Scrambling Is Necessary but Not Sufficient for Chaos

Neil Dowling,[1,*] Pavel Kos,[2] and Kavan Modi[1,3]
[1]*School of Physics & Astronomy, Monash University, Clayton, Victoria 3800, Australia*
[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching*
[3]*Quantum for New South Wales, Sydney 2000 New South Wales Australia*

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We show that out-of-time-order correlators (OTOCs) constitute a probe for local-operator entanglement (LOE). There is strong evidence that a volumetric growth of LOE is a faithful dynamical indicator of quantum chaos, while OTOC decay corresponds to operator scrambling, often conflated with chaos. We show that rapid OTOC decay is a necessary but not sufficient condition for linear (chaotic) growth of the LOE entropy. We analytically support our results through wide classes of local-circuit models of many-body dynamics, including both integrable and nonintegrable dual-unitary circuits. We show sufficient conditions under which local dynamics leads to an equivalence of scrambling and chaos.

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Introduction.—The question of quantum chaos is a long-standing issue. In recent years, a wide plethora of apparently inequivalent notions of quantum chaos have appeared [1–17]. Among them, the most well-known defining features of quantum chaotic models are universal spectral fluctuations, which match those of random matrix theory. They were shown to arise for systems with chaotic semiclassical limits [1,2]. In the absence of this limit, universal fluctuations were subsequently used as a definition of chaos and were recently extensively investigated in many-body settings [3–5,18–29].

Nevertheless, the spectral definition of quantum chaos comes with a few limitations. First, taking the thermodynamic limit is nontrivial, as the discrete spectrum becomes hard to treat and is hard to access experimentally and analytically. The spectrum might not even be defined, for instance, in time-dependent evolution. Therefore, a dynamical indicator of chaos, which is well defined for infinite systems and finite times, is clearly an attractive prospect. A popular candidate is the out-of-time-ordered correlator (OTOC), which measures a notion of scrambling in many-body systems [30–38]. But its definition of "chaos" does not agree with the spectral one [39–44]. On the other hand, a less studied signature of chaos which, in all known examples, agrees with the spectral definition is local-operator entanglement (LOE). It is a well-justified measure of dynamical complexity and quantum chaos [7,8].

In this Letter, we give a novel interpretation for the OTOC by showing that it serves as a probe of LOE. In doing so, we will uncover simple cases where a dynamics is scrambling as signified by an exponential OTOC scaling, yet is not chaotic as demonstrated by the absence of linear LOE entropy growth. Along the way, we derive exact analytical results for a class of many-body local circuits [45,46], including a novel exact computation of the OTOC which is of independent interest. Our results show that scrambling is necessary but not sufficient for quantum chaos.

The LOE is a measure of the complexity scaling of a Heisenberg evolved operator $V_t \coloneqq U_t^\dagger (V \otimes \mathbb{1}_{\bar{B}}) U_t$ for a local operator V [7]. We consider an arbitrary isolated system with finite local Hilbert space dimension. V acts locally on a space $\mathcal{H}_B$, whereas V_t generally has support on the full system $\mathcal{H}_S = \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$. Above, U_t is an arbitrary time evolution operator.

Specifically, the LOE is the entanglement of the Choi state of an initially local, unitary, and traceless Heisenberg operator V_t,

$$|V_t\rangle \coloneqq (V_t \otimes \mathbb{1})|\phi^+\rangle, \tag{1}$$

where $|\phi^+\rangle$ is the maximally entangled state over a doubled space. As this is a pure quantum state, we can analyze its static quantum mechanical properties such as its entanglement (LOE). This entanglement can be computed across any bipartition and for any appropriate metric, such as k-Rényi entanglement entropy.

Despite not being as popular as notions of chaos based on Hamiltonian or Floquet spectra [1,2], LOE is an attractive candidate for a dynamical signature of quantum chaos in the context many-body systems. In particular (i) it classifies the hardness of simulating the operator Heisenberg dynamics with tensor networks [8], (ii) a wide range of studies into physical models support that volume-law LOE signifies nonintegrability, with it scaling at most logarithmically with time for (interacting) integrable systems [8,11,47–51], and (iii) it can be understood as a sensitivity to perturbation, analogous to the classical butterfly effect [17]. Note that the entanglement of *states* in a quantum many-body system is not a signature of chaos, with even free models generally exhibiting a linear growth [52,53]. Further, the LOE should not be confused with the related quantity of the "operator entanglement," which is the entanglement of the full, global unitary evolution operator [54,55]. This quantity generally scales linearly with t irrespective of integrability [48], unless the Hamiltonian is in a localized phase [56].

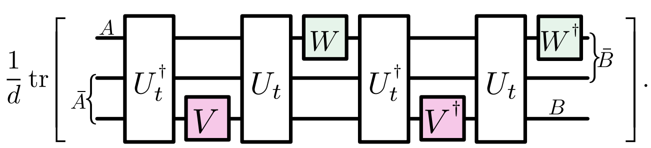
In comparison, OTOC scaling generally indicates operator scrambling, and is defined as a four-point correlator with atypical time ordering [30–33],

$$F(W, V_t) = \frac{1}{d}\mathrm{tr}[W^\dagger V_t^\dagger W V_t], \tag{2}$$

where we compute this expectation value over the maximally mixed state $\rho_\infty = \mathbb{1}/d$. We take V and W to be local unitaries, which wlog are traceless [57]. In this case, the OTOC quantifies how much V_t and W do not commute as a function of time, $\mathrm{Re}[F(W, V_t)] = 1 - \frac{1}{2}\langle [W, V_t]^2\rangle$. The appeal for OTOC stems from a semiclassical argument connecting this equation to classical Poisson brackets, which are in turn related to Lyapunov exponents of a classical process.

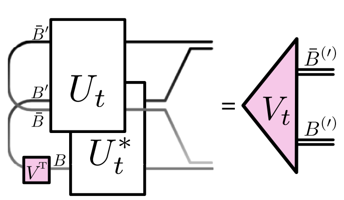
Yet, it remains unclear the precise connection of the OTOC with integrability. In fact, there is controversy in when the OTOC detects chaos in a range of quantum systems without a classical analogue [39,40,43,44] or even with [41,42]. In this work we clarify this confusion, showing that the OTOC probes dynamical chaos, including how it can fail in this purpose, and identifying sufficient conditions when scrambling is equivalent to chaos.

OTOC in terms of local operator Choi state.—We take W in Eq. (2) to be on acting on some potentially large subspace $\mathcal{H}_A$ and V on a local space $\mathcal{H}_B$, with complement spaces defined such that the whole isolated system is $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} = \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$. These spaces are most clearly expressed via a graphical representation of $F(W, V_t)$:



We will use a bracket-prime notation to indicate a doubled space, e.g., prime $\mathcal{H}_{A'}$ represents a copy of the space $\mathcal{H}_A$, while bracketed primes represent a combined double space, $\mathcal{H}_{A^{(\prime)}} := \mathcal{H}_A \otimes \mathcal{H}_{A'}$. For clarity, we rewrite the definition of the Choi state $|V_t\rangle$ [Eq. (1)],

$$|V_t\rangle := \frac{1}{d}$$



(3)

In this setup, V_t can be interpreted as the operator we are interested in, and W as the *probe* to the entanglement of the Choi state of this operator. The first hint of this relation can be seen by rewriting the OTOC in the following.

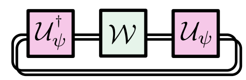
Observation 1.—The OTOC can be expressed in terms of the expectation value of a local unitary, $\mathcal{W} := W \otimes W^*$, with respect to the Choi state of a time evolved Heisenberg operator, $|V_t\rangle$,

$$F(W, V_t) = \langle V_t|(\mathbb{1}_{\bar{A}^{(\prime)}} \otimes \mathcal{W})|V_t\rangle. \tag{4}$$

The proof for this and all following results can be found in [57].

Examining the relation Eq. (4), if $|V_t\rangle$ is maximally entangled in the splitting $A^{(\prime)}: \bar{A}^{(\prime)}$, then the OTOC is equal to zero. Recalling that any maximally entangled state $|\psi\rangle$ corresponds to the Choi state of a unitary matrix $\mathcal{U}_\psi$, one can prove this from Eq. (4) using standard graphical notation:

$$F(W, V_t) = \frac{1}{d}$$



$$= |\mathrm{tr}[W]|^2 = 0. \tag{5}$$

For example, a global Haar random time evolution, $U_t \in \mathbb{H}$, with a large total dimension d, will approximately give a maximally entangled $|V_t\rangle$ on average [57].

Alternatively, if $|V_t\rangle$ is not maximally entangled, $F(W, V_t)$ will generally be nonzero, suggesting that this quantity probes the LOE of V_t, through W. We will now quantify this relationship, showing the necessity of OTOC decay for chaotic LOE behavior.

Chaos implies scrambling.—To investigate the general behavior of the OTOC, we will now compute its average value when a random traceless operator W is sampled, and show that this sampling is typical. To uniformly sample a random matrix with the traceless property, a natural choice is choosing any traceless unitary W and then applying a Haar random unitary channel to it, $W_R = R^\dagger W R$, where $R \in \mathbb{H}$, the Haar measure. We define the averaged OTOC with respect to this traceless probe,

$$G(V_t) := \frac{1}{d}\int_{\mathbb{H}} dR\, \mathrm{tr}[W_R^\dagger V_t^\dagger W_R V_t]. \tag{6}$$

Let us stress that we are *not* averaging over the dynamics, and allow the time-evolution operator U_t to be completely arbitrary. Hinting at a relation between the OTOC and LOE, our results will be framed in terms of $\nu_A(t)$, the (normalized) reduced density matrix of the Choi state $|V_t\rangle$ on (the doubled space) $\mathcal{H}_{A^{(\prime)}} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$,

$$\nu_A(t) := \mathrm{tr}_{\bar{A}^{(\prime)}}[|V_t\rangle\langle V_t|]. \tag{7}$$

We can then use standard techniques adapted from the Weingarten Calculus to arrive at our first main result.

Theorem 2.—The averaged OTOC over Haar random, traceless unitaries W_R [as in Eq. (6)] is equal to

$$G(V_t) = \frac{1}{d_A^2 - 1}(d_A^2 \langle \phi^+ | \nu_A(t) | \phi^+ \rangle - 1), \tag{8}$$

where $|\phi^+\rangle$ is the maximally entangled state across the doubled space $\mathcal{H}_{A^{(t)}}$. Further, this average is typical, with a single shot $F(W_R, V_t)$ exponentially likely in $(d_A \epsilon/8)^2$ to be ϵ–close to $G(V_t)$.

This result presents both an explicit expression for the average OTOC, and that a random OTOC $F(W_R, V_t)$ rarely varies from the average. This is important as the OTOC average $G(V_t)$ features in the rest of this work, and we can be assured that the average case is representative of the typical one.

Notice that the first term in Eq. (8) is proportional to the fidelity between $\nu_A(t)$ and the identity matrix Choi state $|\phi^+\rangle\langle\phi^+|$. In words, this theorem states that the average OTOC is *proportional to the distance between the actual reduced state of* $|V_t\rangle$ *on* $\mathcal{H}_{A^{(t)}}$ *and the state of the identity channel*. Interestingly, considering V_t as a unitary channel, this fidelity is exactly equal to the entanglement fidelity of the reduced channel on $\mathcal{H}_A$, which in turn is proportional to the (efficiently computable) average gate fidelity [63].

Reference [55] reports similar results to Theorem 2. However, there the Haar average is taken for the "bipartite OTOC" over *both* V and W, whose support bipartition the whole system. Our results are distinct, and less restrictive, in allowing the operators to have arbitrary locality, averaging over only one of the unitaries, and most importantly connecting this to LOE (chaos).

To pick apart Theorem 2, consider the example of dynamics consisting of a circuit of swaps $U_t \in \mathbb{S}$. Then, if the operator V is swapped onto a site within the space $\mathcal{H}_A$, the OTOC takes a minimum value,

$$G(V_t)|_{U_t \in \mathbb{S}} = \begin{cases} \frac{-1}{d_A^2 - 1}, & \text{if } V_t \in \mathcal{B}(\mathcal{H}_A) \\ 1, & \text{otherwise,} \end{cases} \tag{9}$$

given that V is taken to be traceless. Similarly, one would (approximately) get this result for $\nu_A(t)$ being (close to) any pure state which is orthogonal to $|\phi^+\rangle\langle\phi^+|$. This is an example of scrambling without chaos: a minimal OTOC is achieved for an integrable dynamics. In fact, it is a simple example of a wide class of local circuit models, which we will analyze later in results 5–7. Alternatively, we saw earlier in Eq. (5) that a maximally entangled $|V_t\rangle$ leads to a small OTOC. This begs the question of what the OTOC tells us if $|V_t\rangle$ is partially entangled? Can we further understand this OTOC average as a quantitative probe to the timescaling of the LOE?

We answer this with the following two bounds, in terms of two different entanglement measures.

Theorem 3.—(Scrambling is necessary for chaos). The OTOC, averaged over traceless unitary operators $W \in \mathcal{B}(\mathcal{H}_A)$, is bounded by the entanglement on $\mathcal{H}_{A^{(t)}}$ of the time-evolved local operator V_t: (A) For geometric measure of entanglement, $E_G(|\phi\rangle) := 1 - \max_{|\psi_{A^{(t)}}\psi_{\bar{A}^{(t)}}\rangle} |\langle \psi_{A^{(t)}}\psi_{\bar{A}^{(t)}}|\phi\rangle|^2$, where the maximum is over all product states $|\psi_{A^{(t)}}\psi_{\bar{A}^{(t)}}\rangle$, $G(V_t)$ satisfies

$$G(V_t) \leq 1 - \frac{d_A^2}{d_A^2 - 1} E_G(|V_t\rangle). \tag{10}$$

(B) For the 2-Rényi entropy $S^{(2)}(\nu) := -\log(\mathrm{tr}[\nu^2])$, $G(V_t)$ satisfies

$$G(V_t) \leq \frac{1}{d_A^2 - 1}(d_A^2 \mathrm{e}^{-\frac{1}{2}S^{(2)}[\nu_A(t)]} - 1). \tag{11}$$

Note that we only used the inequality $\mathcal{D}(\nu_A(t), |\phi^+\rangle\langle\phi^+|) \leq \max_{|\psi\rangle}[\mathcal{D}(\nu_A(t), |\psi\rangle\langle\psi|)]$ for some distance metric $\mathcal{D}$, to arrive at Eq. (10). Therefore it is likely relatively tight for a generic evolution, where V_t does not recohere into a local, pure unitary channel. Indeed, from numerics we notice that Eq. (10) seems to be tighter than Eq. (11). However, in general geometric measures are not practically accessible due to the required optimization over all separable states. On the other hand, the Rényi 2-entropy is. We note that a similar result is derived in Ref. [37], where they lower bound LOE by an extensive sum of OTOCs, in contrast to our bound in terms of a single, average OTOC. The bound (11) is our main result, and will be investigated in the remainder of this work.

The LOE entropy grows at fastest linearly, with strong evidence that this maximal scaling is saturated if and only if the dynamics are chaotic [11,49], compared to logarithmic growth for integrable systems [8,47,48,50,51,64]. Therefore, Eq. (11) gives us a bound on scrambling

$$G(V_t) \lesssim \begin{cases} B\exp(-\alpha t), & \text{if } U_t \text{ chaotic,} \\ Ct^{-\alpha}, & \text{if } U_t \text{ regular.} \end{cases} \tag{12}$$

This should not be confused with the lower bounds on thermally regulated OTOCs from Refs. [31,65], valid for fast-scrambling systems.

Theorem 3 therefore states that fast decay of the OTOC is necessary for chaos. However, the counterargument is not necessarily true, i.e., the bounds in Eq. (12) are not necessarily tight. We will now examine our results through classes of local circuit models, to uncover: (i) When Eq. (11) is saturated, and (ii) when the OTOC decays fast for slowly decaying LOE; i.e., scrambling without chaos.

Application to local circuit models.—"Brickwork" circuits consist of layers of two-body unitary gates which are applied to next-neighbor sites on a lattice (see Fig. 1) [4,53,66–69],

$$U: \mathcal{H}_{\mathrm{i}_1} \otimes \mathcal{H}_{\mathrm{i}_2} \to \mathcal{H}_{\mathrm{o}_1} \otimes \mathcal{H}_{\mathrm{o}_2}. \tag{13}$$

It is necessary to introduce some notation. We take the initial operator V to have support on a single site which we specify wlog to be at $y = 0$, where both the sites and time

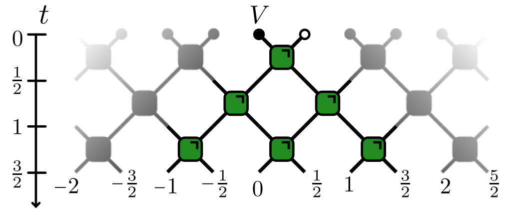
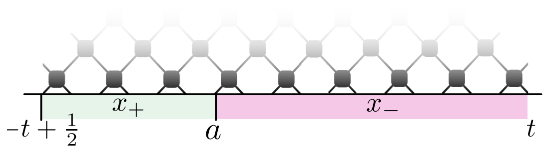


FIG. 1. Brickwork circuit models of dynamics consist of repeated two-site unitaries on a one-dimensional lattice. Time goes from top to bottom, and the light cone for a single-site operator V is shown in green.

steps are labeled with half integers as in Fig. 1. Then the disjoint spaces $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ are labeled by the list of integers of the spins they contain, ℓ_A and $\ell_{\bar{A}}$, respectively. The following results will cover two exclusive cases: when $\mathcal{H}_A$ is connected and contains the left light cone edge ($t \in \ell_A$), or the right edge ($t \in \ell_{\bar{A}}$). This is a technical restraint, related to what is analytically computable for dynamics $\mathbb{D}$ [46]. Finally, we define convenient coordinates $x_\pm \coloneqq t \pm a$, with a the edge of region $\mathcal{H}_{A^{(t)}}$ within the lightcone, such that

$|V_t\rangle =$



In completely general brickwork unitary circuits, both the OTOC and the LOE entropy scale in terms of the same matrix $G(V_t) \sim (\mathcal{T}_{x_+}[U])^{x_-} \sim e^{-\frac{1}{2}S^{(2)}[\nu_A(t)]}$. Informally, $\mathcal{T}_{x_+}[U]$ is a spacetime transfer matrix, representing the contraction of the dual unitary circuit along the light cone direction. This is a well-studied object [11,58,70], and we define it in detail in [57]. This leads to our first result on local circuit OTOC behavior [71].

Observation 4.—When the term $-[1/(d_A^2-1)]$ from Eq. (8) can be neglected, both sides of the inequality Eq. (11) have generically the same leading order scaling for large x_-, but constant x_+.

Further supporting this result, numerical examples of Haar random unitary bricks show similar scaling for both sides of Eq. (11) [57]. In contrast, we will now delve into a specific model where the scaling does not match, in order to highlight that scrambling is distinct from chaos.

Consider the Floquet interacting XXZ model on qubits, consisting of a brickwork dynamics (see Fig. 1) with two-site unitary

$$U_{\mathrm{XXZ}} = \exp\left[-i\left(\frac{\pi}{4}\sigma_x \otimes \sigma_x + \frac{\pi}{4}\sigma_y \otimes \sigma_y + J\sigma_z \otimes \sigma_z\right)\right], \tag{14}$$

where J is a free parameter. We have set the parameter in front of $\sigma_x \otimes \sigma_x$ and $\sigma_y \otimes \sigma_y$ to $\pi/4$ to impose *dual-unitarity* [46,72,73], which we later discuss. Moreover, we additionally specify that $J \neq \pi/4$, as $J = \pi/4$ yields the SWAP circuit as in Eq. (9). This dynamics is not chaotic. In particular, the LOE scales logarithmically with time [51,64]. However, we will see that the OTOC decays exponentially for all times (or is constantly minimal), indicative of strong scrambling.

Theorem 5.—(Scrambling without chaos.) The Floquet dual-unitary XXZ model (14) produces an exponentially decaying OTOC. Concretely, for a single site operator V, Eq. (16) reduces to

$$G(V_t)|_{\mathrm{XXZ}} = \begin{cases} \frac{-1}{d_A^2-1}, & \text{if } t \in \ell_A \\ \beta e^{-\alpha(x_-)} + (1-\beta), & \text{if } t \in \ell_{\bar{A}}. \end{cases} \tag{15}$$

with positive constants α and β reported in [57]. For any V orthogonal to σ_z, the constants are such that $G(V_t)$ decays to a minimal (negative) value.

The fact that the OTOC exhibits (maximal) exponential decay for this clearly integrable model is stark evidence of the distinction between scrambling and chaos. This lays bare the main thesis of this Letter: while the OTOC will always bear witness to chaos, there exists a wide variety of dynamics that are scrambling but not chaotic. In other words, *the decay of the OTOC is necessary but not sufficient for chaos*.

Equation (14) is a particular example of a dual-unitary model [46] (denoted by $\mathbb{D}$), which spread information with maximal velocity [58,74–76]. Using Eq. (8), we can actually compute the average OTOC for this entire class of dynamics. In these local circuits, each brick is unitary in both the space and time direction, which enables analytic computations. Far beyond the trivial swap circuit (9), these models are generically chaotic [5] and include, for example, the (chaotic) self-dual kicked Ising model [20,45,74] and the (integrable) Floquet Heisenberg XXZ model [72,73], as in Eq. (14).

In terms of the (doubled) local, bipartite Hilbert spaces as in Eq. (13), we define the CPTP maps $\mathcal{M}_+ \coloneqq \langle \mathbb{1}|_{\mathrm{i}_1^{(t)}} U^* \otimes U|\mathbb{1}\rangle_{\mathrm{o}_2^{(t)}}$ and $\mathcal{M}_- \coloneqq \langle \mathbb{1}|_{\mathrm{i}_2^{(t)}} U^* \otimes U|\mathbb{1}\rangle_{\mathrm{o}_1^{(t)}}$. These local maps govern the decay of two-point correlations in $\mathbb{D}$ [46]. We can exactly express the OTOC average in terms of these maps.

Theorem 6.—(OTOC in dual unitary circuits) For evolution according to dual unitary circuits $\mathbb{D}$, the average OTOC is exactly

$$G(V_t)|_{\mathbb{D}} = \begin{cases} -\frac{1}{d_A^2-1}, & t \in \ell_A \\ \frac{1}{d_A^2-1}(d_A^2\langle V|\mathcal{M}_+^{x_+}\mathcal{M}_-^{x_+}|V\rangle - 1), & t \in \ell_{\bar{A}}. \end{cases} \tag{16}$$

We stress that our result for $G(t)$ in Eq. (16) relies *only* on the dual unitarity property, both in the chaotic and nonchaotic cases. This is in contrast to previous work computing OTOCs in $\mathbb{D}$, which require the "completely chaotic" assumption [58,75]. Theorem 4 is valid for arbitrary t without additional averaging or conditions, and W may have arbitrarily large support in contrast to the exclusively single site operators considered in Refs. [58,75].

We can further specify the dual unitary dynamics to be completely chaotic [11] ("maximally chaotic" in Ref. [58]), defined by the property that the eigenvectors with eigenvalue one of the transfer matrix $\mathcal{T}$ discussed around Observation 4 are limited to a minimal set [57]. This property generically holds, but is violated if there are additional symmetries (e.g., kicked Ising model) or additional local conservation laws (e.g., trotterized XXZ model). This property leads to a precise equivalence between the LOE and OTOC.

Corollary 7.—For x_- kept fixed, Eq. (16) can be expressed using the known inequalities of Rényi-2 LOE entropy, [57]

$$G(V_t)|_{U_t\in\mathbb{D}} \leq \exp\left[\lim_{x_-\to\infty}\frac{-1}{2}S^{(2)}(\nu_A(t))\right], \tag{17}$$

where equality holds exactly for completely chaotic dual unitary circuits for $|\lambda|\geq d^{-1/2}$ and x_+ large. Here, $|\lambda|$ is the largest nontrivial eigenvalue of $\mathcal{M}_-$ [11].

Notice here that Eq. (17) is exactly equivalent to Eq. (11) for $d_A \gg 1$.

Conclusions and discussion.—In this Letter, we have demonstrated that the out-of-time-ordered correlator probes of the local operator entanglement of the time-evolving operator V_t (results 1–3). This means that formally, scrambling is strictly necessary for chaos. To explore this relationship, we examined OTOCs in dual-unitary circuits (Theorem 6), including an explicit example of an integrable dynamics where the OTOC exponentially decays for all times; maximal scrambling without chaos (Theorem 5). Finally, we also determined generic dual-unitary conditions that defines when LOE and OTOC scaling coincides—including requiring the completely chaotic property (Corollary 7). It would be interesting extend this and determine the class of models which saturate the bound (11). We suspect that the members of this class share other interesting properties.

In our results, the (ultra-)local unitary operator V was left unspecified, but its exact choice may influence computations (cf. Theorem 5). Often, V is averaged over [10,59], but one can take a more subtle approach and define a density operator that encodes all possible OTOCs or local Heisenberg operators [17,60]. We extend our main results to this operator-free setting in [57].

Finally, recently there has been interest in higher point OTOC generalizations, which are thought to probe finer structure of randomness [12,13,59,77]. We can extend present results to connect these to a novel generalization of LOE, but leave this to a future work.

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*neil.dowling@monash.edu

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