Kontsevich-Segal Criterion in the No-Boundary State Constrains Inflation

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We show that the Kontsevich-Segal (KS) criterion, applied to the complex saddles that specify the semiclassical no-boundary wave function, acts as a selection mechanism on inflationary scalar field potentials. Completing the observable phase of slow-roll inflation with a no-boundary origin, the KS criterion effectively bounds the tensor-to-scalar ratio of cosmic microwave background fluctuations to be less than 0.08, in line with current observations. We trace the failure of complex saddles to meet the KS criterion to the development of a tachyon in their spectrum of perturbations.

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Introduction.—Future experiments promise to tighten the upper bound on the tensor-to-scalar ratio r of CMB fluctuations down to around 10^{-3} [1,2]. It is therefore of great experimental and theoretical interest to understand whether quantum gravity can produce inflationary models with a level of primordial gravitational waves above this.

The value of r predicted by inflationary theory is related to the total amount of displacement experienced by the inflaton field as it rolls down the scalar potential. While the first string theory models of inflation predicted an unobservably low tensor contribution to the CMB anisotropies, further studies have suggested that r might lie in the detectable range after all [3,4]. Still, many suspect that the quantum completion of inflation implies a theoretical upper bound on r and it would be very interesting to understand where it is.

Recently, Kontsevich and Segal (KS) in [5] have advanced an interesting criterion that complex metrics should satisfy in order to qualify as backgrounds for physically meaningful quantum field theories. Witten [6] subsequently explored whether this criterion might be employed to select physically sensible saddles of gravitational path integrals (see Refs. [7,8] for earlier work in this direction). The idea behind this is that only those backgrounds g on a D-manifold M should be considered (or are "allowable") on which an arbitrary quantum field theory could be defined. Concretely one takes this to mean that the path integral for all free p-form matter on (M, g) should converge, or that

$$\mathrm{Re}\left(\sqrt{g} g^{\mu_1\nu_1} \cdots g^{\mu_p\nu_p} F_{\mu_1\cdots\mu_p} F_{\nu_1\cdots\nu_p}\right) > 0 \qquad (1)$$

for all $p \in \{0, \ldots, D\}$ and all *real-valued* antisymmetric p tensors F on M. For metrics that are diagonal in a real basis [9] with diagonal elements λ_i, this is equivalent [5] to the requirement that

$$\sum_{i=1}^{D} |\arg \lambda_i| < \pi, \qquad (2)$$

where $\arg \in (-\pi, \pi]$.

This criterion has passed several nontrivial checks [6]. For example, it eliminates pathological wormhole solutions with vanishing action, but it does allow for the complexified Kerr solutions that correctly encode the thermodynamic properties of rotating black holes. Yet it remains unclear whether the KS criterion is necessary or sufficient. Regarding necessity, recently solutions were found [10,11] which violate the KS criterion but nonetheless appear to describe physically sensible saddles. Regarding sufficiency, clearly not all sensible quantum field theories are covered by those of free p forms. Hence more work is needed both to refine the KS criterion and to better understand its physical implications, especially in the context of inflation (for recent studies see Refs. [12–16]).

To this end we study the implications of the KS criterion for our understanding of the quantum gravitational origin of inflation. We assume the universe to be in the Hartle-Hawking no-boundary state [17] and we consider this wave function in a variety of single field, slow-roll models of inflation. In each of these models the semiclassical no-boundary wave function (NBWF) is specified by $O(4)$-invariant, complex solutions of the Einstein equations. Loosely speaking, these complex saddles describe the nucleation and subsequent quasiclassical evolution of an expanding universe with an early phase of inflation.

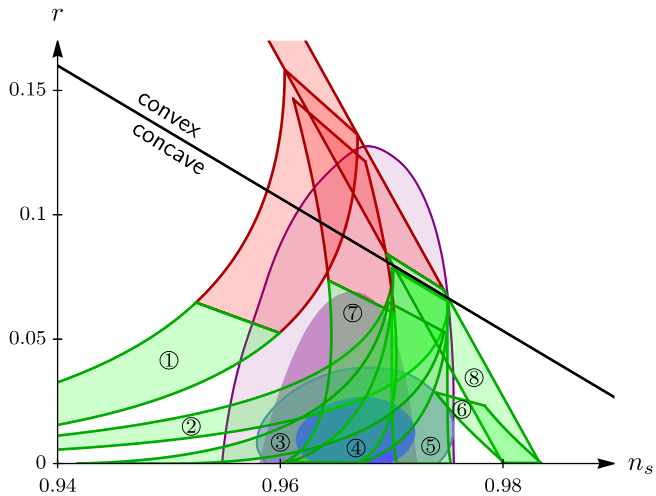

FIG. 1. The Kontsevich-Segal criterion applied to the no-boundary state selects those inflationary models that predict CMB fluctuations with a low tensor-to-scalar ratio $r \lesssim 0.08$. Shown are the predictions for the scalar tilt n_s and r in eight different slow-roll models of inflation (see Table I). Indicated in green are inflationary trajectories of 50 to 60 *e-folds* that are associated with no-boundary saddles that satisfy the KS criterion. Shown in red are inflationary universes that are ruled out by KS. The observational constraints from the 2018 Planck TT, TE, EE + lowE + lensing analysis are indicated in purple. Finally, the blue region shows the combined constraints of Planck and the 2018 BICEP/Keck data and BAO.

However, while the original NBWF implies an inflationary origin, the theory allows for a vast range of inflationary potentials. That is, even though the no-boundary prior favors some potentials over others (e.g., [18]), it does not exclude slow-roll potentials entirely. We find that the KS criterion does exactly this. In the next sections, we show that the KS criterion in conjunction with no-boundary initial conditions acts as a selection mechanism on inflationary potentials. Specifically, the criterion predicts that universes with a significant number of *e-folds* emerge from a concave patch of the potentials. This in turn sets an upper bound on the tensor-to-scalar ratio of CMB fluctuations, consistent with observations. Figure 1 summarizes our findings. We now describe how we arrive at these.

Allowable no-boundary saddles.—We consider the Hartle-Hawking state in $D = 4$ in a minisuperspace model consisting of Einstein gravity minimally coupled to a homogeneous scalar field with potential V. When it exists, we assume that the semiclassical no-boundary amplitude $\Psi(b, \chi)$ of a round three-sphere with volume $\propto b^3$ filled with a uniform scalar field of value χ is specified by an $O(4)$-symmetric saddle living on the four-ball $M = B^4$ and satisfying the KS criterion. That is, we follow [19] but include KS.

We adopt the following ansatz for the saddle-point geometries and field profile,

$$g_{\mu\nu} dx^\mu dx^\nu = dr^2 + a(r)^2 d\Omega_3^2, \qquad \phi = \phi(r), \tag{3}$$

where the scale factor a and scalar field ϕ take the values (b, χ) on the boundary $\partial M = S^3$. The saddle-point equations of motion (EOM) are (with $M_{\rm pl} = 1$)

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{a^2} + \frac{1}{3}\left(\frac{(\phi')^2}{2} - V(\phi)\right),$$
$$0 = \phi'' + 3\frac{a'}{a}\phi' - V'(\phi). \tag{4}$$

The coordinate r runs from the center of the ball at $r = 0$ to an end point at $r = v$. Its range is determined by the boundary conditions of regularity at the center,

$$a(r) = r + \mathcal{O}(r^3), \qquad \phi(r) = \phi_0 + \mathcal{O}(r^2) \quad \text{as } r \to 0, \tag{5}$$

together with the conditions that

$$a(v) = b, \qquad \phi(v) = \chi. \tag{6}$$

These boundary conditions generally imply that ϕ_0 and v are complex [20,21] and hence that the solutions $[a(r), \phi(r)]$ are complex too. Note that this need not be at odds with the assumption underlying the KS criterion that one integrates over real matter field fluctuations, since complex saddle-point solutions can arise as an approximation to an integral over real $\{\phi\}$.

The r-coordinate runs along a curve $\gamma(\ell): 0 \to v$ in the complex plane. Along this curve the metric reads

$$ds^2 = \gamma'(\ell)^2 d\ell^2 + a[\gamma(\ell)]^2 d\Omega_3^2. \tag{7}$$

We say that a given solution $[a(r), \phi(r), v]$ obeys the KS criterion if there exists a curve γ such that the induced metric (7) satisfies (2) along its entire length:

$$|\arg \gamma'^2| + 3|\arg a(\gamma)^2| < \pi. \tag{8}$$

The boundary value problem (4)–(6) has two complex boundary conditions in (6) and equally many free parameters in (ϕ_0, v). Hence it has a discrete solution set. Furthermore for given (b, χ), each solution $[a(r), \phi(r), v]$ is fourfold degenerate: the tuples $[a(-r), \phi(-r), -v]$, $[a(r^*)^*, \phi(r^*)^*, v^*]$ and $[a(-r^*)^*, \phi(-r^*)^*, -v^*]$ are also solutions. Either they all satisfy the KS criterion or none of them do, as it should be because the physical predictions of the four saddles are identical (see the discussion). In all the models we will consider, we find an $O(4)$-symmetric solution to (4)–(6) for all (b, χ), but that solution does not necessarily satisfy KS.

Considering the solution with v in the first quadrant, our strategy to verify the KS criterion is based on the construction of an "extremal curve" $\gamma_{\rm e}$ that saturates the inequality (8) and lies in the first quadrant (cf. [13]):

$$\arg \gamma_{\rm e}'^2 + 3|\arg a(\gamma_{\rm e})^2| = \pi. \tag{9}$$

From the known behavior (5) of the scale factor near the origin it follows that $\lim_{\ell\to 0}\arg\gamma_{\mathsf{e}}(\ell)=\pi/8$. Note that γ_{e} is required to always be right moving. Also, the curves everywhere satisfying (8) and starting at $r=0$ are constrained to remain below γ_{e}. Therefore if $\mathrm{Im}\gamma_{\mathsf{e}}<\mathrm{Im}v$ when $\mathrm{Re}\gamma_{\mathsf{e}}=\mathrm{Re}v$, there is no allowable $\gamma\colon\ 0\to v$. Conversely if $\mathrm{Im}\gamma_{\mathsf{e}}\geq\mathrm{Im}v$ when $\mathrm{Re}\gamma_{\mathsf{e}}=\mathrm{Re}v$, we expect by continuity there to exist an allowable curve $\gamma\colon\ 0\to v$, obtained from (9) by decreasing the right-hand side.

For any given (b,χ) this procedure allows us to determine whether there is an $O(4)$-invariant no-boundary saddle that meets the KS criterion [22]. A systematic analysis for all (b,χ) thus divides the minisuperspace in two regions. One range of configurations will be associated with saddles that are physically meaningful, according to the KS criterion. The semiclassical amplitude of these configurations is specified by the usual Hartle-Hawking saddle. But in regions of superspace where the KS criterion fails, the original "vanilla" NBWF will be strongly modified. Specifically, the KS criterion strongly suppresses the semiclassical NBWF in this regime, by excluding the contribution from what would have been the dominant saddle in the absence of the KS criterion. This in turn sharpens the predictions of the theory [23].

We now carry out the above analysis, first in a particular model that is analytically solvable and then in a representative class of slow-roll inflation models.

A solvable model.—Consider Einstein gravity minimally coupled to a scalar subject to the potential

$$V(\phi)=\Lambda\cosh\left(\sqrt{\frac{2}{3}}\phi\right),\qquad \Lambda>0. \tag{10}$$

Note that this potential does not have standard slow-roll patches, since $\eta=V''/V=2/3$ everywhere. Rather it is the combination of the cosmological constant and the scalar field in the lower regions of this potential that can drive exponential expansion.

A change of coordinates $dr\to d\tau/a$, together with an overall rescaling so that

$$ds^2=\frac{\sqrt{3/2}}{\Lambda}\left(\frac{d\tau^2}{a(\tau)^2}+a(\tau)^2d\Omega_3^2\right), \tag{11}$$

and the introduction of new variables

$$x=\sqrt{\frac{3}{2}}a^2\cosh\left(\sqrt{\frac{2}{3}}\phi\right),\quad y=\sqrt{\frac{3}{2}}a^2\sinh\left(\sqrt{\frac{2}{3}}\phi\right) \tag{12}$$

results in a quadratic Euclidean action for (x,y) [24,25]:

$$S=\frac{\sqrt{6}\pi^2}{\Lambda}\int d\tau\left[\frac{1}{2}(\dot{y}^2-\dot{x}^2)+x-3\right]. \tag{13}$$

The EOM with boundary conditions $x(0)=y(0)=0$, $x(v)=X$, $y(v)=Y$, where (X,Y) are specified by (b,χ) through (12), are solved by

$$x(\tau)=-\frac{\tau^2}{2}+A\tau, \tag{14}$$

$$y(\tau)=B\tau, \tag{15}$$

where

$$A=\frac{1}{v}\left(X+\frac{v^2}{2}\right),\qquad B=\frac{Y}{v}, \tag{16}$$

while the Hamiltonian constraint determines the possible values of the end point v in the complex τ plane. The solution of interest in the first quadrant is given by

$$v=\sqrt{C+D}+\sqrt{C-D}, \tag{17}$$

where

$$C=6-X,\qquad D=\sqrt{X^2-Y^2}. \tag{18}$$

The equation for the extremal curve in these coordinates reads [26]

$$\arg\gamma_{\mathsf{e}}'^2+2\arg a(\gamma_{\mathsf{e}})^2=\pi, \tag{19}$$

where, via (12), $a=[2(x^2-y^2)/3]^{1/4}$.

In the regime $C\geq D$, which essentially corresponds to $bH(\chi)<\mathcal{O}(1)$ where $H(\chi)=\sqrt{V(\chi)/3}$, v in (17) is seen to lie on the positive real axis. In this regime the metric (11) is purely Euclidean on the segment $[0,v]$. Hence the semiclassical amplitude of all such configurations (b,χ) is given by a no-boundary solution that meets the KS criterion.

In the regime $C\leq -D$ on the other hand, which corresponds to $bH>\mathcal{O}(1)$ and $b\chi>6^{3/4}$ when $\chi\ll 1$, v is purely imaginary. From (19) it follows there can be no curve that connects the origin to v along which the induced metric satisfies (2). Thus the KS criterion, taken at face value, appears to strongly suppress the semiclassical amplitude of this part of the minisuperspace, by excluding what would have been the leading saddle.

Finally, we have the intermediate regime $|C|<D$, which corresponds to $bH>\mathcal{O}(1)$ and $b\chi<6^{3/4}$ when $\chi\ll 1$, and which includes the de Sitter (dS) solution with $\chi=0$. Here v is neither real nor imaginary but complex. A solution for γ_{e} in (19) is given by $\gamma_{\mathsf{e}}'=i/a(\gamma_{\mathsf{e}})^2$, which upon integration gives the relation

$$\frac{1}{3}(\gamma_{\mathsf{e}}^2-A\gamma_{\mathsf{e}}-6B^2-12)\sqrt{\gamma_{\mathsf{e}}^2-4A\gamma_{\mathsf{e}}+24}$$
$$-4AB^2\tanh^{-1}\left(\frac{\gamma_{\mathsf{e}}-2A}{\sqrt{\gamma_{\mathsf{e}}^2-4A\gamma_{\mathsf{e}}+24}}\right)=i\sqrt{6}\ell+\mathrm{const}, \tag{20}$$

where the constant is determined by setting $\gamma_{\mathsf{e}}(0)=0$. To proceed, we set $\gamma_{\mathsf{e}}=v$ in (20) and equate the real parts of both sides. This yields a curve $\chi_\star(b)$ that indicates those points which the KS criterion marginally allows. Points lying above this curve in the (b, χ) plane are excluded while points below it are allowable. Asymptotically this critical line behaves as

$$\chi_\star(b) = \frac{6^{1/4}}{b\sqrt{\log b}}\left(1 + \mathcal{O}\left(\frac{1}{\log b}\right)\right) \quad \text{as } b \to \infty. \tag{21}$$

On the other hand, one can examine the set of classical histories predicted by Ψ in this model [20,25,27]. These are the curves $p_\alpha = \partial_\alpha \mathrm{Im} S$, where S is the action of the complex saddle and p_α, $\alpha \in \{a, \phi\}$, are the canonical momenta. In a 2D minisuperspace model of this kind, this is a one-parameter set of curves, or histories, in the (b, χ) plane. In the regime $|C| < D$, these histories are characterized by the relation

$$\chi_{\mathsf{classical}}(b) = c\frac{6^{3/4}}{b}\left(1 + \mathcal{O}(1/b)\right) \quad \text{as } b \to \infty \tag{22}$$

with $c \in [0, 1)$ labeling the history, where $c = 0$ corresponds to dS space [the dependence on b can be inferred from (4) after rotating $r \to it$]. Comparison with (21) shows that, strikingly, every classical history except empty dS exits the domain of allowability at some point. That is, the no-boundary state augmented by the KS criterion predicts that classical evolution does not continue forever in this model. It would be interesting to better understand whether this is a peculiar property of this particular model or a more general prediction of the KS criterion in conjunction with no-boundary conditions. In a realistic cosmology, however, this would require one to take into account the coupling of the inflaton to other forms of matter in order to evaluate the wave function well after inflation ends—and indeed at the present stage of evolution.

Slow-roll inflation.—We now turn to the no-boundary saddles that appear in slow-roll models of inflation [20,21]. We are especially interested in regions of the minisuperspace where the scale factor is large in local Hubble units, $bH(\chi) \gg 1$ with $H(\chi) \approx \sqrt{V(\chi)/3}$ in the slow-roll regime. Based on our results above, we expect that as the potential becomes flatter, or more precisely, as the background $a(r)$ approaches the form

$$a_{\mathsf{dS}}(r) = \frac{1}{H}\sin(Hr) \tag{23}$$

with H constant, more *e-folds* $\log bH(\chi)$ will be allowable.

The saddles $[a(r), \phi(r), v]$ that correspond to configurations (b, χ) along a slow-roll trajectory are complex deformations of the so-called real tunneling instanton (23) that describes the quantum creation of empty dS. In its familiar representation the dS saddle consists of half of a four-sphere of radius $1/H$, along the segment $[0, \pi/2H]$ of the real r axis, glued to the expanding branch of Lorentzian dS space along a segment parallel to the imaginary r axis [28]. No-boundary saddles associated with slow-roll inflationary universes typically involve half of a deformed S^4 with an approximate radius $1/H(|\phi_0|)$, which transitions to a slow-roll attractor in the imaginary r direction. Importantly, $a(r)$ and $\phi(r)$ are purely real along neither segment except if the inflaton starts out at an extremum of V. Instead, along the approximately Lorentzian direction, the imaginary parts decay in a way dictated by the real parts, whose evolution is governed by the usual slow-roll approximation, viz. $\mathrm{Im}a \sim (\mathrm{Re}a)^{-2}$, $\mathrm{Im}\phi \sim (\mathrm{Re}a)^{-3}$ [21]. It is this mere approximate reality of the fields that can cause the KS criterion to fail for certain configurations (b, χ) [29].

We proceed by solving the equations governing the background (4)–(6) and the extremal curve (9) numerically [22]. A trustworthy analysis of the KS criterion requires exponential numerical precision. This can be seen even from the pure dS saddle, where the extremal curve asymptotes to the vertical line $\mathrm{Re}r = \pi/2H$ on which the end points $v(b)$ are located, as

$$\mathrm{Re}\left(\frac{\pi}{2H} - \gamma_{\mathsf{e}}\right) = \mathcal{O}\left(\frac{1}{H}\exp(-3H\mathrm{Im}\gamma_{\mathsf{e}})\right) \quad \text{as } \mathrm{Im}\gamma_{\mathsf{e}} \to \infty. \tag{24}$$

Interestingly, this is the sort of level of detail through which no-boundary saddle-point geometries in the large-volume regime encode the fine details of the quantum origin of inflation. Hence, physically the required accuracy stems from the fact that KS is a global criterion on complex saddles that probes the quantum nature of inflation, even at late times.

We would like to determine in which models saddles corresponding to inflationary histories with $N_e = \mathcal{O}(50-60)$ *e-folds* meet the KS criterion. To identify these models we first pick a potential and fix χ to its value at the end of inflation, where $\varepsilon = (V'/V)^2/2$ or $|\eta| = |V''|/V$ are equal to unity. Then we vary $\log bH$ between 50 and 60. Finally we use the method described above to verify whether the no-boundary saddle corresponding to these configurations (b, χ) is allowable.

We carried out this procedure for most of the inflationary potentials discussed in the 2018 Planck analysis [30]. We ensured all the numerics are trustworthy by dialing up the precision of our numerical algorithm [22] and observing convergence in the results. Note that the KS criterion does not depend on the overall scale of the potentials [22], which may thus be adjusted to match the observed amplitude of CMB fluctuations.

We summarize some of our results in Table I, where for eight one-parameter potentials we list the ranges of parameter values $f, \mu, \ldots$ for which the KS criterion applied to saddles with $N_e = 60$ *e-folds* is satisfied. As an example, consider the power-law potentials $V \propto \phi^p$. Whereas the KS criterion allows inflationary histories with $N_e = 60$ for

TABLE I. Families of slow-roll potentials in which we subjected the no-boundary instantons giving rise to 60 *e-folds* of inflation to the KS criterion. In the allowable column we list the ranges of parameter values $f, \mu, \ldots$ that specify potentials in which configurations (b, χ) at the end of 60 *e-folds* of inflation, prepared by no-boundary conditions, satisfy the KS criterion. The "disallowable" column lists parameter values for which the criterion is not satisfied.

No.	V/Λ	Allowable	Disallowable
①	$1+\cos(\phi/f)$	[2, 6.09)	[6.09, 10]
②	$1-\phi^2/\mu^2$	$[10^{1/2}, 10^4]$	
③	$1-\phi^4/\mu^4$	$[10^{-1}, 10^2]$	
④	$1-\exp(-q\phi)$	$[10^{-3}, 10^3]$	
⑤	$1-\mu^2/\phi^2$	$[10^{-6}, 10^3]$	
⑥	$1+\alpha\log\phi$	$[10^{-3}, 10]$	
⑦	$[1-\exp(-\sqrt{2}\phi/\sqrt{3\alpha})]^2$	$[10^{-1}, 93.9)$	$[93.9, 10^4]$
⑧	ϕ^p	$[1/2, 1.05)$	$[1.05, 7/2]$

$p \lesssim 1.05$, for larger values of p we find that all slow-roll saddles exit the regime of allowability before the end of inflation [31]. In general, the table shows that the KS criterion selects those universes in the no-boundary state that emerge on a *concave* patch of the scalar slow-roll potential, with an additional model-dependent pressure towards lower values of r. This in turn favors small-field models of inflation [32].

Figure 1 gives a representation of these KS constraints in terms of predictions for two key observables associated with the spectrum of CMB fluctuations generated by inflation. The figure shows the values of the scalar tilt n_s and the tensor-to-scalar ratio r predicted by the eight different inflationary models listed in the table above. (Encircled numbers in Fig. 1 correspond to the number of the model in the table.) Inflationary universes that remain allowable by KS for 50–60 *e-folds* are indicated in green whereas those for which the KS criterion fails before the end of inflation are shown in red. Superposed on these theoretical predictions are the observational constraints following from the 2018 Planck TT, TE, EE + lowE + lensing analysis [30] (68% and 95% confidence levels), indicated in purple, and, in blue, the constraints with the combined 2018 BICEP/Keck data and BAO added [33]. We see that in this set of models, which we believe to be representative, the KS criterion translates into an upper bound on the tensor-to-scalar ratio of $r \lesssim 0.08$.

Discussion.—We have given strong evidence that the semiclassical no-boundary wave function, augmented with the Kontsevich-Segal criterion, selects inflationary models with a relatively low tensor-to-scalar ratio $r \lesssim 0.08$ in the microwave background anisotropies. This upper bound on r is in accordance with the current observational constraints, yet it leaves room for a future detection of gravitational waves from inflation.

Our results indicate that the KS criterion can be viewed as a refinement of the no-boundary theory of the quantum state that sharpens its predictions. In models of inflation with larger values of r, no-boundary saddles fail to satisfy the KS criterion when the universe becomes large. One might wonder what can possibly cause the KS criterion to fail during the quasiclassical slow-roll phase. Among the p-form criteria in (1) it turns out that the "0-form" criterion $\mathrm{Re}\sqrt{g} > 0$ fails. This criterion, which is related to the convergence of the path integral of a massive scalar on $(M = B^4, g)$, is saturated by the extremal curve described by (9) whereas the higher-form criteria are not. The failure of allowability during inflation can thus be attributed to the late-time development of a tachyon in the spectrum of scalar perturbations around the Hartle-Hawking solution. This being said, our analysis indicates that the KS criterion does probe the fine details of the quantum origin of inflation, for the latter are encoded precisely in the exponentially small corrections to the late-time saddle-point geometries that determine whether or not they satisfy the criterion. Indeed our results lend further credence to the raison d'être of quantum cosmology, namely, that a quantum gravitational completion of inflation can have verifiable observational consequences.

As an aside, we note that the alternative tunneling wave function of the Universe [34], constructed via a gravitational path integral [35], fails to meet the KS criterion. The semiclassical tunneling wave function involves gravitational instantons that belong to the fourfold degenerate family of no-boundary solutions that we discussed below (8). When evaluating their action, however, one chooses the opposite sign for $\mathrm{Re}\sqrt{g}$ along the curve $\gamma\colon 0 \to v$ compared to what the KS criterion demands, viz. $\mathrm{Re}\sqrt{g} < 0$ instead of $\mathrm{Re}\sqrt{g} > 0$. This is consonant with the observation that the "naive" wave function of fluctuations in the tunneling state appears to be non-normalizable (cf. [7]). Instead it appears that a well-behaved wave function of fluctuations in the tunneling state would have to be based on a complexified integration contour for matter field fluctuations [35], thereby evading the KS criterion altogether.

Ultimately, the utility of quantum cosmology lies in the fact that a theory of the quantum state combined with the structure of the low-energy scalar potential yields a cosmological measure that specifies a theoretical prior for observations (see e.g., [36–38]). In this Letter we have considered but the simplest quantum completion of inflation, in which an observable phase of slow-roll emerges directly from a no-boundary origin. It would be interesting to study the implications of the KS criterion in more elaborate models of initial conditions. For example, one could take a more expansive view and conceive of the range of models of inflation as different slow-roll patches in a landscape potential. In this context, the no-boundary amplitude of different backgrounds and fluctuations implies a relative weighting over different landscape regions and hence over cosmological observables that differentiate between regions. Crucially, predictions for observations follow from conditional probabilities. The "bare" no-boundary weighting favors backgrounds starting at a low value of the potential, followed by only a few *e-folds* of slow-roll inflation. However, no-boundary probabilities conditioned on a sufficiently accurate description of our observational situation favor slow-roll backgrounds originating on a flat plateau-like patch of the scalar potential where the conditions for eternal inflation hold [18,39]. In future work we intend to extend our analysis into this regime and determine whether our findings are sharpened or modified in this more elaborate setting.

It would also be interesting to understand how the KS criterion relates to the swampland program. At first sight there appears to be a certain tension between both approaches, because KS appears to favor near-de Sitter saddles whereas the swampland points towards a short-lived inflationary phase. On the other hand, both considerations seem to align on a relatively low tensor-to-scalar ratio. It would be very interesting to study whether KS and the swampland are somehow two different ways of saying the same thing, or whether they are genuinely at odds with one another.

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[9] Metrics that satisfy the criterion can always be diagonalized in a real basis, but the metrics we will consider in the following are already in diagonal form from the get-go. Notice that the criterion (1) should hold pointwise on M. Further, the criteria arising from $p = q$ and $p = (D - q)$-forms are equivalent: The former has q minus signs in the sum $|\pm \arg \lambda_1 \pm \arg \lambda_2 \pm \cdots \pm \arg \lambda_D| < \pi$ while the latter has $D - q$. Finally, the $p = 0$ criterion reads $\mathrm{Re}\sqrt{g} > 0$ and can be motivated from the convergence of the path integral for a massive scalar on (M, g).

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[23] The semiclassical, KS-corrected NBWF need not vanish in the regime where KS excludes the leading saddle, since there may be another instanton with less symmetry and a

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[29] That this is consistent follows from the following argument: Suppose $a(x+i\ell)$ is real and increasing for all ℓ at a fixed x. Then $a'(x+i\ell)$ is negative imaginary. To solve for the extremal curve we write $\gamma_{\mathrm{e}}(\ell)=x-\varepsilon(\ell)+i\ell$ and expand at large ℓ, assuming ε is small, and initially positive, so that $a(\gamma(\ell))\approx a(x+i\ell)-a'(x+i\ell)\varepsilon$. With this one checks that the allowability criterion (2) becomes $\varepsilon'\approx -3(-\mathrm{Im}(a')/a)\varepsilon$, so that indeed ε is decreasing, perhaps to zero, but does not change sign.

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