Asymptotic Quantum Many-Body Scars

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We consider a quantum lattice spin model featuring exact quasiparticle towers of eigenstates with low entanglement at finite size, known as quantum many-body scars (QMBS). We show that the states in the neighboring part of the energy spectrum can be superposed to construct entire families of low-entanglement states whose energy variance decreases asymptotically to zero as the lattice size is increased. As a consequence, they have a relaxation time that diverges in the thermodynamic limit, and therefore exhibit the typical behavior of exact QMBS, although they are not exact eigenstates of the Hamiltonian for any finite size. We refer to such states as *asymptotic* QMBS. These states are orthogonal to any exact QMBS at any finite size, and their existence shows that the presence of an exact QMBS leaves important signatures of nonthermalness in the rest of the spectrum; therefore, QMBS-like phenomena can hide in what is typically considered the thermal part of the spectrum. We support our study using numerical simulations in the spin-1 XY model, a paradigmatic model for QMBS, and we conclude by presenting a weak perturbation of the model that destroys the exact QMBS while keeping the asymptotic QMBS.

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Introduction.—Quantum many-body scars (QMBS) [1–4] in non-integrable quantum lattice models of any dimension are one of the paradigms for the weak violation of the eigenstate thermalization hypothesis (ETH) [5,6], according to which all local properties of energy eigenstates in the middle of the spectra of nonintegrable models coincide with those of a thermal Gibbs density matrix at a suitable temperature [7–10]. QMBS are isolated energy eigenstates that are outliers in many respects, e.g., in the expectation value of a local observable or in the entanglement entropy. Numerous instances of lattice models featuring exact towers of QMBS at finite size have been discovered [11–23]. Most of these results have also been understood via unified frameworks or systematic construction recipes [3,13,17,20,24–30].

A question that has been less explored is whether the presence of a finite-size QMBS affects the properties of the rest of the spectrum. Reference [31] pointed out the existence of low-entanglement states in the PXP model which exhibit slow relaxation even though they are orthogonal to the known exact QMBS: the energy variance of such states is independent of system size and thus their fidelity relaxation time does not decrease [32]. This is a remarkable phenomenology to be contrasted with that of short-range correlated states, whose energy variance grows with system size, whereas the fidelity relaxation time decreases.

Are there even more drastic examples of slowly relaxing states [33], for instance, with an energy variance decreasing with system size, which would lead to a relaxation time that *diverges polynomially* in the thermodynamic limit (TL)? Slow relaxation of hydrodynamic origin is ubiquitous in systems with continuous symmetries, where it occurs at a diverging timescale known as the *Thouless time* [34–37], and is related to diffusion or subdiffusion [38–43]. The interpretation of QMBS as an unconventional nonlocal symmetry [29,44] motivates the search for such slow relaxation. Long-lived quasiparticles, e.g., the phonons of a superfluid with Beliaev decay [45], also induce slow relaxation. QMBS are associated with quasiparticles with specific momenta and infinite lifetime [4], hence it is natural to look for long-lived quasiparticles at neighboring momenta.

In this Letter we address these questions by considering the spin-1 XY model featuring exact QMBS at any finite size [14] and show that it is possible to construct slowly relaxing low-entanglement initial states that exhibit QMBS-like features, but nevertheless are orthogonal to the exact QMBS. They have an energy variance that goes to zero in the TL and asymptotically display the typical dynamical phenomenology of a QMBS, i.e., the lack of thermalization; hence we refer to such initial states as *asymptotic* QMBS. Our work widens the range of initial states that qualitatively exhibit a nonthermalizing phenomenology and motivates the search for nonthermal features in regions of the spectrum where entanglement signatures do not make them evident.

The model and the exact QMBS.—We consider a one-dimensional spin-1 chain of length L even, and consider a spin-1 XY model with external magnetic field and axial anisotropy:

$$H = J\sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + h\sum_j S_j^z + D\sum_j (S_j^z)^2 + J_3\sum_j (S_j^x S_{j+3}^x + S_j^y S_{j+3}^y), \quad (1)$$

where S_j^α, with $\alpha = x, y, z$, are the spin-1 operators on site j. We use open boundary conditions (OBC) for the numerical simulations and periodic boundary conditions (PBC) for some of the analytical results. This model with OBC has been numerically shown to be nonintegrable; the last term breaks a hidden nonlocal symmetry [4,14,46].

The Hamiltonian in Eq. (1) exhibits QMBS for any finite value of L [14]. In order to see that, we define the fully polarized state $|\Downarrow\rangle = |- - \cdots - -\rangle$ with all spins in the eigenstate of S_j^z with eigenvalue -1, and the operator

$$J_k^+ = \frac{1}{2}\sum_{j=1}^{L} e^{ikj}(S_j^+)^2. \quad (2)$$

The scar states read

$$|n,\pi\rangle = \frac{1}{\sqrt{N_{n,\pi}}}(J_\pi^+)^n|\Downarrow\rangle, \quad (3)$$

where $N_{n,\pi}$ is a normalization constant. The state satisfies the energy eigenvalue equation $H|n,\pi\rangle = (-Lh + 2nh + LD)|n,\pi\rangle$ and for generic values of h and D it lies in the middle of the Hamiltonian spectrum. Its existence is related to quantum interference effects, similar to those that are responsible for the existence of η-pairing states in the Hubbard model [16].

Moreover, it is possible to consider the reduced density matrix $\rho_{A,n,\pi}$ of $|n,\pi\rangle$ defined on half the system (conventionally, the region A is $1 \le j < L/2$), and to compute its entanglement entropy, $S_{n,\pi} = -\mathrm{tr}[\rho_{A,n,\pi}\log\rho_{A,n,\pi}]$. The explicit calculation has been done in Ref. [14], and it shows that it scales as $\log L$, displaying a mild logarithmic violation of an entanglement area law; see Supplemental Material [47] and Ref. [48] for details. QMBS are easily found numerically by plotting the entanglement entropy S_{E_i} of ρ_{A,E_i}, the reduced density matrix of the eigenstate $|E_i\rangle$, as a function of energy. Indeed, almost all the eigenstates appear to satisfy the ETH and are characterized by an S_{E_i} that is only a function of the energy E_i; they have a higher amount of entanglement than the QMBS states, which indeed violate ETH.

A family of states obtained by deforming the exact QMBS.—We now consider other initial states for the dynamics of the model in Eq. (1); they read as follows:

$$|n,k\rangle = \frac{1}{\sqrt{N_{n,k}}}J_k^+(J_\pi^+)^{n-1}|\Downarrow\rangle, \quad (4)$$

where $N_{n,k}$ is a normalization constant, and they coincide with the exact QMBS in Eq. (3) when $k = \pi$. When $k \neq \pi$ and is an integer multiple of $(2\pi/L)$, they are *orthogonal* to the exact QMBS: the relation $\langle n,k|n',\pi\rangle = \delta_{n,n'}\delta_{k,\pi}$ for any $1 \le n,\ n' \le L-1$ is proved in the Supplemental Material [47]. Models where such classes of multimagnon states are exact eigenstates have been studied in [49], however, for $k \neq \pi$ these are *not* eigenstates of the spin-1 XY model. It is easy to show that the average energy of these states does not depend on k and reads $\langle n,k|H|n,k\rangle = -Lh + 2nh + LD$ [47].

Furthermore, the entanglement of the states in Eq. (4) scales with system size as a subvolume law. For a quick proof, since $|n,k\rangle \propto J_k^+|n-1,\pi\rangle$, we note that J_k^+ can be straightforwardly expressed as a matrix product operator (MPO) of bond dimension $\chi = 2$ [3,50,51], hence the half-subsystem entanglement entropies of $|n-1,\pi\rangle$ and $|n,k\rangle$ can differ at most of an additive term log 2. In other words, since the operator J_k^+ can be split in two terms, one acting on $j < L/2$ and one on $j \ge L/2$, it is possible to show [47] that the total number of Schmidt states in $|n,k\rangle$ is at most twice than that in $|n-1,\pi\rangle$.

To further characterize the states in Eq. (4), we compute the variance of the energy ΔH^2 under the Hamiltonian H in PBC, and as we show in the Supplemental Material [47], we obtain

$$\Delta H^2 = 4\left[J^2\cos^2\left(\frac{k}{2}\right) + J_3^2\cos^2\left(\frac{3k}{2}\right)\right]. \quad (5)$$

Among the states defined in Eq. (4), the $|n,\pi\rangle$ are the only eigenstates of the Hamiltonian, because $\Delta H^2 = 0$ only for $k = \pi$. When $k \neq \pi$, $|n,k\rangle$ must be a linear superposition of the energy eigenstates of H, which are mostly in a window centered around the same energy of $|n,\pi\rangle$ and in a width of about ΔH. When $k \neq \pi$ is chosen to be an integer multiple of $(2\pi/L)$, $|n,\pi\rangle$ is not part of this set of states due to orthogonality. Since $|n,\pi\rangle$ numerically appear to be the only exact QMBS of H [14], we conclude that such states $|n,k\rangle$ must be a linear superposition of “thermal” eigenstates, i.e., those that are typically said to satisfy ETH, having an entanglement entropy and expectation values of local observables that are smooth functions of energy.

We have numerically verified this statement using the python-based package QuSpin [52]: we diagonalize the Hamiltonian (1) and compute the bipartition entanglement entropy S_{E_i} and the average square magnetization $S_{E_i}^{z2} = (1/L)\sum_j\langle(S_j^z)^2\rangle$ of all eigenstates. Subsequently, we compute the scalar product of the state $|n,k\rangle$ with all eigenstates for $n = L/2$ and $k = \pi - (2\pi/L)$ and look at the properties of the eigenstates with whom the overlap is not zero. The results are reported in Fig. 1, and support our thesis.

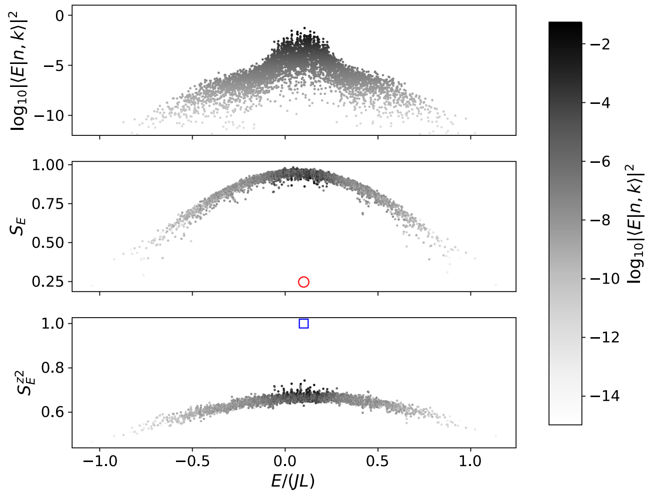

FIG. 1. Top: Squared overlap of $|n,k\rangle$ for $n = L/2$ and $k = \pi - (2\pi/L)$ with the eigenstates $|E_i\rangle$ of Hamiltonian (1) with zero magnetization, $S_z = 0$; the parameters of the simulation are $\{J, h, D, J_3\} = \{1, 0, 0.1, 0.1\}$ and $L = 10$. The information on $|\langle E_i|n,k\rangle|^2$ is also encoded in the color code of the marker of all panels using a logarithmic scale, see color bar. Middle and bottom: We plot the data of the top panel in a diagram with the energy E on the abscissa and the bipartition entanglement entropy S_E or the average square magnetisation $S^{z2}(E)$ of the eigenstate on the ordinate, respectively. For the entanglement entropy, we use the natural logarithm and we divide the result by $L/2$ to obtain an intensive quantity. The state $|n,k\rangle$ has overlap only with states whose S_{E_i} or $S^{z2}_{E_i}$ lies on the continuous thermal curve. The red circle and the blue square highlight the regions of the plots where the QMBS $|n = L/2, \pi\rangle$ appear: the absence of any gray mark means that the scalar product is compatible with the numerical zero.

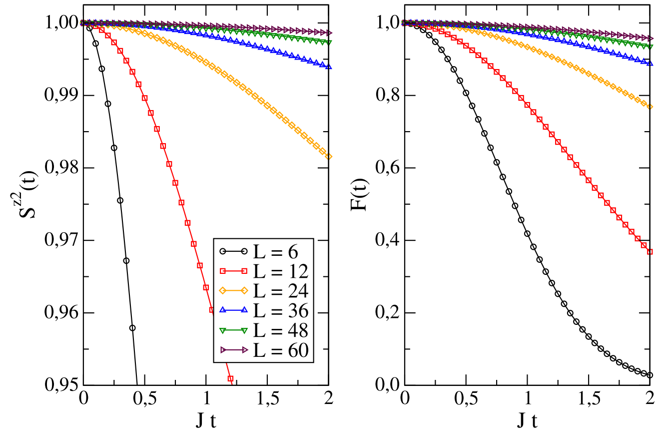

FIG. 2. The properties of the state $e^{-iHt}|n,k\rangle$ for $n = L/2$ and $k = \pi - 2\pi/L$ as a function of time for various system sizes L. Left: time evolution of the squared magnetisation $S^{z2}(t)$. Right: time evolution of the fidelity with the initial state $F(t)$.

Dynamics and asymptotic QMBS.—The dynamical properties of the states $|n,k\rangle$ for large system sizes depend on how we approach $L \to \infty$. If the limit is taken while the momentum k is held fixed, then the variance is finite in the TL (see Ref. [31] for examples in the PXP model). Loosely speaking, we can invoke the well-known energy-time uncertainty relation, linking the typical timescale of the dynamics τ of a quantum state to the fluctuations of the energy:

$$\tau \geq \frac{\hbar}{2\Delta H}, \tag{6}$$

to claim that for these states the dynamics is frozen up to a given timescale τ that is independent of L and that afterwards an evolution towards thermal equilibration takes place [47]. To be more precise, the energy variance ΔH^2 of the initial state determines the fidelity relaxation time $\tau \sim 1/\Delta H$ [53], since the fidelity $F(t) = |\langle\Psi|e^{-iHt}|\Psi\rangle|^2$ of an initial state $|\Psi\rangle$ decays at short times as $\sim \exp(-\Delta H^2 t^2)$; τ is a lower bound for the relaxation time of local observables [10,54].

Another class of states can be obtained by approaching the TL while letting k flow to π. This can be done by setting $k = \pi + (2\pi/L)m$, with the coefficient $m \in \mathbb{Z}$ kept constant while $L \to \infty$. In this case the energy variance scales as $\Delta H^2 \sim (J^2 + 9J_3^2)(k-\pi)^2$ and tends to zero as $1/L^2$. We refer to this second class of states as *asymptotic QMBS* of the model, since according to (6), the typical relaxation timescale of their dynamics scales as $\tau \sim L$, i.e., the system is frozen for timescales that *increase* polynomially with the system size. On the contrary, low entanglement states, by virtue of their diverging variance [33], are typically expected to lose fidelity on timescales that decrease with system size, and the expectation values of typical observables relax in timescales that do not change drastically with system size [10,55–61]. Hence the dynamics of this class of states asymptotically approaches QMBS-like behavior even though they are *not* exact QMBS of the system at finite size, and moreover they are *orthogonal* to all the exact QMBS $|n,\pi\rangle$. To the best of our knowledge, this phenomenology has never been discussed before.

We support the previous statements with a numerical simulation of the dynamics of the states $|n,k\rangle$ under the action of H using a time-evolving block decimation code based on a matrix-product-state representation of the state obtained via the ITensor library [62,63]. We consider in particular the state $|n = L/2, k = \pi - 2\pi/L\rangle$ for several system sizes up to $L = 60$ and truncation error 10^{-12}. We then compute the observable $S^{z2}(t) = (1/L)\sum_j \langle (S^z_j)^2\rangle_t$ and the fidelity of the time-evolved state with the initial state $F(t) = |\langle n,k|e^{-iHt}|n,k\rangle|^2$. The results, reported in Fig. 2, show in both cases an important slow-down of the dynamics as the size increases. In the Supplemental Material we show that the data concerning the fidelity can be collapsed via a rescaling of time by a factor of L [47], which suggests the divergence of the relaxation time in the TL. The result on the fidelity $F(t)$ shows undoubtedly that the time-evolved state maintains an overlap with the initial state that increases with L and it implies the freezing of the state. In the Supplemental Material we complement this analysis by contrasting it with the typical dynamics of other states [47]; we also analyze states obtained by acting on the exact QMBS with $(J_k^+)^m$, i.e., creating multiple quasiparticles of momenta close to π, and we argue that they should also be asymptotic QMBS as long as m does not scale with L [47].

Slow relaxation and nonthermalness in the middle of the energy spectrum.—Two properties make the asymptotic QMBS particularly interesting: (a) they have a limited amount of entanglement, i.e., a subvolume law, but an extensive amount of energy; (b) they have an energy variance ΔH^2 that drops fast enough to zero in the TL. Any state that satisfies these conditions is guaranteed to have a long relaxation time, both in the fidelity and in the observables, while having an average energy that lies in the middle of the Hamiltonian spectrum. Note that both (a) and (b) are necessary features that make the behavior of asymptotic QMBS atypical. While any linear superposition of thermal eigenstates with small energy variance relaxes slowly, it typically has a large entanglement [33]. On the other hand, a typical low-entanglement state has an energy variance that increases with system size [33].

It is tempting to think that the existence of asymptotic QMBS should imply some kind of "non-thermalness" [31] or ETH-violation in the "thermal" states orthogonal to the exact QMBS, even at finite system size. Note that ETH consists of two parts [6,9,64], pertaining to diagonal and off-diagonal matrix elements of a local operator in the energy eigenbasis. The diagonal matrix elements control the late-time expectation values of observables, and the existence of asymptotic QMBS does not imply any violation of diagonal ETH since we expect them to eventually thermalize for any finite system size. On the other hand, the timescale of relaxation is controlled by both the energy variance of the initial state and the off-diagonal matrix elements [60]. It is plausible that our result entails a violation of off-diagonal ETH at least in a part of the Hamiltonian spectrum.

Asymptotic QMBS without exact QMBS.—Our definition of asymptotic QMBS is based on a deformation of the tower of exact QMBS supported at finite size; it is not clear whether asymptotic QMBS can exist in models without any exact QMBS or at energies distant from those of the exact QMBS.

We now show that it is possible to weakly perturb the Hamiltonian H in a way that destroys all exact QMBS, but such that the perturbed model maintains the asymptotic QMBS. As an example, we consider $H' = H + V$ with $V = (J_z/L)\sum_j S_j^z S_{j+1}^z$, which is still a nontrivial local perturbation since its spectral norm $\|V\|_\infty$ corresponding to its largest singular value is subextensive and scales as $O(1)$. Using the PYTHON-based QuSpin package [52], we numerically diagonalize H' and compute the entanglement entropy S_{E_i} and the average square magnetization $S^{z2}(E_i)$ for all eigenstates. The plots, in Fig. 3, do not indicate the presence of any exact QMBS.

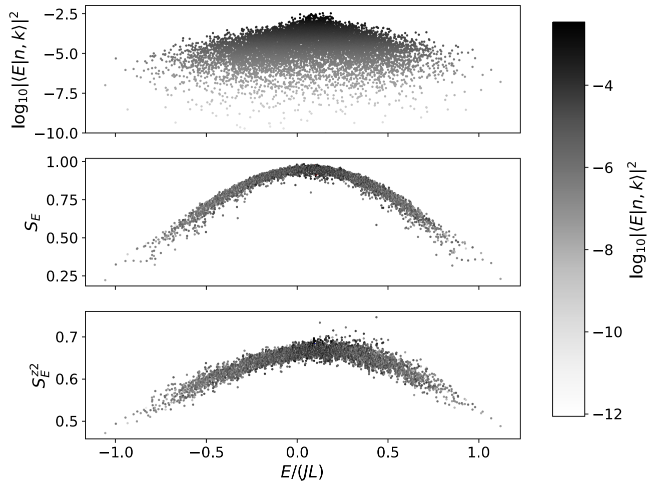

FIG. 3. Properties of the eigenstates of Hamiltonian H' in the zero magnetization sector $S_z = 0$; the parameters of the simulation are $\{J, h, D, J_3, J_z\} = \{1, 0, 0.1, 0.1, 1\}$ and $L = 10$. Top: Squared overlap of $|n,\pi\rangle$ for $n = L/2$ with the eigenstates $|E_i\rangle$ of Hamiltonian H' with zero magnetisation, $S_z = 0$. The information on $|\langle E_i|n,\pi\rangle|^2$ is also encoded in the color code of the marker of all panels using a logarithmic scale, see color bar. Middle and bottom: We plot the data of the top panel in a diagram with the energy E on the abscissa and the bipartition entanglement entropy S_E or the average square magnetisation $S^{z2}(E)$ of the eigenstate on the ordinate, respectively. The state $|n,\pi\rangle$ has overlap only with states whose S_{E_i} or $S^{z2}(E_i)$ lies on a continuous curve. In the Supplemental Material [47] we show the entire spectrum and show that the model does not have any QMBS (here the spectrum is incomplete because we plot only state that have a non-negligible overlap with $|n,\pi\rangle$).

We now consider the state $|n,\pi\rangle$ of Eq. (3), which is an exact QMBS of H but *not* an eigenstate of H'. Using the ITensor library [62,63], we compute $S^{z2}(t)$ and the fidelity $F(t)$ for the time-evolved state $|\Psi(t)\rangle = e^{-iH't}|n,\pi\rangle$; the results are in Fig. 4. The plots display the phenomenology of an asymptotic QMBS in a Hamiltonian that does not show any exact QMBS at finite size, and the $F(t)$ curves exhibit a collapse when time is rescaled by a factor $\sqrt{L}$ [47], indicating a diverging relaxation time. This behavior can be directly attributed to the fact that the variance of the state $|n,\pi\rangle$ under the Hamiltonian H' scales as $\sim 1/L$ when n is a finite fraction of L, as it is proven in the Supplemental Material [47].

Conclusions.—In this Letter we revisited the paradigmatic one-dimensional spin-1 XY model that supports exact QMBS at finite size, and we explored the properties of the rest of the spectrum. We showed that it is possible to construct other states, dubbed asymptotic QMBS, with little entanglement and whose relaxation time diverges

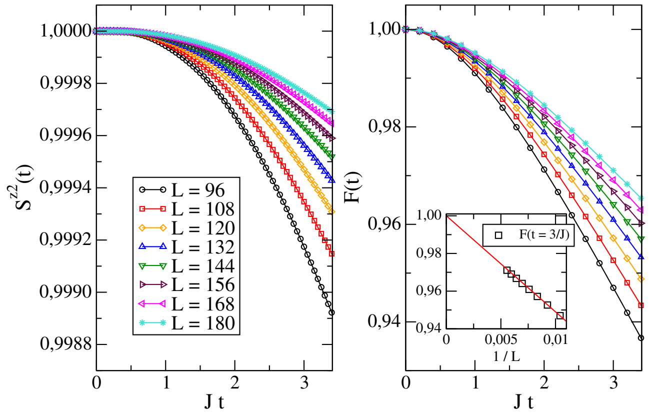

FIG. 4. The properties of the state $e^{-iH't}|n,\pi\rangle$ for $n = L/2$ as a function of time; the parameters of the Hamiltonian employed in the simulation are the same of Fig. 3. Left: time evolution of the squared magnetization $S^{z2}(t)$; right: time evolution of the fidelity with the initial state $F(t)$. The inset shows the scaling as a function of size of the values of $F(t = 3/J)$; we find a scaling to 1 as $1/L \to 0$.

polynomially in the thermodynamic limit. These asymptotic QMBS indicate the existence of slowly relaxing modes and novel long-lived quasiparticles in systems with exact QMBS; it would be interesting to understand their relations to analogous slowly relaxing modes of hydrodynamic origin.

Remarkably, asymptotic QMBS are linear combinations of thermal eigenstates whose entanglement entropy and average squared magnetization are "smooth" functions of energy; we leave for future work the investigation of a possible violation of off-diagonal ETH [65–71].

Asymptotic QMBS with similar properties can also be constructed in higher dimensional spin-1 XY models [47], but other extensions would also be interesting, considering first the exhaustive algebra of local Hamiltonians that have the same exact QMBS $|n,\pi\rangle$ [29]. Second, they likely can always be constructed in Hamiltonians with simple quasiparticle towers of exact QMBS [11–13,17–19,25,72]. Third, there are many different types of exact QMBS [3], e.g., with non-local "quasiparticles" [13,27,73], or with nonisolated states [24,32]; they could appear in gauge theories [74,75] or Floquet systems [28,76–78]. Are there asymptotic QMBS in these models?

Finally, one could also consider deformations of Hamiltonians with exact QMBS (a problem that we partially addressed in the final part of this Letter), and ask what are the conditions for a Hamiltonian to display an asymptotic QMBS without any exact QMBS.

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