Gauge-Invariant Double Copies via Recursion Relations

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We prove by construction that all tree-level amplitudes in pure (super)gravity can be expressed as termwise, gauge-invariant double copies of those of pure (super-)Yang-Mills obtained via on-shell recursion. These representations are far from unique: varying the recursive scheme leads to a wide variety of distinct but equally valid representations of gravitational amplitudes, all realized as double copies.

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Introduction.—The rich connections between scattering amplitudes in gauge theory and gravity have been a source of tremendous progress in our understanding of both theories. Among the most seminal of these is so-called color-kinematic duality [1], which states that gravitational scattering amplitudes may be represented as "double copies" of those of Yang-Mills theory, provided the latter is represented in terms of color-kinematic satisfying ("dual") numerators with denominators (typically) built from scalar φ^3 field theory (see, e.g., [2–5]). The existence of such numerators was first conjectured, but can be proven at tree level in a number of ways [6–12], with much evidence suggesting that color-kinematic duality should continue to the level of loop integrands (see, e.g., [13–23]). The potential form, structure, and scope of these numerators, as well as the theoretical origins of this story more generally have been the subject of a great deal of research (see, e.g., [24–30]).

Prior to the discovery of color-kinematic duality, on-shell recursion relations [31–33] for tree-level scattering amplitudes led to similarly great leaps in our understanding of gravitational and gauge-theory amplitudes [34–41]. Some of this Letter connected directly to results derived from string and twistor string theory (e.g., [42–46]).

In this Letter, we show that on-shell recursion relations directly lead to representations of amplitudes in color-dressed, four-dimensional Yang-Mills theory (YM) and gravity (GR) that may be expressed in the form

$$\mathcal{A}^{\text{YM}}(1,\dots,n)=\sum_{\vec{a}\in\mathfrak{S}(A)}\sum_{\Gamma}\frac{c_{\alpha\beta}^{\vec{a}}\,n(\Gamma_{\alpha\beta}^{\vec{a}})}{D(\Gamma_{\alpha\beta}^{\vec{a}})}\delta^{2\times2}(\lambda\cdot\tilde{\lambda})$$

$$\mathcal{A}^{\text{GR}}(1,\dots,n)=\sum_{\vec{a}\in\mathfrak{S}(A)}\sum_{\Gamma}\frac{n(\Gamma_{\alpha\beta}^{\vec{a}})\,n(\Gamma_{\alpha\beta}^{\vec{a}})}{D(\Gamma_{\alpha\beta}^{\vec{a}})}\delta^{2\times2}(\lambda\cdot\tilde{\lambda}) \qquad (1)$$

for any choice $\{\alpha,\beta\}\subset[n]$ of the external legs, where $A\coloneqq[n]\backslash\{\alpha,\beta\}$, $\mathfrak{S}(A)$ denotes permutations of the labels in A and Γ indexes on-shell diagrams arising in recursion. The realization that on-shell recursion takes this form constitutes a novel proof that all tree amplitudes can be realized as double copies.

For Yang-Mills theory, the form (1) will be seen to be somewhat quixotic, as the color-kinematic dual "numerators" will be simply *defined* to be the product of $D(\Gamma_{\alpha\beta}^{\vec{a}})$ and a more familiar gauge-invariant, on-shell function [47]; as such, the real novelty arises in the identification of the *denominators* $D(\Gamma_{\alpha\beta}^{\vec{a}})$, which we define recursively. It is worth pointing out that because the color factors appearing in (1) are entirely independent, these numerators are only "color-kinematic dual" in a rather trivial sense: neither satisfies any identities.

The existence of formulas such as (1) follows from the on-shell diagrammatic interpretation of on-shell recursion in YM. Ignoring factors of color and momentum conservation, the double-copy follows from the fact that for any primitive [48] on-shell diagram Γ, the on-shell functions f_Γ of gravity and Yang-Mills differ by a simple Jacobian factor $J(\Gamma)$ depending on the graph

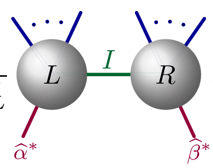
$$\mathfrak{f}_{\Gamma}^{\text{GR}}=J(\Gamma)(\mathfrak{f}_{\Gamma}^{\text{YM}})^2. \qquad (2)$$

This general fact (see, e.g., [49–53]) is a simple consequence of the definition of an on-shell function and the relationship between the 3-particle S matrices of the two theories: an on-shell function may be defined as the product of amplitudes evaluated on the *residue* $1/J(\Gamma)$ of the scalar graph, which puts all internal lines on-shell; squaring an on-shell function in YM gives the correct product of 3-particle amplitudes in GR but squares also $1/J(\Gamma)$, which must be corrected by the numerator of (2).

Tree-level, on-shell recursion for YM and GR.—The starting point for on-shell (BCFW) recursion [33] is to consider an amplitude as meromorphic function of external momenta and deform the momenta of any two particles labeled $\{\alpha,\beta\}$ according to

$$p_\alpha \mapsto \hat{p}_\alpha(z) \coloneqq p_\alpha + z\lambda_\alpha\tilde{\lambda}_\beta, \quad p_\beta \mapsto \hat{p}_\beta(z) \coloneqq p_\beta - z\lambda_\alpha\tilde{\lambda}_\beta, \tag{3}$$

where $p_a \eqqcolon \lambda_a\tilde{\lambda}_a$ are spinor-helicity variables [54]; this deformation preserves momentum conservation and keeps each particle on-shell. (For superamplitudes, the shift (3) is accompanied by a shift in the Grassmann parameters $\tilde{\eta}_\alpha \mapsto \tilde{\eta}_\alpha + z\tilde{\eta}_\beta$ that encode the on-shell coherent states [55].) Note that the choice $\{\alpha,\beta\}$ is distinct from the choice $\{\beta,\alpha\}$: they differ by parity. At tree level, amplitudes have poles at finite z corresponding to factorization channels, the residues of which we may represent diagrammatically as

$$\mathcal{A}_L(\hat{\alpha}^*,\cdots,I)\frac{1}{p_L^2}\mathcal{A}_R(I,\cdots,\hat{\beta}^*) \eqqcolon \frac{1}{p_L^2}$$


, (4)

where $1/p_L^2$ is the off-shell propagator being cut, with the left and right amplitudes evaluated with $p_{\alpha^*,\beta^*} \coloneqq p_{\alpha,\beta}(z^*)$ on the location of the pole $z^* = p_L^2/\langle\alpha|(p_L)|\beta]$ and summed over the states I that can be exchanged. Importantly, the deformed legs must necessarily be on opposite sides of the factorization channel for the simple reason that $\hat{p}_\alpha + \hat{p}_\beta = p_\alpha + p_\beta$ is z-independent.

Provided there are no poles at infinity, Cauchy's theorem allows us to write an amplitude as a sum over all residues of the form (4) [56]. For YM or GR, this will be the case provided the deformed momenta are chosen judiciously according to their helicity (see, e.g., [57]), while amplitudes in maximally supersymmetric ($\mathcal{N}=4$) YM (sYM) or ($\mathcal{N}=8$) GR (sGR) will be free of poles at infinity regardless of which legs are chosen. Because tree-level amplitudes in pure (or any degree of less supersymmetric) YM and GR are identical to those of sYM and sGR for appropriately restricted sets of external states [55], we may therefore without loss of generality consider the case of maximally supersymmetric YM and GR, ensuring that amplitudes are free of poles at infinity for any choice of legs $\{\alpha,\beta\}$. Upon truncation of the external states to those appropriate, our results are equally valid for YM and GR with any degree of less or no supersymmetry.

Notice that the channels (4) allow for arbitrary distributions of the other $(n-2)$ legs $A \coloneqq [n]\backslash\{\alpha,\beta\}$. Thus, on-shell recursion results in a sum of terms of the form

$$\mathcal{A} = \sum_{\substack{\vec{a}\in\mathfrak{S}(A)\\(\vec{a}_L,\vec{a}_R)=\vec{a}}} \mathcal{A}_L(\hat{\alpha}^*,\vec{a}_L,I)\frac{1}{p_{\alpha a_L}^2}\mathcal{A}_R(I,\vec{a}_R,\hat{\beta}^*)$$
$$\eqqcolon \sum_{\vec{a}\in\mathfrak{S}(A)} \mathcal{A}(\alpha,\vec{a},\beta), \tag{5}$$

where $\mathcal{A}(\alpha,\vec{a},\beta) \coloneqq \mathcal{A}(\alpha,a_1,\ldots,a_{n-2},\beta)$ are *partial amplitudes* involving external momenta with *specific ordering*.

As amplitudes in color-dressed YM and gravity are fully permutation-invariant (due to Bose symmetry), any choice of legs $\{\alpha,\beta\}$ may be taken; and any particular ordering of the other legs $\vec{a}\in\mathfrak{S}(A)$ will suffice to generate the full amplitude upon summation over permutations of the labels $\vec{a}$. Thus, we may without loss of generality focus our attention on the determination of the partial amplitude $\mathcal{A}(1,2,\ldots,n-1,n)$.

It is important to note that in *neither theory* (sYM nor sGR) is the partial amplitude unique: not only does it depend on the legs chosen, but also the specific sequence of choices made for iterated recursion. Of course, the so-called "primitive" amplitudes of YM—those partial amplitudes that have been stripped of their color tensors—do enjoy many scheme independent properties. But for our purposes, it is more natural to use "partial amplitude" to denote a term arising in the recursion with a particular ordering of their external legs, *including their color tensors*, which are recursively defined in terms of 3-particle amplitudes.

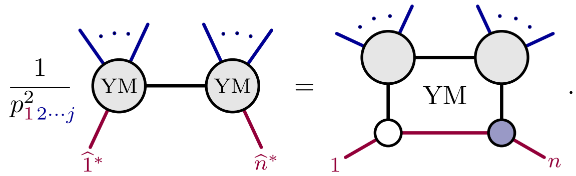
One particularly convenient recursion scheme would be to always choose the first and last leg of every iteratively recursed amplitude, and use the same parity of bridge at each stage of recursion. In the case of Yang-Mills, this results in partial amplitudes dressed by the color factors appearing in the familiar representation of Del Duca–Dixon–Maltoni [58]. Letting $\mathcal{A}^{\mathrm{YM}}(\alpha,\vec{a},\beta) \eqqcolon c_{\alpha\beta}^{\vec{a}}A^{\mathrm{YM}}(\alpha,\vec{a},\beta)$, it is easy to see that the recursion (5) separates color and kinematics cleanly so that, upon recursing iteratively down to factorizations involving only three-point amplitudes, we find

$$c_{\alpha\beta}^{\vec{a}} \coloneqq \sum_{e_i} c^{\alpha,a_1,e_1}c^{e_1,a_2,e_2}\cdots c^{e_{-1},a_{n-2},\beta}, \tag{6}$$

where ${c_a}^c{}_b$ are the structure constants of some Lie algebra (into which we may freely absorb any coupling constant), and $\{a,b,c\}$ are (adjoint) color labels for the gluons. These color tensors are all independent under Jacobi relations, and the so-called "primitive" ordered amplitudes of YM turn out to be gauge-invariant, local, dihedrally symmetric, and to enjoy Kleiss-Kuijf relations [59]. (All of these properties can be deduced from the Jacobi identity and Bose symmetry of color-dressed amplitudes alone.) Besides gauge invariance, none of these properties will be enjoyed by the partial amplitudes of gravity, the meaning of which will depend strongly on how recursion is implemented (analogously to how the specific color tensors involved in partial amplitudes in YM are recursively defined via a particular recursion scheme).

Because this recursion scheme results in the same color-factor $c_{\alpha}{}^{\vec{a}}{}_{\beta}$ coefficient for every term, it is common to factor it out entirely and focus on the *color-stripped* partial amplitude primitive $A^{\rm YM}(1,2,\ldots,n-1,n)$.

On-shell diagrammatics of YM and GR.—For color-stripped partial amplitudes in YM, there exists a powerful diagrammatic manifestation of recursion relations following from the simple fact that



. (7)

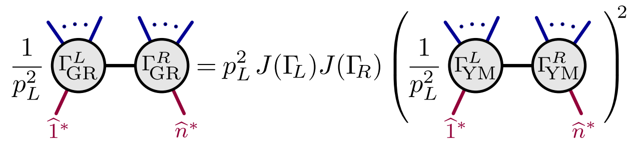
Here, we have introduced "flat" vertices to denote *ordered* partial amplitude primitives. The right-hand side represents an *on-shell function* of YM: the product of (color-stripped) amplitudes at the vertices, summing over all the on-shell, internal states that can be exchanged between them. These functions are extremely well understood: they are classified combinatorially, and all their functional relations can be understood to arise homologically from an auxiliary Grassmannian "positive" geometry (see, e.g., [47,60]).

Applying recursion successively results in a representation of color-stripped YM partial amplitudes as sums over on-shell functions encoded by *specific* on-shell diagrams $\{\Gamma\}$,

$$A^{\rm YM}(1,2,\ldots,n-1,n)=\sum_{\Gamma}\mathfrak{f}^{\rm YM}_{\Gamma}=:\sum_{\Gamma}\hat{\mathfrak{f}}^{\rm YM}_{\Gamma}\delta^{2\times2}(\lambda\cdot\tilde{\lambda}), \qquad (8)$$

where the sum is over on-shell diagrams $\{\Gamma\}$ of the form (7) involving *exclusively* three-point vertices. The N^kMHV-degree of an amplitude is encoded by the graph according to $k=2n_B+n_W-n_I-2$, where $n_B(n_W)$ denotes the number of blue(white) vertices and n_I the number of internal lines. Even for maximally helicity violating (MHV) amplitudes, there are vastly more on-shell *diagrams* than on-shell functions as diagrams related by mergers and square moves leave on-shell functions unchanged *in YM* [47]. On-shell diagrams in gravity enjoy only the square move as an (unmodified) equivalence relation [49].

Color-kinematic denominators for gravity.—For amplitudes in GR, there is no simple analog of (7) (see, e.g., [49,50]), but, supposing that there is some diagrammatic representation for partial amplitudes in GR in terms of on-shell diagrams of YM, we may recursively conclude that



$$=: D(\Gamma)\big(\hat{\mathfrak{f}}^{\rm YM}_{\Gamma}\big)^2\delta^{2\times2}(\lambda\cdot\tilde{\lambda})\,. \qquad (9)$$

The fact that BCFW recursion for GR can be expressed in the form (9) is reasonably well-known [49–53]. The precise form of $D(\Gamma)$ depends both on the graph Γ and the recursion scheme followed. If we always choose the first and last labels for subsequent recursion, so that every diagram appearing is of the form $\Gamma_{\alpha}{}^{\vec{a}}{}_{\beta}\coloneqq\Gamma_L[\hat{\alpha}^*,\vec{a}_L,I]\otimes\Gamma_R[I,\vec{a}_R,\hat{\beta}^*]$, then $D(\Gamma)$ will be

$$D(\Gamma^{\vec{a}}_{\alpha\beta})=p^2_{\alpha a_L}D(\Gamma^{\vec{a}_L}_{\hat{\alpha}^*\,I})D(\Gamma^{\vec{a}_R}_{I\;\hat{\beta}^*}). \qquad (10)$$

(Here, we have used notation $\Gamma_{\alpha}{}^{\vec{a}}{}_{\beta}$ to emphasize the different roles of legs involved in any particular diagram arising via successive recursion, and also to highlight the similarity these factors will have with color tensors.)

We call these factors color-kinematic *denominators* because if we let $n(\Gamma)\coloneqq D(\Gamma)\hat{\mathfrak{f}}^{\rm YM}_{\Gamma}$ then individual terms appearing in the recursion of an amplitude in YM take the form

$$\frac{c^{\vec{a}}_{\alpha\beta}n(\Gamma^{\vec{a}}_{\alpha\beta})}{D(\Gamma^{\vec{a}}_{\alpha\beta})}=c^{\vec{a}}_{\alpha\beta}\hat{\mathfrak{f}}^{\rm YM}_{\Gamma^{\vec{a}}_{\alpha\beta}}, \qquad (11)$$

while terms in gravity are given by the *double-copy*

$$\frac{n(\Gamma^{\vec{a}}_{\alpha\beta})n(\Gamma^{\vec{a}}_{\alpha\beta})}{D(\Gamma^{\vec{a}}_{\alpha\beta})}=D(\Gamma^{\vec{a}}_{\alpha\beta})(\hat{\mathfrak{f}}^{\rm YM}_{\Gamma^{\vec{a}}_{\alpha\beta}})^2. \qquad (12)$$

Illustrations of on-shell double copies.—Arguably the simplest amplitudes in either theory are the so-called $(\mathrm{N}^{k=0})$MHV amplitudes [42,46,61]. On-shell recursion results in a single term for ordered partial amplitudes in either theory:

$$\begin{aligned}\mathcal{A}^{\rm YM}_{\rm MHV}(1,\ldots,n)&=c_1{}^{\vec{a}}{}_n\frac{\delta^{2\times4}(\lambda\cdot\tilde{\eta})\delta^{2\times2}(\lambda\cdot\tilde{\lambda})}{\langle12\rangle\langle23\rangle\cdots\langle n-1n\rangle\langle n1\rangle}\\&=:c_1{}^{\vec{a}}{}_n\frac{n(\Gamma^{\rm MHV}_{1\;n})}{D(\Gamma^{\rm MHV}_{1\;n})}\delta^{2\times2}(\lambda\cdot\tilde{\lambda})\\&=:c_1{}^{\vec{a}}{}_n\mathrm{PT}(1\cdots n)\delta^{2\times2}(\lambda\cdot\tilde{\lambda}),\end{aligned} \qquad (13)$$

where the denominators are determined recursively. In the default recursion scheme (10), we find

$$D(\Gamma^{\mathrm{MHV}}_{1\,n}) \coloneqq p^2_{n-1n}\prod_{j=4}^{n-1}\frac{\langle 1|(2\cdots j-2)|(j-1)|j\rangle}{\langle 1j\rangle},$$

$$n(\Gamma^{\mathrm{MHV}}_{1\,n}) \coloneqq D(\Gamma^{\mathrm{MHV}}_{1\,n})\frac{\delta^{2\times 4}(\lambda\cdot\tilde{\eta})}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1n\rangle\langle n1\rangle}$$
$$= \frac{[32]}{\langle 23\rangle\langle n1\rangle^2}\prod_{j=4}^{n-1}\frac{\langle 1|(2\cdots j-1)|j]}{\langle 1j\rangle}\delta^{2\times 4}(\lambda\cdot\tilde{\eta}); \qquad (14)$$

this representation immediately allows us to write the corresponding expressions for GR as a double-copy:

$$\mathcal{A}^{\mathrm{GR}}_{\mathrm{MHV}}(1,\ldots,n) = \frac{n(\Gamma^{\mathrm{MHV}}_{1\,n})n(\Gamma^{\mathrm{MHV}}_{1\,n})}{D(\Gamma^{\mathrm{MHV}}_{1\,n})}\delta^{2\times 2}(\lambda\cdot\tilde{\lambda})$$
$$= D(\Gamma^{\mathrm{MHV}}_{1\,n})(\mathrm{PT}(1\cdots n))^2\delta^{2\times 2}(\lambda\cdot\tilde{\lambda}). \qquad (15)$$

Unlike the case of YM, these partial amplitudes in GR are noncyclic and involve nonlocal poles. Only upon summing over all $(n-2)!$ orderings of $\{2,\ldots,n-1\}$ do we recover a local, permutation-invariant amplitude. We have checked that this formula agrees with the closed-form expression of Hodges [62] through $n=12$ particles.

It is worth noting that this representation of MHV amplitudes in gravity (15) is *identical* (upon a rotation of labels) to that found in [40]. And as with [40], the use of the "bonus relations" stemming from the good large-z behavior of amplitudes in GR [55] allows us to rewrite (15) as a sum over $(n-3)!$ terms [38]:

$$\hat{\mathcal{A}}^{\mathrm{GR}}_{\mathrm{MHV}} = \sum_{\vec{a}\in\mathfrak{S}(3,\ldots,n-1)}\frac{\langle n1\rangle\langle 23\rangle}{\langle n2\rangle\langle 13\rangle}D(\Gamma^{2\vec{a}}_{1\,n})(\mathrm{PT}(12\vec{a}n))^2. \qquad (16)$$

[Here, $\hat{\mathcal{A}}$ is used analogously to (9): it simply denotes the amplitude divided by the momentum-conserving δ function $\delta^{2\times 2}(\lambda\cdot\tilde{\lambda})$.]

For higher $\mathrm{N}^{k>0}$MHV degrees, on-shell recursion typically involves a sum over terms, each represented in YM by a particular, primitive on-shell diagram. The simplest nontrivial example is the 6-particle NMHV amplitude, which involves three terms to represent the ordered amplitude. Following the recursion scheme described above, the three on-shell diagrams $\{\Gamma_1,\Gamma_2\Gamma_3\}$ that result are given in Table I, where we have also indicated the numerators $n(\Gamma_i)$ and denominators $D(\Gamma_i)$ of each. Thus, we may write the ordered, partial NMHV amplitude primitive in YM as

$$\mathcal{A}^{\mathrm{YM}}_{6,1}(1,\ldots,n) = c^{\vec{a}}_{1\,6}\left(\frac{n(\Gamma_1)}{D(\Gamma_1)}+\frac{n(\Gamma_2)}{D(\Gamma_2)}+\frac{n(\Gamma_3)}{D(\Gamma_3)}\right)\delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) \qquad (17)$$

and the corresponding partial amplitude in GR as the double-copy

$$\mathcal{A}^{\mathrm{GR}}_{6,1}(1,\ldots,n) = \left(\frac{n(\Gamma_1)^2}{D(\Gamma_1)}+\frac{n(\Gamma_2)^2}{D(\Gamma_2)}+\frac{n(\Gamma_3)^2}{D(\Gamma_3)}\right)\delta^{2\times 2}(\lambda\cdot\tilde{\lambda}). \qquad (18)$$

In both cases, these partial amplitudes must be summed over the $(n-2)!$ orderings of the legs $\{2,\ldots,5\}$. Although all diagrams involved in the representation of a single partial amplitude involve the same color tensor, this is not in any conflict with (1). We have checked this expression against the those obtained by the string-based results of Kawai-Lewellen-Tai [63].

The numerators listed in Table I involve Grassmann δ functions involving the $\tilde{\eta}$'s that label the external states of each supermultiplet [55]; they are defined by

$$\delta^{3\times 4}(C_a\cdot\tilde{\eta}) \coloneqq \delta^{2\times 4}(\lambda\cdot\tilde{\eta})\delta^{1\times 4}([aa+1]\tilde{\eta}_{a-1} + [a+1a-1]\tilde{\eta}_a + [a-1a]\tilde{\eta}_{a+1}). \qquad (19)$$

TABLE I. On-shell, gauge-invariant contributions to the 6-point NMHV partial amplitudes of Yang-Mills and gravity. The Grassmann δ functions $\delta^{3\times 4}(C_i\cdot\tilde{\eta})$ appearing in these numerators are defined in (19).

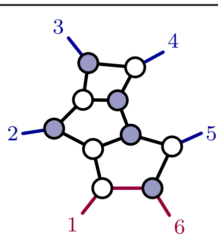
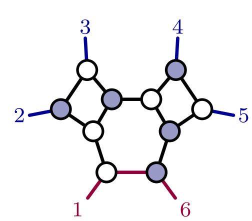
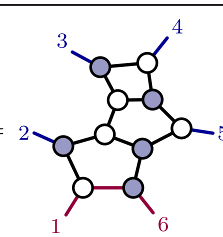
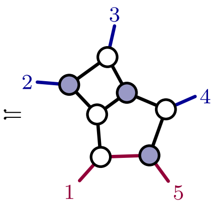
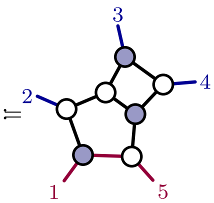
Γ_i	$\Gamma_1 \coloneqq$ 	$\Gamma_2 \coloneqq$ 	$\Gamma_3 \coloneqq$ 
$J(\Gamma_i)$	$s_{61}s_{234}s_{34}D(\Gamma_1)$	$s_{61}s_{23}s_{45}D(\Gamma_2)$	$s_{61}s_{345}s_{34}D(\Gamma_3)$
$1-1D(\Gamma_i)$	$s_{56}\{\langle 1|(2)|(34)|5\rangle/\langle 15\rangle\}$ $\{\langle 1|(23)|(4)|(3)|2]/\langle 1|(43)|2]\}$	$s_{123}\{\langle 1|(2)|(3)|(45)|6]/\langle 1|(23)|6]\}$ $\{\langle 1|(23)|(4)|(5)|6]/\langle 1|(45)|6]\}$	$s_{12}\{[2|(34)|(5)|6]/[26]\}$ $\{\langle 5|(4)|(3)|(45)|6]/\langle 5|(34)|6]\}$
$1-1n(\Gamma_i)$	$\{\langle 12\rangle\langle 34\rangle[56]\delta^{3\times 4}(C_3\cdot\tilde{\eta})/$ $s_{234}\langle 1|(56)|2][34]\langle 51\rangle\langle 61\rangle\}$	$\{[23]\langle 45\rangle\delta^{3\times 4}(C_5\cdot\tilde{\eta})/$ $\langle 1|(23)|6]^2[45]\langle 23\rangle\}$	$\{\langle 12\rangle[34][56]\delta^{3\times 4}(C_1\cdot\tilde{\eta})/$ $s_{345}\langle 5|(34)|6]\langle 34\rangle[26][61]\}$

TABLE II. Alternative recursion schemata resulting in distinct ordered, partial amplitudes $\mathcal{A}^{\mathrm{GR}}(1,2,3,4,5)$.

Γ_i	$\Gamma_a \coloneqq$ 	$\Gamma_b \coloneqq$ 
$J(\Gamma_i)$	$s_{51}s_{23}D(\Gamma_a)$	$s_{51}s_{34}D(\Gamma_b)$
$D(\Gamma_i)$	$s_{45}\{\langle 1\|(2)\|(3)\|4\rangle/\langle 14\rangle\}$	$s_{12}\{\langle 2\|(3)\|(4)\|5\rangle/\langle 25\rangle\}$
$n(\Gamma_i)$	$\{[23][45]/\langle 14\rangle\langle 23\rangle\langle 51\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$	$\{[12][34]/\langle 34\rangle\langle 25\rangle\langle 51\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$

Notice that $\delta^{3\times 4}(C_a\cdot\tilde{\eta})$ is invariant under permutations of both the set $\{a-1,a,a+1\}$ and its complement. This will turn out to have interesting consequences as we discuss in the forthcoming work [64].

Although the expression (17) may seem unusual, it is worth observing that, for example,

$$\frac{n(\Gamma_2)}{D(\Gamma_2)} = \frac{\delta^{2\times 4}(\lambda\cdot\tilde{\eta})\delta^{1\times 4}([56]\tilde{\eta}_4 + [64]\tilde{\eta}_5 + [45]\tilde{\eta}_6)}{[56]\langle 12\rangle\langle 3|(45)|6]s_{123}[4|(56)1\rangle\langle 23\rangle[45]}, \qquad (20)$$

is simply the (momentum-space version of the) familiar R invariant $\mathcal{R}[1,3,4,5,6]$ (see, e.g., [65]).

Up to minor conventional differences, the recursion scheme used to construct denominators (10) generally reproduces the form of amplitudes in GR as they were derived in [39]. We have implemented in this in MATHEMATICA and have verified agreement against Kawai-Lewellen-Tai [63] through the 10-particle $\mathrm{N}^3\mathrm{MHV}$ amplitude. With BCFW we can recursively construct gauge-invariant double copies for all n, thus concluding the proof. These tools will be made available in a forthcoming, public package for tree amplitudes more generally [66] (see also [67]).

Nonuniqueness of dual denominators.—As emphasized above, even restricting ourselves to successively choosing the first last legs for all iterated recursions, variability emerges from the chiral asymmetry of the BCFW deformation (3). Even for five particles, choosing $\{\alpha,\beta\}$ to be $\{1,5\}$ versus $\{5,1\}$ (conjugating the shifting rule) results in two distinct diagrams Γ_a and Γ_b as shown in Table II corresponding to the *distinct* primitives $n(\Gamma_i)^2/D(\Gamma_i)$

$$\mathcal{A}_a^{\mathrm{GR}}(1,\ldots,5) \coloneqq \frac{\langle 1|(2)|(3)|4\rangle[45]\delta^{2\times 8}(\lambda\cdot\tilde{\eta})}{\langle 14\rangle\langle 45\rangle(\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 51\rangle)^2}\delta^{2\times 2}(\lambda\cdot\tilde{\lambda});$$

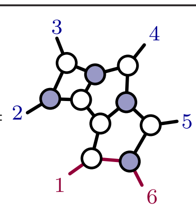
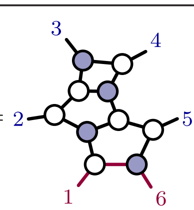
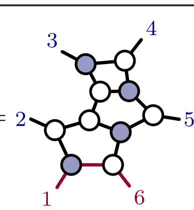
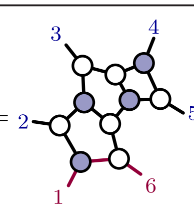
$$\mathcal{A}_b^{\mathrm{GR}}(1,\ldots,5) \coloneqq \frac{\langle 2|(3)|(4)|5\rangle[12]\delta^{2\times 8}(\lambda\cdot\tilde{\eta})}{\langle 12\rangle\langle 25\rangle(\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle)^2}\delta^{2\times 2}(\lambda\cdot\tilde{\lambda}).$$

Nevertheless, it is easy to verify that the sum over their permuted images agree:

$$\mathcal{A}^{\mathrm{GR}} = \sum_{\vec{a}\in\mathfrak{S}([2,\ldots,4])}\mathcal{A}_a^{\mathrm{GR}}(1,a_1,\ldots,a_{-1},5) = \sum_{\vec{a}\in\mathfrak{S}([2,\ldots,4])}\mathcal{A}_b^{\mathrm{GR}}(1,a_1,\ldots,a_{-1},5).$$

This variability only proliferates for higher multiplicity, as evidenced by the four examples for 6-point MHV given in Table III, where the alternative expressions were found by merely varying the parity of the recursive choice made at each step. For example, Γ_a represents the default scheme—a consistent choice of parity—leading to an instance of (14); while for Γ_c, the opposite parity was chosen at the first stage, relative to all subsequent recursions. More

TABLE III. Alternative recursion schemata resulting in distinct ordered, partial amplitudes $\mathcal{A}^{\mathrm{GR}}(1,2,3,4,5,6)$.

Γ_i	$\Gamma_a \coloneqq$ 	$\Gamma_b \coloneqq$ 	$\Gamma_c \coloneqq$ 	$\Gamma_d \coloneqq$ 
$J(\Gamma_i)$	$s_{61}s_{23}s_{234}D(\Gamma_a)$	$s_{61}s_{34}s_{234}D(\Gamma_b)$	$s_{61}s_{34}s_{345}D(\Gamma_c)$	$s_{61}s_{23}s_{345}D(\Gamma_d)$
$D(\Gamma_i)$	$s_{56}\{\langle 1\|(2)\|(3)\|4\rangle/\langle 14\rangle\}$ $\{\langle 1\|(23)\|(4)\|5\rangle/\langle 15\rangle\}$	$s_{56}\{\langle 2\|(3)\|(4)\|5\rangle/\langle 25\rangle\}$ $\{\langle 1\|(2)\|(34)\|5\rangle/\langle 15\rangle\}$	$s_{12}\{\langle 2\|(3)\|(4)\|5\rangle/\langle 25\rangle\}$ $\{\langle 2\|(34)\|(5)\|6\rangle/\langle 26\rangle\}$	$s_{12}\{\langle 2\|(3)\|(45)\|6\rangle/$ $\langle 26\rangle\}\{\langle 3\|(4)\|(5)\|6\rangle/\langle 36\rangle\}$
$n(\Gamma_i)$	$\{\langle 1\|(23)\|4]\[23\][56]/$ $\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 61\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$	$\{\langle 5\|(34)\|2][34][56]/$ $\langle 15\rangle\langle 25\rangle\langle 34\rangle\langle 61\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$	$\{\langle 2\|(34)\|5][12][34]/$ $\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 61\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$	$\{\langle 6\|(45)\|3][12][45]/$ $\langle 26\rangle\langle 36\rangle\langle 45\rangle\langle 61\rangle\}\delta^{2\times 4}(\lambda\cdot\tilde{\eta})$

generally, the equivalence of expressions upon distinct variations gives powerful identities among not merely the partial amplitudes in YM, but even among individual on-shell functions appearing in $N^{k>0}$MHV amplitudes. We will explore the scope of these possibilities and their consequences in a forthcoming work [64].

Conclusions and futures directions.—The number of terms generated by BCFW to represent the n particle N^kMHV amplitude for a specific ordering is given by a Narayana number $\{1/[n-3]\}\binom{n-3}{k+1}\binom{n-3}{k}$. It is natural to suppose that including a sum over $(n-2)!$ permutations of leg labels would result in as many more terms in the expression for gravity. This turns out to not be the case—as evidenced, for example, by the more compact expression for MHV amplitudes (16).

Interestingly, for higher multiplicity and N^kMHV degree, considerations of Grassmannian geometry of the $\tilde{\eta}$ coefficients expose *even more* symmetry than would result from mere permutation invariance of amplitudes in GR [64]. For the 10-particle N^3MHV amplitude, for example, we require only 343 252 distinct superfunctions—about 20 times fewer than the naïve estimate of $(175\times 8!)$. Moreover, the contributions that appear are found to satisfy a number of novel functional relations, some of which can be demonstrated using bonus relations or by equating formulas resulting from different recursion schemata, but we have also stumbled into yet further relations that remain to be understood. We explore some of these aspects of gravitational amplitudes in [64]. This additional, geometric structure hints at the possibility of a broader geometric story, perhaps analogous to the "gravituhedron" described in [52].

While the existence of color-kinematic dual numerators remains conjectural beyond tree level, there is a great deal of evidence from specific examples that the double-copy should generalize to loop integrands (in some form or other) [13–16,68–70]. In Ref. [49], Heslop and Lipstein gave evidence at one loop that the obvious extension of on-shell recursion for loop integrands for sYM [71] works also for sGR. It is natural to wonder if this Letters more generally, and if loop amplitude integrands for sGR continue to be generated as a double-copy of those for color-dressed sYM.

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