ll \epsilon \ll l_z$. This is a thickening of the interval $[l_z/2, l_z/2 + w]$, but since V acts as either the identity or fermion parity in the thickened regions, and all terms in H_Δ are fermion parity even, and hence not affected. Carrying out the same procedure with H'_Δ yields a commuting projector Hamiltonian with the same ground state.

Let us now analyze $H_{x>0}$ in the low energy field theory. Away from the range $l_z/2 < z < l_z/2 + w$ it is just H_Δ, while for z in this range its terms are given by conjugating those of H_Δ by the slowly spatially varying U(1) rotation v. Hence, these terms may be approximated by the action of a uniform U(1) rotation. More precisely, for any such local operator A of diameter d localized at some z coordinate z_A, we have by assumption that

$$\|\tilde{V} A \tilde{V}^{-1} - \exp[\mathrm{i} f(z_A) Q] A \exp[-\mathrm{i} f(z_A) Q]\| < \frac{Cd}{w} \|A\|. \tag{3}$$

Hence

$$H_{x>0} = H_{\text{Weyl}} + H_{\text{twisted}} + O(1/w)$$

where

$$H_{\text{twisted}} = \int d\vec{r} \Delta(z) \psi_1(\vec{r}) \psi_2(\vec{r}) + \text{H.c.}$$

with $\Delta(z)$ being a complex number of amplitude Δ whose phase winds by 2π as for $z \in [l_z/2, l_z/2 + w]$. Since the ground states of both $H_{x>0}$ and $H_{\text{Weyl}} + H_{\text{twisted}}$ are gapped, for sufficiently large w the extra term of order $(1/w)$ above will have no effect, so that $H_{x>0}$ can be continuously connected to $H_{\text{Weyl}} + H_{\text{twisted}}$ without closing the gap.

But the situation with $H_{\text{Weyl}} + H_{\text{Majorana}}$ on one side of the interface and $H_{\text{Weyl}} + H_{\text{twisted}}$ is precisely a fundamental 2π vortex in the order parameter of the Majorana mass term, located at the effectively one dimensional interface $x = 0$, as illustrated in Fig. 2. As derived in the supplementary material, a fundamental vortex hosts a gapless chiral mode with chiral central charge $1/2$. Since this is a topological property, the same is true of the interface between $H_{x<0}$ and $H_{x>0}$, and also $H'_{x<0}$ and $H'_{x>0}$. We thus have a situation where an interface between two $2d$ local commuting projector Hamiltonians hosts a gapless chiral mode. If we fold the system at the interface, stacking the two gapped phases on top of each other, this yields a local commuting projector Hamiltonian with a chiral edge mode, which is believed to be impossible [20]. Hence we have derived a contradiction.

Generalization to discrete subgroups $\mathbb{Z}_{2N}$.—Consider a theory of $N_f \neq 0 \bmod 2N$ flavors of same helicity Weyl

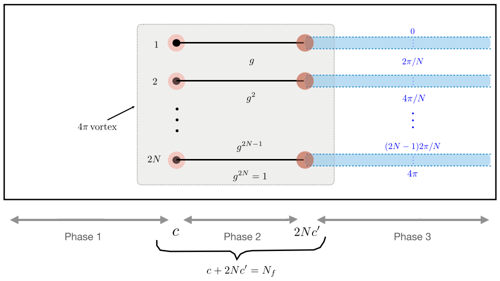

FIG. 3. An effective 4π vortex in Δ constructed from fusing $2N$ defects of a fundamental vortex of $\mathbb{Z}_{2N} \subset \mathrm{U}(1)$ symmetry. Dimensionally reduced phases 1 and 3 can be described within the low energy field theory of N_f Weyl fermions with Majorana mass term.

fermions, but instead of the full U(1) symmetry suppose that only a $\mathbb{Z}_{2N}$ subgroup is preserved. We will show that in this case the $\mathbb{Z}_{2N}$ action cannot be realized by shallow depth quantum circuits in a graded tensor product Hilbert space. The argument is again by contradiction, and is illustrated in Fig. 3.

Again the geometry is dimensionally reduced to $2d$ along the z direction. We now stack $2N$ defects of the generator g of the Z_{2N} symmetry; as illustrated in Fig. 3 the state to the left of the defects is the original ground state (labeled "phase 1"), whereas the state to the right of the defects (labeled "phase 2") can be obtained from the ground state by multiplying by the truncated unitary actions of $g, g^2, \ldots, g^{2N} = 1$ in the indicated regions (this is where we use the shallow circuit assumption, as shallow circuits can be truncated). As noted in the U(1) case, phase 1 is trivial and hence has a commuting projector parent Hamiltonian, and by essentially the same argument in the U(1) case, phase 2 is trivial and has a commuting projector Hamiltonian as well.

Now, as we cross the branch cut surfaces emanating from each defect, the phase of Δ abruptly advances by $2\pi/N$. This is a discontinuity in Δ and cannot be studied in the low energy theory of the Weyl fermion and Majorana mass term; however, we can imagine instead thickening the branch cut surfaces and advancing the phase slowly by $2\pi/N$ across these thickened surfaces. This is labeled "phase 3" in Fig. 3 and can be studied using the low energy field theory. The $1d$ interface between an abrupt branch cut (in phase 2) and a slow one (in phase 3) is not necessarily gapped, and may carry some chiral central charge c'. However, since the state immediately above and below the branch cuts can be disentangled into a product state, each such interface is effectively an interface between two-dimensional fermionic invertible gapped states, and hence c' is an integer multiple of $1/2$. Since the $2N$ different branch cuts are all related by symmetry, the value of c' is the same for all.

Now we simply note that the gray shaded region encompassing phase 2 in Fig. 3 is effectively a 4π vortex in the Majorana mass term, when viewed from far away. This means that its total chirality must be N_f. This chirality is a sum of the contributions of the interfaces between phases 2 and 3, which contribute $2Nc'$, and some contribution c from the defects (note that c is *not* necessarily a half integer multiple of $2N$). Hence $c + 2Nc' = N_f$, so $c = N_f - 2Nc'$. Thus if N_f is not an integer multiple of N, then c must be nonzero modulo N, and hence nonzero. This is a contradiction for the same reason as in the U(1) proof, namely the fact that one cannot have a chiral edge mode between two $2d$ phases described by commuting projector Hamiltonians (phases 1 and 2 in Fig. 3).

In this section we considered the situation where there is a discrete $\mathbb{Z}_{2N}$ chiral symmetry rather than a $\mathrm{U}(1)_A$ symmetry. As long as $N \geq 2$, this chiral symmetry still prohibits any fermion bilinear mass operator in the Weyl fermion Hamiltonian, though some higher order fermion operators are allowed. These higher order terms are obviously perturbatively irrelevant under renormalization group (RG) flow, hence there will still be a $\mathrm{U}(1)_A$ emergent symmetry in the infrared. In the past few years, it has been gradually recognized that when the $\mathrm{U}(1)_A$ symmetry is broken down to its discrete subgroups on the lattice scale, Weyl fermions with emergent IR $\mathrm{U}(1)_A$ symmetry can be regularized as essentially a three-dimensional spatial lattice model, with the proper combination of short range interaction, flavor number, and flavor symmetries [21–27]. These results were obtained from various ways of demonstrating the absence of the 't Hooft anomaly of the discrete axial symmetries. It is worth noting that the absence of any 't Hooft anomaly is a stronger result than ours, which states that the symmetry cannot be implemented through a finite depth quantum circuit. For example, if $N = 2$, i.e. if we have a $\mathbb{Z}_4$ subgroup of $\mathrm{U}(1)_A$, then the symmetry cannot be realized by finite depth circuits for any odd N_f, while it has been shown that the $\mathbb{Z}_4$ axial symmetry is completely anomaly free when $N_f = 0$ mod 16 [26,28–31], and hence the symmetry should be realizable on-site in that case. We note that if the symmetry can be realized by a locality preserving unitary [32] in the case $N_f = 1$, then a standard argument would show that for any even N_f the symmetry should be realizable by a shallow circuit [33].

Summary and discussion.—In this Letter, we showed that $\mathrm{U}(1)_A$ symmetry cannot be realized by a quasilocal charge Q. For the case of $\mathbb{Z}_{2N}$ and $N_f \neq 0$ mod $2N$, we also showed that the unitaries generating the group action cannot be quasilocal shallow depth circuits. In this case, there is still the possibility that these unitaries could be locality preserving but not shallow depth, i.e. they could be nontrivial quantum cellular automata (QCA); however, for a continuous connected group like $U(1)_A$ this possibility is unlikely [34]. We note that while in-cohomology SPT phases always have a boundary action of symmetry by shallow-depth circuits [35], there exists a beyond cohomology SPT phase where the boundary action is a nontrivial QCA [36]. It would be interesting to relate the present work to this classification of boundary symmetry actions.

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