Graded Quasiperiodic Metamaterials Perform Fractal Rainbow Trapping

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(Received 18 May 2023; accepted 27 September 2023; published 23 October 2023)

The rainbow trapping phenomenon of graded metamaterials can be combined with the fractal spectra of quasiperiodic waveguides to give a metamaterial that performs fractal rainbow trapping. This is achieved through a graded cut-and-project algorithm that yields a geometry for which the effective projection angle is graded along its length. As a result, the fractal structure of local band gaps varies with position, leading to broadband "fractal" rainbow trapping. We demonstrate this principle by designing an acoustic waveguide, which is characterised using theory, simulation and experiments.

DOI: 10.1103/PhysRevLett.131.177001

Introduction.—Graded metamaterials have become popular for several significant applications, such as wave energy harvesting [1,2] and machine hearing [3–5]. They have heterogeneous microstructures that are slowly varied (or "graded") to yield different effective properties in different spatial regions. Graded metamaterials were first developed in optics, to realize so-called "rainbow trapping" [6]. This is the phenomenon of different frequencies being reflected at different positions, due to a local effective band gap that is slowly shifted by a monotonic gradient function. This principle has since been applied in many other physical settings, such as plasmonics [7], acoustics [8], elasticity [9,10], seismology [11] and water waves [12].

Another exciting but as-yet-unrelated direction for metamaterial physics is the move beyond periodic microstructures into the realm of quasiperiodic metamaterials. Quasicrystals are structures that are ordered and deterministic but nonperiodic. While predicting the transmission spectra of quasiperiodic systems is a notoriously challenging and long-standing problem [13–15], they have some very appealing properties. In particular, quasiperiodic metamaterials often have a fractal structure of many large spectral gaps [16–19]. This work designs graded metamaterials that exploit these large gaps.

Quasicrystals may also have some beneficial robustness properties. Their large spectral gaps are retained under periodic approximations [20,21], for example. Similarly, localized modes (such as edge modes) in quasiperiodic waveguides may benefit from innate robustness with respect to imperfections [22–24]. However, it is not generally clear if this is linked to underlying topological properties [13,25,26] or even if any robustness benefits exist when the system is appropriately normalized [27].

In this Letter, we take advantage of the complex, "fractal" structure of many spectral gaps that is typical of a quasicrystal to produce an acoustic metamaterial that performs broadband fractal rainbow trapping. The quasicrystal literature contains many variants of the famous "Hofstadter butterfly" [28]. These are plots that show how the spectrum varies when a parameter is modulated [22,27,29,30]. They typically show a self-similar and fractal collection of spectral gaps that shift up or down and open or close as the parameter is varied (sometimes resembling a butterfly). Given that such a complex collection of (sometimes large) band gaps is an appealing prospect for broadband wave control, the aim of this work is to design a graded quasiperiodic metamaterial that leverages this to perform fractal rainbow trapping. Note that the metamaterials developed here are not fractal metamaterials, as studied by, e.g., [31,32], since the geometry is not fractal. Instead, it is the *spectrum* that is fractal.

There have been a variety of attempts to optimize the performance of graded metamaterials [33–35]. This is typically achieved by modifying the gradient function applied to a conventional periodic metamaterial, however, this is a challenging problem [36]. Other approaches have included using arrays with topologically protected edge modes [37] and creating symmetry-broken systems which have zero group velocity modes inside the Brillouin zone (which was shown to lead to longer interaction times and an

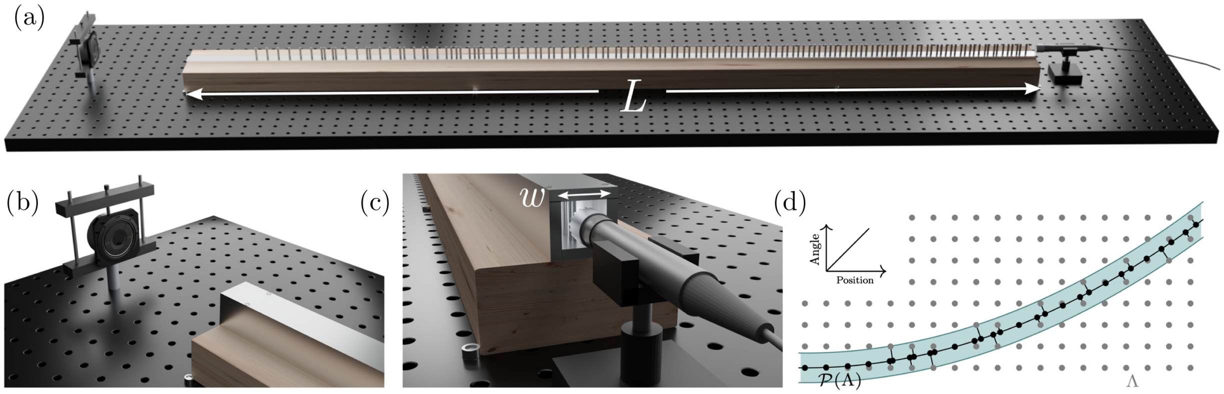

FIG. 1. Experimental schematic and cut-and-project algorithm: (a) 127 acoustically rigid rods (of diameter 4 mm) are placed at positions determined by the graded cut-and-project algorithm, forming an array of length $L = 1.5$ m, shown without the enclosing waveguide. (b),(c) End views of source (loudspeaker) and receiver (microphone) positions, respectively, with enclosing waveguide shown with a square-cross section of width $w = 2$ cm. (d) Schematic of the graded cut-and-project algorithm, which projects a square lattice Λ onto a quadratic curve. The rod positions and sample geometry are further detailed in Supplemental Material [41].

increase in the harvested energy) [38]. While notions of topology have been developed for quasicrystals and can be applied to the setting considered in this work [13,25,26], it is unclear how they relate to rainbow effects. Instead, the main opportunity arising from graded quasicrystals is the ability to perform rainbow effects over a broadband frequency range. For conventional graded periodic metamaterials, typically just one or two band gaps are modulated within the operating frequency range and the bandwidth can be increased either by widening the band gap or extending the array. Conversely, quasicrystals are well known to have fractal collections of many large band gaps, leading to broadband effects with relatively short arrays [22,24,39].

A crucial feature of quasicrystals, which is central to our work, is that they can be obtained by taking incommensurate projections of higher-dimensional periodic structures [40]; the angle of this projection often serves as the canonical spectrum-modulating parameter. We will grade the quasicrystal by grading the projection angle, achieved by projecting onto a curved line (see Fig. 1). This is used to design our acoustic metamaterial, which has scatterers placed at the points specified by the curved cut-and-project algorithm. A wave traveling through this system experiences the fractal sequence of effective band gaps shown in Fig. 2—a typical butterfly-type plot. The tendency of these gaps to shift upward as the angle increases leads to a fractal rainbow trapping effect.

While some rainbow-trapping-type effects have been observed in nonperiodic optical materials previously [42], it is unclear what mechanism was responsible for this or how these effects could be controlled. Our graded cut-and-project algorithm overcomes this, giving a fractal rainbow trapping effect that can be understood easily by studying the modulated spectral bands.

Graded cut and project.—The quasicrystals considered in this work are one-dimensional lattices that are obtained by projecting a two-dimensional square lattice. Suppose we have a periodic lattice $\Lambda \subset \mathbb{R}^2$ and a curve $\Gamma \subset \mathbb{R}^2$. If a point in Λ is within a distance $w > 0$ of the curve Γ, we project it onto Γ at the closest point (if this is not unique, we take all equidistant points) to give the projected lattice

$$\mathcal{P}(\Lambda) = \left\{ z \in \Gamma \,\middle|\, \begin{array}{l} \exists\, y \in \Lambda \text{ such that } \|z-y\|_2 < w \\ \text{and } \quad \|z-y\|_2 = \min_{\zeta \in \Gamma} \|\zeta - y\|_2 \end{array} \right\}. \tag{1}$$

The set $\mathcal{P}(\Lambda)$ is a set of points in $\Gamma \subset \mathbb{R}^2$ that are distributed along the curve Γ, meaning it is fundamentally a one-dimensional set embedded in $\mathbb{R}^2$. There are many ways to project $\mathcal{P}(\Lambda)$ into one dimension; for this work we choose to do so by just retaining the first coordinate of each point.

Predicting the transmission spectra of quasiperiodic systems is notoriously challenging and has been a long-standing problem for spectral theorists [13–15]. Taking advantage of the cut-and-project operator is a promising

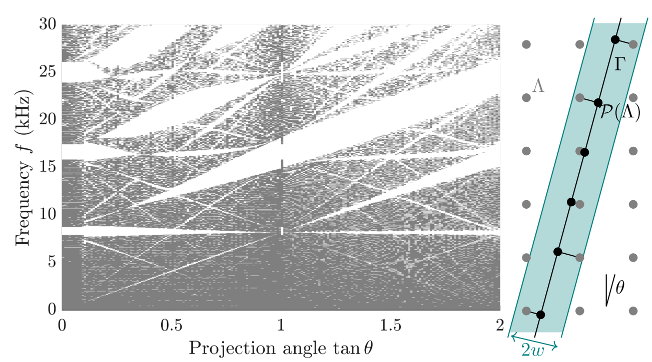

FIG. 2. The spectral "butterfly." Varying the cut-and-project angle θ of a square lattice causes the band gaps to shift and open or close. Any points in the square periodic lattice Λ that are within a distance w of the straight line Γ are projected onto the line, yielding the quasicrystal $\mathcal{P}(\Lambda)$.

strategy [43,44], however, approximating the quasicrystal by a periodic material (often known as a *supercell*) remains the most prevalent approach [20,21].

We use a projected quasicrystal $\mathcal{P}(\Lambda)$ to place scatterers within an acoustic waveguide (see Fig. 1). If two points are close enough that the scatterers would touch or overlap, we remove the second one. A simple theoretical one-dimensional model can be used to obtain an initial characterization of this system. For a frequency ω, this model is given by $u''(x) + \omega^2 c(x)^{-2} u(x) = 0$. We assume the wave speed $c(x)$ varies between two piecewise constant values along the array, such that it takes a larger value in a neighborhood of each of the scatterers (due to the local increase in impedance). See Supplemental Material [41] for more details.

The obvious choice for the curve Γ is a straight line, i.e., $\Gamma \coloneqq \{y = \cos(\theta)x\}$. In Fig. 2 we show the spectra of the projected quasicrystal as the angle θ varies, computed using the theoretical 1D model. These are Bloch spectra, as the 200 values of θ are all chosen such that $\tan\theta = m/100$ is rational. We observe a complex pattern of band gaps, that displays exotic properties reminiscent of the spectral "butterflies" associated with quasiperiodic systems. The fractality of this set is due to fact that as $\tan\theta \in \mathbb{Q}$ approaches an irrational number (and the irrationals are dense in the rationals) the corresponding unit cell becomes arbitrarily large, leading to a countable collection of gaps. We have estimated the box-counting dimension of this fractal as approximately 2.12. More details are given in Supplemental Material [41].

The propensity for gaps to sweep upward in Fig. 2 as $\tan\theta$ increases is reminiscent of the mechanism that leads to rainbow trapping in conventional graded metamaterials. If we design a graded quasicrystal for which the effective projection angle gradually increases along its length, then the local effective band gaps will shift upward. This means that, at least above 12 kHz, the higher a frequency is, the further it will be able to travel before it experiences a given band gap. We achieve this grading of the projection angle by projecting onto a quadratic curve Γ, as depicted in Fig. 1(d). The choice of a quadratic means that the angle varies in direct proportion to the position in the array.

In Fig. 2, gaps appear around 8, 16, and 24 kHz for many different projection angles. This has been inherited from the original periodic lattice, which has band gaps at these frequencies. Since our projection algorithm (1) preserves the average separation distances when the angle is small, the first part of the array has strong effective gaps at these frequencies. We detail this through additional experiments on separate parts of the array in Supplemental Material [41].

The discontinuities that are visible in Fig. 2 are due to discontinuities in the number of points in $\mathcal{P}(\Lambda)$ that lie within a given interval. This is due to points in the square lattice Λ leaving or entering the $2w$-wide projection strip as the angle θ varies as well as our decision to remove points that are too close to each other. This is particularly visible at $\tan\theta = 1$ ($\theta = \pi/4$), where the projected crystal is very different from nearby angles (but turns out to be the same as at $\theta = 0$).

Experimental and numerical results.—We experimentally verify the fractal rainbow effect using two experimental procedures, performed using the experimental setup shown in Fig. 1. We form an acoustic waveguide in air, using aluminum plates to ensure sound hard boundaries, with square a cross-section of width $w = 2$ cm and length $L = 1.5$ m. The waveguide consists of a bottom plate and a surrounding "hood" that encases the bottom plate (split into three sections—see Supplemental Material [41]), that can be made into two configurations: "free" (no scatterers) or "scattering" (scatterers present). In the latter configuration 127 aluminum rods, of diameter 4 mm, are placed into holes machine drilled into the bottom plate at positions dictated by the graded cut-and-project algorithm. The holes are milled to 20 μm positional precision and are of length $w + h$, where h is the thickness of the bottom plate, such that there is a tight fit within the waveguide.

A transmission experiment first confirms the expected band gap that originates at 8 kHz; a loudspeaker (Visaton SC 8 N) is fixed at a distance of 175 mm from the end of the sample and emits a single-cycle Gaussian pulse, centerd at 15 kHz. A microphone (Brüel and Kjær type 4966 1/2-in free-field, with preconditioning amplifier) is placed at the exit of the waveguide, with the signal received recorded on an oscilloscope (Siglent SDS2352X-E). Acoustic data were recorded with a sampling frequency of 5 $\mathrm{MSa\,s^{-1}}$, with 50 averages taken. The experiment was conducted in both the free and scattering configurations and the spectra obtained by means of the fast Fourier transform. A ratio of the Fourier amplitudes with and without the scatterers then gives us a measurement of the transmission t. The commercial finite element method (FEM) solver COMSOL Multiphysics [45] is used to simulate the experiment, with the problem being reduced to two dimensions. As such, the

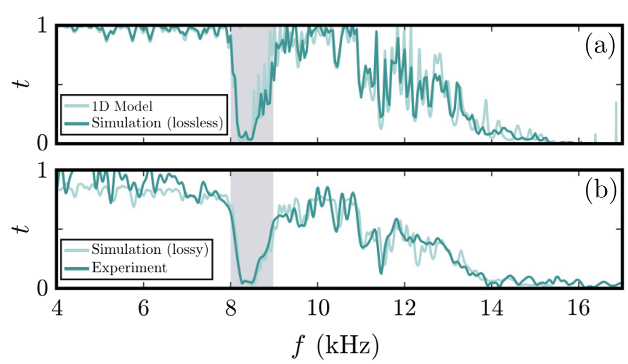

FIG. 3. Comparison of the transmission spectra: (a) the 1D model and lossless FEM simulations; (b) lossy FEM simulation (thermoviscous physics) and experimental results.

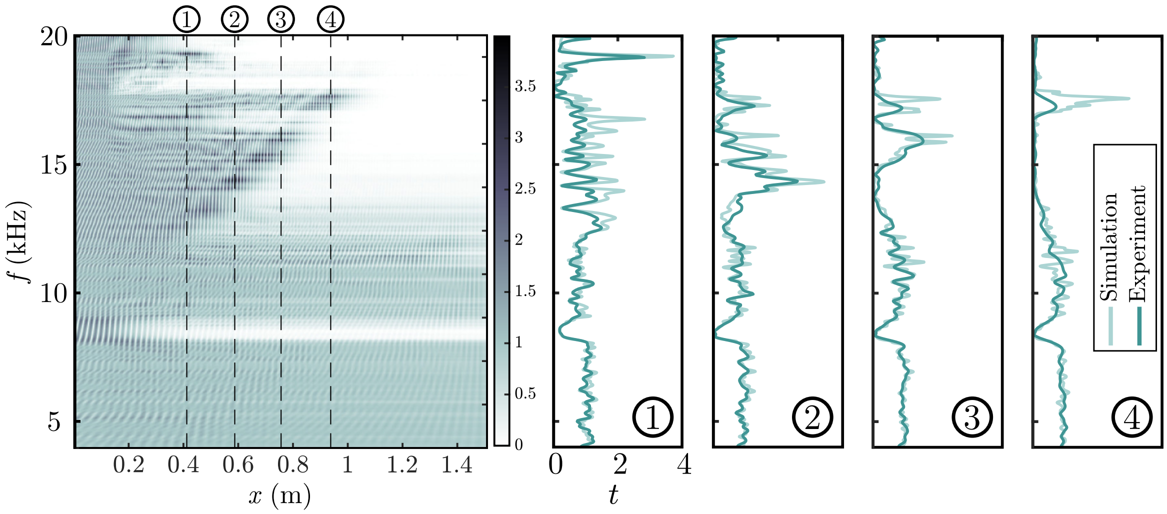

FIG. 4. Demonstration of the fractal rainbow effect. FEM simulations showing frequency spectra as a function of position. The numbered lines show the measurement positions (see Supplemental Material [41] for details), with the corresponding experimental comparisons shown in the numbered plots (same frequency axis).

computational domain is now a rectangular waveguide ($L \times w$) with the rods being approximated by voids with sound hard boundaries. Both lossless and lossy simulations are performed, with thermoviscous physics driving the loss mechanism. Comparisons between theory and lossless simulations, and lossy simulations and experiment can be seen in Fig. 3. It is clear thermoviscous losses play a role within the waveguide, and are thus adopted in all subsequent numerical models.

A second experiment was conducted to confirm the fractal rainbow effect: four small holes, of diameter 1.3 mm were drilled into one side of the hood at positions $x = 0.41$, 0.58, 0.76 0.94 m (schematic shown in Supplemental Material [41]). A needle microphone (Brüel and Kjær type 4182 probe microphone) was inserted into the holes, halfway between the outer walls of the waveguide and the center lines of the rods. A repeat of the above methodology was conducted, with the ratio of the Fourier amplitude at these four positions confirming the localization of acoustic energy as per the theoretical and numerical predictions, shown in Fig. 4. Since t is proportional to the ratio of pressure amplitudes, with and without the quasicrystal, its magnitude may exceed unity when the pressure field undergoes localization by the scatterers within the waveguide. It is clear that, within the 12–18 kHz range, there is a linear relationship between the frequency and the position of reflection, demonstrating the occurrence of fractal rainbow trapping.

Conclusions.—This work shows that it is possible to combine the fields of graded metamaterials and quasicrystals to create devices that perform fractal rainbow trapping. This greatly enlarges the graded metamaterial design space and takes advantage of quasicrystals' exceptional properties. In particular, the dense, Cantor-set-like patterns of many nearby spectral gaps typical of quasicrystals can be exploited to increase the operating bandwidth in graded metamaterial applications, such as energy harvesting. Since the one-dimensional model used here is also a toy problem for other wave regimes (such as photonics, linear elasticity, and water waves, under suitable assumptions), fractal rainbow trapping could also be performed in other settings, based on the same projected geometry.

Our results pose several open questions. In this work, we projected a square lattice onto a quadratic curve, giving a linear relationship between effective projection angle and position. Can the performance be optimized by tuning this relationship? How does their performance compare with conventional periodic graded metamaterials? Given the subtleties of drawing calibrated comparisons between graded metamaterials [36], this requires a careful systematic study. Additionally, it remains to be seen if the topologically protected edge modes that have been observed in Harper-type quasicrystals [24,29,30] can be exploited to perform topological rainbow trapping [37] in quasicrystals. By leveraging the quasicrystals whose spectra are already well understood, we can unlock many new graded metamaterials very quickly.

B. D. is supported by the Engineering and Physical Sciences Research Council (EPSRC) through a Research Fellowship with Grant No. EP/X027422/1. G. J. C. gratefully acknowledges financial support from the Royal Commission for the Exhibition of 1851 in the form of a Research Fellowship. T. A. S. gratefully acknowledges financial support from DSTL. R. V. C.'s work was funded by UK Research and Innovation (UKRI) under the UK government's Horizon Europe funding guarantee [Grant No. 10033143].

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