# Exact New Mobility Edges between Critical and Localized States

Xin-Chi Zhou,[1,2,*] Yongjian Wang,[3,4,*] Ting-Fung Jeffrey Poon,[1,2,*] Qi Zhou,[5,†] and Xiong-Jun Liu[1,2,6,‡]

[1]*International Center for Quantum Materials, School of Physics, Peking University, Beijing 100871, China*

[2]*Hefei National Laboratory, Hefei 230088, China*

[3]*School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China*

[4]*School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE, Beijing Normal University, Beijing 100875, China*

[5]*Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China*

[6]*International Quantum Academy, Shenzhen 518048, China*

(Received 16 January 2023; revised 25 August 2023; accepted 29 September 2023; published 25 October 2023)

The disorder systems host three types of fundamental quantum states, known as the extended, localized, and critical states, of which the critical states remain being much less explored. Here we propose a class of exactly solvable models which host a novel type of exact mobility edges (MEs) separating localized states from robust critical states, and propose experimental realization. Here the robustness refers to the stability against both single-particle perturbation and interactions in the few-body regime. The exactly solvable one-dimensional models are featured by a quasiperiodic mosaic type of both hopping terms and on-site potentials. The analytic results enable us to *unambiguously* obtain the critical states which otherwise require arduous numerical verification including the careful finite size scalings. The critical states and new MEs are shown to be robust, illustrating a generic mechanism unveiled here that the critical states are protected by zeros of quasiperiodic hopping terms in the thermodynamic limit. Further, we propose a novel experimental scheme to realize the exactly solvable model and the new MEs in an incommensurate Rydberg Raman superarray. This Letter may pave a way to precisely explore the critical states and new ME physics with experimental feasibility.

DOI: 10.1103/PhysRevLett.131.176401

*Introduction.*—Anderson localization (AL) is a fundamental and ubiquitous quantum phenomenon that quantum states are exponentially localized due to disorder [1]. Scaling theory shows that all noninteracting states are localized in one and two dimensions with arbitrarily small disorder strength [2,3], while in three dimensions (3D), the localized and extended states can coexist at finite disorder strength, and be separated by a critical energy $E_c$, dubbed the mobility edge (ME). The ME leads to various fundamental phenomena, such as metal-insulator transition by varying the particle number density or disorder strength [4]. Moreover, a system with ME exhibits strong thermoelectric response, enabling application in thermoelectric devices [5–7]. An important feature of ME between extended and localized states is that it is stable, and can survive under perturbations and interactions [8–12].

Unlike in randomly disordered system, the extended-AL transition and ME can exist in a 1D system with quasiperiodic potential [13–30]. This result has triggered many experimental studies in realizing quasiperiodic systems with ultracold atoms [11,31–40] and other systems like photonic crystals, optical cavities, and superconducting circuits [41–46]. More importantly, quasiperiodic systems can host a third type of states called critical states [46–51]. The critical states are extended but non-ergodic, locally scale invariant and fundamentally different from the localized and extended states in spectral statistics [52–54], multifractal properties [55–57], and dynamical evolution [58–60]. With interactions, the single-particle critical states may become a many-body critical (MBC) phase [29,61,62] that interpolates the thermal and many-body localized phase [63,64]. However, unlike localized and extended states, confirming critical states is more subtle and requires arduous numerical calculations like finite-size scaling. It remains unclear what generic mechanism leads to the critical states. Therefore, it is highly important to develop exactly solvable models with critical states being unambiguously determined and fully characterized. Moreover, similar to the ME for extended and localized states, are there new MEs separating critical from localized states [25], in particular, in experimentally feasible models? A definite answer to this fundamental question is yet elusive but may be provided by resolving the following issues. First, one can develop exactly solvable models with analytic MEs between critical and localized states. Further, one needs to confirm that such new MEs are robust, e.g., in the presence of perturbation and/or interactions. Finally, the proposed exactly solvable models are feasible in experimental realization.

In this Letter, we propose a class of exactly solvable 1D models featured with mosaic type quasiperiodic hopping coefficients and on-site potential, and obtain *unambiguously* critical states and robust exact MEs. The new MEs fundamentally distinct from those in previous exact solvable models [23]. The localization and critical features of all quantum states in the spectra are precisely determined by extending Avila's global theory [65], enabling an accurate characterization of the critical states and new MEs. We further confirm the robustness of MEs against single-particle perturbation and interactions in the few-body regime. The robustness is rooted in a profound mechanism unveiled with our exactly solvable models that the critical states are protected by incommensurately distributed zeros (IDZs) of mosaic hopping terms in thermodynamic limit. Finally, we propose a novel scheme with experimental feasibility to realize and detect the exact MEs in *Raman superarray* of Rydberg atoms.

*Model.*—We propose a class of quasiperiodic mosaic models as pictorially shown in Fig. 1(a), and described by

$$H = \sum_j (t_j a_j^\dagger a_{j+1} + \mathrm{H.c.}) + \sum_j V_j n_j, \tag{1}$$

where the particle number operator $n_j = a_j^\dagger a_j$, with $a_j^\dagger(a_j)$ the creation (annihilation) operator at site $j$, and the key ingredients in the models are that both the quasiperiodic hopping coefficient $t_j$ and on-site potential $V_j$ are mosaic, with

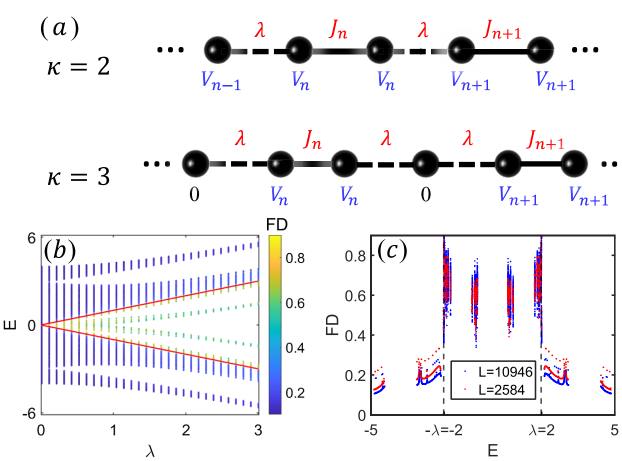


FIG. 1. (a) The 1D quasiperiodic mosaic model with $\kappa = 2$ and $\kappa = 3$. The black solid and dashed lines denote the quasiperiodic hopping ($t_j = J_n$) and constant hopping ($t_j = \lambda$), respectively. The black sphere denotes a lattice site, and $V_n$ is the quasiperiodic mosaic potential. Here $J_n = V_n = 2t_0\cos(2\pi\kappa\alpha n)$, $n$ is an integer. (b) Fractal dimension (FD) of different eigenstates as a function of corresponding $E$ and $\lambda$ for $L = 2584$. The red lines denote the MEs $E_c = \pm\lambda$. (c) FD versus $E$ with fixed $\lambda = 2.0$ for $L = 2584$ (red dots) and $L = 10\,946$ (blue dots). The dashed lines represent the MEs. The $t_0$ is set to 1.

$$\{t_j, V_j\} = \begin{cases} \{\lambda, 2t_0\cos[2\pi\alpha(j-1)+\theta]\}, & j = 1 \bmod \kappa, \\ 2t_0\cos(2\pi\alpha j+\theta)\{1,1\}, & j = 0 \bmod \kappa, \\ \{\lambda, 0\}, & \text{else}. \end{cases} \tag{2}$$

Here $\kappa$ is an integer and $\kappa \geq 2$. $\lambda$ and $\theta$ denote hopping coefficient and phase offset, respectively. We take $t_0 = 1$, and set $\theta = 0$ and $\alpha = \lim_{n\to\infty}(F_{n-1}/F_n) = (\sqrt{5}-1)/2$ without affecting generality, with $F_n$ being Fibonacci numbers. For finite system one may choose the system size $L = F_n$ and $\alpha = F_{n-1}/F_n$. To facilitate our discussion we focus on the minimal model for $\kappa = 2$ in main text. The results with $\kappa > 2$ are in the Supplemental Material [66]. We shall prove that the minimal model has exact energy-dependent MEs separating localized states and critical states, which are given by

$$E_c = \pm\lambda. \tag{3}$$

Before showing the rigorous proof, we present numerical verification. For this exactly solvable model, the different types of states can be identified by the fractal dimension (FD), defined for $m$th eigenstate $|\psi_m\rangle = \sum_{j=1}^{L} u_{m,j} a_j^\dagger |vac\rangle$ as $\mathrm{FD} = -\lim_{L\to\infty} \ln(\sum_j |u_{m,j}|^4)/\ln(L)$. The FD tends to 1 and 0 for the extended and localized states, respectively, while $0 < \mathrm{FD} < 1$ for critical states. Fig. 1(b) shows FD as a function of $\lambda$ for different eigenstates of eigenvalues $E$. The red lines starting from band center denote the MEs $E_c = \pm\lambda$, across which FD changes from values close to 0.5 to values close to 0, indicating a critical-to-localization transition predicted by the analytic results. Particularly, we fix $\lambda = 2.0$ and show FD of different eigenstates in Fig. 1(c) as a function of the corresponding eigenvalues for different sizes. The dashed lines in the figure are the MEs $E_c = \pm\lambda = \pm 2.0$. One can observe that FD tends to 0 for all states in energy zones with $|E| > \lambda$ with increasing the system size, implying that those states are localized. On the contrast, in energy zones with $|E| < \lambda$, FD is far different from 0 and 1, and nearly independent of the system size. A more careful finite size scaling for FD can be found in [66].

*Rigorous proof.*—The MEs of the models in Eqs. (1)–(2) can be analytically obtained by computing Lyapunov exponent (LE) $\gamma_\epsilon$ in combination with IDZs of hopping coefficients, which provides the unambiguous evidence of the critical zone. Denote $T_i$ as the transfer matrix at site $i$, i.e., $(C_{i+1}, C_i)^\top = T_i(C_i, C_{i-1})^\top$ and $\mathcal{T}_i = T_i T_{i-1}\cdots T_1$. The LE $\gamma_0$ for a state with energy $E$ is computed via $\gamma_\epsilon(E) = \lim_{m\to\infty}\int d\theta \ln\|\mathcal{T}_m(\theta + i\epsilon)\|/(2\pi m)$, where $\|\cdot\|$ denotes the matrix norm and $\epsilon$ is imaginary part of complexified $\theta$. We extend Avila's global theory [65] to singular cocycles, and show that $\gamma_0 = \lim_{\epsilon\to\infty}\gamma_\epsilon = \kappa^{-1}\ln\|\lim_{\epsilon\to\infty}\mathcal{T}_\kappa(\theta + i\epsilon)\|$ [69]. For $\kappa = 2$,

$$\mathcal{T}_2(\theta+i\epsilon)=\frac{1}{\lambda M}\begin{pmatrix} E-M & -M \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} E-M & -\lambda \\ M & 0 \end{pmatrix},$$

where $M=2\cos(2\pi\alpha+\theta+i\epsilon)$ and the LE is

$$\gamma_0(E)=\frac{1}{2}\ln\left||E/\lambda|+\sqrt{(E/\lambda)^2-1}\right|. \quad (4)$$

If $|E|>|\lambda|$, then $\gamma_0(E)>0$ and the state with energy $E$ is localized with the localization length $\xi(E)=\gamma_0^{-1}$. If $|E|<|\lambda|$, then $\gamma_0(E)=0$ and the state can be either extended or critical, which belong to absolutely continuous (AC) spectrum or singular continuous (SC) spectrum, respectively [70]. There are two basic approaches to exclude the AC spectrum: one is introducing unbounded spectrum [25] and the other is introducing zeros of hopping terms in Hamiltonian [71,72]. For our model, there exists a sequence of sites $\{2j_k\}$ such that $t_{2j_k}\to 0$ in the thermodynamics limit, so there is no AC spectrum [73], and all the eigenstates with $|E|\leq|\lambda|$ are all critical. In summary, vanishing LEs and IDZs of hopping unambiguously determine the critical region for $|E|\leq|\lambda|$ and positive LEs determine the localized region for $|E|>|\lambda|$. Therefore $E=\pm\lambda$ mark critical energies separating localized states and critical states, manifesting MEs.

*Mechanism of critical states.*—The emergence of MEs and critical states has a universal underlying mechanism unveiled from the exact results. Namely, it is due to IDZs of hopping coefficients [74] in the thermodynamic limit and vanishing the LE. Such zeros in $t_j$ effectively divide system into weakly coupled subchains, which cannot support extended states, and leaving the localized or critical states depending on the corresponding LEs. To confirm the abundance of critical states in the whole spectrum, we first consider the special case of $\lambda\to 0$ [Fig. 1(a)], where the model consists a series of dimmers, each with a $2\times 2$ matrix with all elements being $J_j=V_j=2\cos(2\pi\alpha j)$. The eigenvalues are $E_1=2J_j$ and $E_2=0$, corresponding to localized states and a zero-energy flat band with degeneracy equal to half of the system size, respectively. Note that a linear combination of the zero modes are also eigenstates of the model, which can be either localized, extended, or critical. Further inclusion of $\lambda$ hybridizes the zero-energy flat-band modes and localized modes, yielding the critical states and MEs between them and localized ones. This mechanism also explains emergence of critical states starting from band center. Moreover, the number of critical states equals to that of localized states under the exactly solvable condition, as verified by the numerical counting. This zero hopping coefficient mechanism also explains a novel feature that the critical zone is robust against single-particle perturbation which deviates the model from exactly solvable condition. We consider an extra mosaic on-site potential term $V_p$ as perturbation,

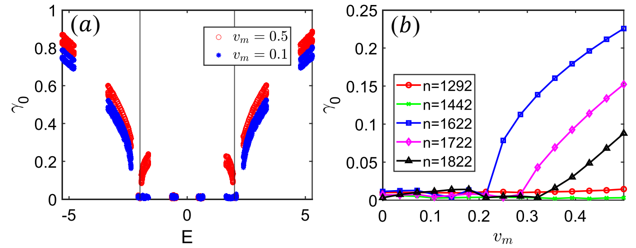


FIG. 2. Numerical LEs $\gamma_0$ for perturbation strength $v_m$. (a) $\gamma_0$ as a function of eigenvalues $E$ for different $v_m$. Solid lines are original MEs. (b) $\gamma_0$ as a function of $v_m$ for eigenstates $|\psi_n\rangle$ from the band center ($n=1292$) to the states nearby original MEs ($n=1822$). Critical states nearby MEs can be driven to localized states while critical zones nearby band center remains unchanged. The other parameters are $\lambda=2.0$ and $L=2584$.

which represents the mismatch between hopping and potentials in Eq. (2), and is relevant to real experiment,

$$H_p=H+\sum_j V_j^p n_j, \quad (5)$$

where $V_j^p=v_m\cos(2\pi\alpha j+\theta)$ for even-$j$ sites and $V_j^p=v_m\cos[2\pi\alpha(j-1)+\theta])$ for odd-$j$ sites. Figure 2 shows numerically calculated LEs for different $v_m$, with $\lambda=2$. The strong mismatch potentials can localize the critical states near MEs while those near the band center remain unchanged [Fig. 2(a)]. A closer inspection of LEs shows that localizing critical states requires finite $v_m>v_m^c$, which depends on the position of the critical state [Fig. 2(b)]. These results manifest the robustness of critical zones against perturbations. In comparison, the celebrated Aubry-André (AA) model exhibit a self-duality point at $V=2t$, where all states are critical [13]. However, those critical states are not robust and become localized for infinitesimal perturbations. The present study unveils a generic mechanism to obtain robust critical zones protected by the zeros of hopping coefficients with vanishing LEs in the thermodynamic limit, which are not removed by the perturbations. This also shows a nontrivial regime that while Avila's global theory cannot give analytical MEs, the MEs exist.

*Robustness against interactions.*—We further demonstrate the robustness of MEs in the presence of interactions by studying a few-body Hamiltonian given by

$$H=H_0+U\sum_j n_j n_{j+1}, \quad (6)$$

where $H_0$ is the Hamiltonian of Eq. (1) for few hard-core bosons with $\langle n_j\rangle\leq 1$, and $U$ is the strength of neighboring interactions. This model can be simulated with Rydberg atoms (see details in next section). We propose normalized participation ratio (NPR) to detect the MEs in the few-body system. The NPR of an eigenstate $|\psi_m\rangle = \sum_c u_{m,c}|c\rangle$ is defined as $\mathrm{NPR} = 1/(V_H \sum_c |u_{m,c}|^4)$, where $\{c\}$ is the computational basis and $V_H$ is the size of the Hilbert state [75]. When $U = 0$, the few-body states are product states of single particle orbitals, which fall into three categories [8,9] depending on whether particles occupy localized (critical) orbitals or occupy mixed orbitals. Denote the maximum energy of critical (localized) orbitals as $\lambda$ ($E_{\max}$), where $E_{\max}$ is the maximum energy of spectrum, we can construct the maximally allowed energy for $N_c$ ($N_l$) particles filled in critical (localized) orbitals as $E_{N_c,N_l} = (N_c\lambda + N_l E_{\max})/N_p$ with $N_p = N_c + N_l$. Then $E_{N_c,N_l}$ locates the transition of different types of few-body states, where NPR changes discontinuously. For instance, when $N_p = 3$, the maximal allowed energy for a mixed state with 2 particles filled in critical orbitals and 1 particle filled in localized orbitals is $E_{2,1}$. Figure 3(a) shows that NPR changes sharply at $E_{N_c,N_l}$ as expected.

The sharp discontinuities near $\pm E_{N_c,N_l}$ persist for $U \neq 0$ for the few-body regime, manifesting the robustness of MEs against few-body interactions. Figure 3(b) shows that for $U = 1.4$, $E_{N_c,N_l}$ can still identify the NPR transition. This is because only those states with at least two particles occupying neighboring sites can be influenced by the interaction, and the portion of such states is of order $\mathcal{O}(L^{-1})$, with $L$ the system size. Thus for relatively large $L$, most eigenstates remain product states of single-particle orbitals, except for those perturbed by interaction. Note that the localized orbitals have zero contribution to NPR in large $L$ limit, and the number of critical orbitals determine NPR

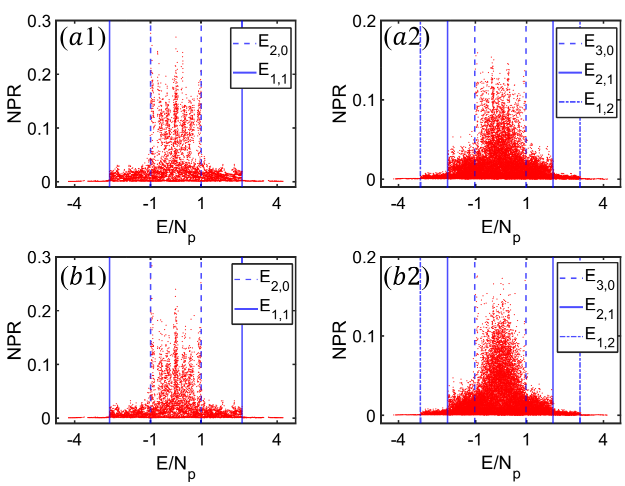


FIG. 3. Normalized participation ratio (NPR) as a function of energy density $E/N_\mathrm{p}$ of few hardcore bosons with $U = 0$ (upper panel) and $U = 1.4$ (lower panel), with $\lambda = 1.0$ and $t_0 = 1.0$. Critical energy $E_{N_c,N_l}$ is the maximal allowed energy for $N_c$ particles filled in critical orbitals and $N_l$ particles filled in localized orbitals. (a1)(b1) $N_\mathrm{p} = 2$, $L = 120$. (a2)(b2) $N_\mathrm{p} = 3$, $L = 60$. The critical energy $E_{2,0}(E_{3,0})$ is equal to the single particle ME $E = \lambda$ for two(three)-boson case.

of the few-body states. Thus the NPR exhibits sharp transitions across critical energies of $E_{N_c,N_l}$, which are related to single particle MEs, showing the robustness of MEs and critical zones in the few-body case. This novel result motivate us to realize the exactly solvable model and observe our predictions with Rydberg atoms arrays [67,76,77] which are natural platforms to simulate hard-core bosons [78,79].

*Experimental realization.*—Finally we propose an experimental scheme dubbed *incommensurate Raman superarray* of Rydberg atoms [Figs. 4(a1) and 4(a2)] to realize the model in Eq. (1) with $\kappa = 2$. The realization of the Hamiltonian is equivalent to the realization of a two-leg lattice model, with even (odd) sites mapped to the sites on $A(B)$-leg as shown in Fig. 4(a2), whose Hamiltonian reads $H = \sum_j (J_j a_j^\dagger b_j + \lambda a_j^\dagger b_{j+1} + \mathrm{H.c.}) + \sum_j V_j (a_j^\dagger a_j + b_j^\dagger b_j)$. This idea can be generalized to models with larger $\kappa$ by introducing more legs in supperarray. The Hamiltonian can be realized by Rydberg atoms using three key ingredients. (i) The AB-leg superarray has an effective Zeeman splitting

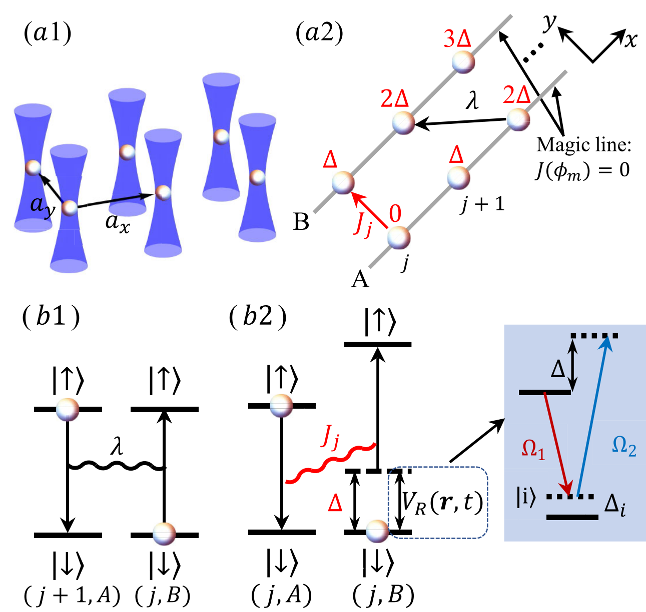


FIG. 4. Realization of the quasiperiodic mosaic model with $\kappa = 2$ in incommensurate Raman superarrays of Rydberg atoms. (a1) Rydberg atoms trapped in optical tweezers to form a two-leg array. (a2) The equivalent two-leg lattice model. The incommensurate and constant hopping coefficients are denoted as $J_j$ and $\lambda$. An effective Zeeman splitting gradient (red) modifies energy difference between Rydberg states. The Rydberg atoms are tapped along the magic angle $\phi_m$, at which the amplitude of dipolar interaction vanishes. (b1) The intrinsic dipole-dipole interaction induce constant hopping coupling $\lambda$. (b2) The interleg exchange coupling between $A_j$ and $B_j$ is suppressed by the Zeeman detuning. A two-photon Raman process (see inset) compensates this energy penalty and induces the laser-assisted dipole-dipole interaction. The compensated exchange coupling realizes incommensurate hopping coupling $J_j$.

gradient along the $x$ direction (a2); (ii) two types of nearest neighbour couplings, with constant hopping coupling $\lambda$ simulated by intrinsic dipole-exchange interaction and quasiperiodic hopping coupling $J_j$ induced by laser-assisted dipolar interaction [68]; and (iii) an on-site incommensurate chemical potential $V_j$. Two Rydberg states $|\downarrow\rangle = |70, S\rangle$ and $|\uparrow\rangle = |70, P\rangle$ are chosen to simulate empty and occupied states at each site. As illustrated in Fig. 4(b1), the intrinsic dipole-dipole interactions between two Rydberg states lead to an exchange coupling, which maps to constant hopping $\lambda$ of the hard-core bosons [78,79].

The AB-leg superarray and laser-assisted dipole-dipole interactions altogether realize the incommensurate hopping $J_j$. The inter-leg exchange couplings between $A_j$ and $B_j$ sites are suppressed by a large energy detuning $\Delta$, but can be further restored by applying the Raman coupling potential $V_R \propto \cos(4\pi\alpha j_x)\cos(\pi j_y)e^{i(\omega_2-\omega_1)t}\sigma^x_{j_x,j_y} + \mathrm{H.c.}$, which is generated by two Raman beams $\Omega_{1,2}$ with frequency difference $\omega_1 - \omega_2 \approx \Delta$ such that the Zeeman splitting can be compensated by a two-photon process [Fig. 4(b2)]. Further, the spatial modulation of the Raman potential determines the incommensurate strength of the induced exchange couplings as $J_j = 2t_0 \cos 4\pi\alpha j$ [66]. To prohibit laser-assisted exchange couplings along $x$ direction, we use the angular dependence of dipole-dipole interaction $V_{\mathrm{dd}} = d^2(1 - 3\cos^2\phi)/R^3$ with $d$ and $R$ being the dipole moment and distance between two Rydberg levels, respectively. There exists a "magic angle" $\phi_m = \arccos(1/\sqrt{3}) \approx 54.7°$ [67,80], along which the exchange coupling vanishes. By arranging the Rydberg atoms along this angle, we avoid coupling in the $x$ direction. Finally, the incommensurate mosaic potential $V_j$ can be realized via ac Stark shift [66]. Combining those ingredients together we reach the target model. The non-interacting critical states and MEs can be observed from the spectrum when a single hard-core boson is excited in this scheme, while the critical energies depicted in Fig. 3 will be observed when several hard-core bosons are excited in experiment.

*Conclusion and discussion.*—We have proposed a class of exactly solvable 1D incommensurate mosaic models to realize new and robust MEs separating critical states from localized states, and suggested a novel experimental realization through incommensurate Raman superarrays of Rydberg atoms. The robust critical states and MEs originate from a combination of incommensurately distributed zeros of hopping coefficients in the thermodynamic limit and the zero LEs, which can be analytically obtained for the proposed models and in agreement with the numerical studies. We note that these two features, serving as a generic mechanism, can provide the guidance to construct a broad class of analytic models hosting robust critical states. Moreover, we demonstrate the robustness of MEs and propose NPR as a new probe to detect the MEs in the few-body regime. A future intriguing issue is to explore the interacting effects in the finite filling regime, which might lead to exotic many-body new MEs. Our Letter broadens the concept of MEs and provides a feasible lattice model that hosts exact MEs and unambiguous critical zones with experimental feasibility.

We thank Yucheng Wang for fruitful discussions. This work was supported by National Key Research and Development Program of China (2021YFA1400900 and 2020YFA0713300), the National Natural Science Foundation of China (Grants No. 11825401, No. 12261160368, No. 12071232, and No. 12061031), and the Innovation Program for Quantum Science and Technology (Grant No. 2021ZD0302000). Q. Z. was also supported by the Science Fund for Distinguished Young Scholars of Tianjin (No. 19JCJQJC61300) and Nankai Zhide Foundation.

*These authors contribute equally to this work.
†qizhou@nankai.edu.cn
‡xiongjunliu@pku.edu.cn

[1] P. W. Anderson, Absence of diffusion in certain random lattices, Phys. Rev. **109**, 1492 (1958).
[2] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Scaling theory of localization: Absence of quantum diffusion in two dimensions, Phys. Rev. Lett. **42**, 673 (1979).
[3] D. J. Thouless, Electrons in disordered systems and the theory of localization, Phys. Rep. **13**, 93 (1974).
[4] F. Evers and A. D. Mirlin, Anderson transitions, Rev. Mod. Phys. **80**, 1355 (2008).
[5] R. Whitney, Most efficient quantum thermoelectric at finite power output, Phys. Rev. Lett. **112**, 130601 (2014).
[6] C. Chiaracane, M. T. Mitchison, A. Purkayastha, G. Haack, and J. Goold, Quasiperiodic quantum heat engines with a mobility edge, Phys. Rev. Res. **2**, 013093 (2020).
[7] K. Yamamoto, A. Aharony, O. Entin-Wohlman, and N. Hatano, Thermoelectricity near Anderson localization transitions, Phys. Rev. B **96**, 155201 (2017).
[8] X. Li, S. Ganeshan, J. H. Pixley, and S. Das Sarma, Many-body localization and quantum nonergodicity in a model with a single-particle mobility edge, Phys. Rev. Lett. **115**, 186601 (2015).
[9] R. Modak and S. Mukerjee, Many-body localization in the presence of a single-particle mobility edge, Phys. Rev. Lett. **115**, 230401 (2015).
[10] S. Nag and A. Garg, Many-body mobility edges in a one-dimensional system of interacting fermions, Phys. Rev. B **96**, 060203(R) (2017).
[11] F. A. An, K. Padavić, E. J. Meier, S. Hegde, S. Ganeshan, J. H. Pixley, S. Vishveshwara, and B. Gadway, Interactions and mobility edges: Observing the generalized Aubry-André model, Phys. Rev. Lett. **126**, 040603 (2021).
[12] T. Kohlert, S. Scherg, X. Li, H. P. Lüschen, S. Das Sarma, I. Bloch, and M. Aidelsburger, Observation of many-body

localization in a one-dimensional system with a single-particle mobility edge, Phys. Rev. Lett. **122**, 170403 (2019).
[13] S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, Ann. Isr. Phys. Soc. **3**, 133 (1980).
[14] S. Das Sarma, S. He, and X. C. Xie, Mobility edge in a model one-dimensional potential, Phys. Rev. Lett. **61**, 2144 (1988).
[15] D. J. Boers, B. Goedeke, D. Hinrichs, and M. Holthaus, Mobility edges in bichromatic optical lattices, Phys. Rev. A **75**, 063404 (2007).
[16] J. Biddle, B. Wang, D. J. Priour Jr, and S. Das Sarma, Localization in one-dimensional incommensurate lattices beyond the Aubry-André model, Phys. Rev. A **80**, 021603 (R) (2009).
[17] J. Biddle and S. Das Sarma, Predicted mobility edges in one-dimensional incommensurate optical lattices: An exactly solvable model of Anderson localization, Phys. Rev. Lett. **104**, 070601 (2010).
[18] S. Ganeshan, J. H. Pixley, and S. Das Sarma, Nearest neighbor tight binding models with an exact mobility edge in one dimension, Phys. Rev. Lett. **114**, 146601 (2015).
[19] X. Deng, S. Ray, S. Sinha, G. V. Shlyapnikov, and L. Santos, One-dimensional quasicrystals with power-law hopping, Phys. Rev. Lett. **123**, 025301 (2019).
[20] X. Li, X. Li, and S. Das Sarma, Mobility edges in one dimensional bichromatic incommensurate potentials, Phys. Rev. B **96**, 085119 (2017).
[21] H. Yao, H. Khoudli, L. Bresque, and L. Sanchez-Palencia, Critical behavior and fractality in shallow one-dimensional quasiperiodic potentials, Phys. Rev. Lett. **123**, 070405 (2019).
[22] N. Roy and A. Sharma, Fraction of delocalized eigenstates in the long-range Aubry-André-Harper model, Phys. Rev. B **103**, 075124 (2021).
[23] Y. Wang, X. Xia, L. Zhang, H. Yao, S. Chen, J. You, Q. Zhou, and X.-J. Liu, One dimensional quasiperiodic mosaic lattice with exact mobility edges, Phys. Rev. Lett. **125**, 196604 (2020).
[24] S. Roy, T. Mishra, B. Tanatar, and S. Basu, Reentrant localization transition in a quasiperiodic chain, Phys. Rev. Lett. **126**, 106803 (2021).
[25] T. Liu, X. Xia, S. Longhi, and L. Sanchez-Palencia, Anmalous mobility edges in one-dimensional quasiperiodic models, SciPost Phys. **12**, 027 (2022).
[26] J. Fraxanet, U. Bhattacharya, T. Grass, M. Lewenstein, and A. Dauphin, Localization and multifractal properties of the long-range Kitaev chain in the presence of an Aubry-André-Harper modulation, Phys. Rev. B **106**, 024204 (2022).
[27] M. Gonçalves, B. Amorim, E. V. Castro, and P. Ribeiro, Hidden dualities in 1D quasiperiodic lattice models, SciPost Phys. **13**, 046 (2022).
[28] M. Gonçalves, B. Amorim, E. V. Castro, and P. Ribeiro, Critical phase in a class of 1D quasiperiodic models with exact phase diagram and generalized dualities, arxiv:2208.07886 [Phys. Rev. Lett. (to be published)].
[29] Y. Wang, L. Zhang, S. Niu, D. Yu, and X.-J. Liu, Realization and detection of non-ergodic critical phases in optical Raman lattice, Phys. Rev. Lett. **125**, 073204 (2020).
[30] Y. Wang, L. Zhang, W. Sun, T.-F. J. Poon, and X.-J. Liu, Quantum phase with coexisting localized, extended, and critical zones, Phys. Rev. B **106**, L140203 (2022).
[31] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Anderson localization of a non-interacting Bose-Einstein condensate, Nature (London) **453**, 895 (2008).
[32] L. Fallani, J. E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Ultracold atoms in a disordered crystal of light: Towards a Bose glass, Phys. Rev. Lett. **98**, 130404 (2007).
[33] C. D'Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio, and G. Modugno, Observation of a disordered bosonic insulator from weak to strong interactions, Phys. Rev. Lett. **113**, 095301 (2014).
[34] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Observation of many-body localization of interacting fermions in a quasirandom optical lattice, Science **349**, 842 (2015).
[35] P. Bordia, H. P. Lüschen, S. S. Hodgman, M. Schreiber, I. Bloch, and U. Schneider, Coupling identical one-dimensional many-body localized systems, Phys. Rev. Lett. **116**, 140401 (2016).
[36] P. Bordia, H. P. Lüschen, S. Scherg, S. Gopalakrishnan, M. Knap, U. Schneider, and I. Bloch, Probing slow relaxation and many-body localization in two-dimensional quasi-periodic systems, Phys. Rev. X **7**, 041047 (2017).
[37] H. P. Lüschen, P. Bordia, S. Scherg, F. Alet, E. Altman, U. Schneider, and I. Bloch, Observation of slow dynamics near the many-body localization transition in one- dimensional quasiperiodic systems, Phys. Rev. Lett. **119**, 260401 (2017).
[38] H. P. Lüschen, S. Scherg, T. Kohlert, M. Schreiber, P. Bordia, X. Li, S. D. Sarma, and I. Bloch, Single-particle mobility edge in a one-dimensional quasiperiodic optical lattice, Phys. Rev. Lett. **120**, 160404 (2018).
[39] F. A. An, E. J. Meier, and B. Gadway, Engineering a flux-dependent mobility edge in disordered zigzag chains, Phys. Rev. X **8**, 031045 (2018).
[40] Y. Wang, J.-H Zhang, Y. Li, J. Wu, W. Liu, F. Mei, Y. Hu, L. Xiao, J. Ma, C. Chin, and S. Jia, Observation of interaction-induced mobility edge in an atomic Aubry-André wire, Phys. Rev. Lett. **129**, 103401 (2022).
[41] Y. Lahini, R. Pugatch, F. Pozzi, M. Sorel, R. Morandotti, N. Davidson, and Y. Silberberg, Observation of a localization transition in quasiperiodic photonic lattices, Phys. Rev. Lett. **103**, 013901 (2009).
[42] R. Merlin, K. Bajema, Roy Clarke, F.-Y. Juang, and P. K. Bhattacharya, Quasiperiodic GaAs-AlAs heterostructures, Phys. Rev. Lett. **55**, 1768 (1985).
[43] D. Tanese, E. Gurevich, F. Baboux, T. Jacqmin, A. Lemaître, E. Galopin, I. Sagnes, A. Amo, J. Bloch, and E. Akkermans, Fractal energy spectrum of a polariton gas in a Fibonacci quasiperiodic potential, Phys. Rev. Lett. **112**, 146404 (2014).
[44] V. Goblot, A. Štrkalj, N. Pernet, J. L. Lado, C. Dorow, A. Lemaître, L. Le Gratiet, A. Harouri, I. Sagnes, S. Ravets, A. Amo, J. Bloch, and O. Zilberberg, Emergence of criticality

through a cascade of delocalization transitions in quasi-periodic chains, Nat. Phys. **16**, 832 (2020).
[45] P. Roushan *et al.*, Spectroscopic signatures of localization with interacting photons in superconducting qubits, Science **358**, 6367 (2017).
[46] H. Li, Y.-Y Wang, Y.-H Shi, K. Huang, X. Song, G.-H Liang, Z.-Y Mei, B. Zhou, H. Zhang, J.-C Zhang, S. Chen, S.-P. Zhao, Y. Tian, Z.-Y Yang, Z. Xiang, K. Xu, D. Zheng, and H. Fan, Observation of critical phase transition in a generalized Aubry-André-Harper model with superconducting circuits, npj Quantum Inf. **9**, 40 (2023).
[47] Y. Hatsugai and M. Kohmoto, Energy spectrum and the quantum Hall effect on the square lattice with nextnearest-neighbor hopping, Phys. Rev. B **42**, 8282 (1990).
[48] J. H. Han, D. J. Thouless, H. Hiramoto, and M. Kohmoto, Critical and bicritical properties of Harper's equation with next-nearest-neighbor coupling, Phys. Rev. B **50**, 11365 (1994).
[49] Y. Takada, K. Ino, and M. Yamanaka, Statistics of spectra for critical quantum chaos in one-dimensional quasiperiodic systems, Phys. Rev. E **70**, 066203 (2004).
[50] F. Liu, S. Ghosh, and Y. D. Chong, Localization and adiabatic pumping in a generalized Aubry-André-Harper model, Phys. Rev. B **91**, 014108 (2015).
[51] J. Wang, X.-J. Liu, X. Gao, and H. Hu, Phase diagram of a non-Abelian Aubry-André-Harper model with p-wave superfluidity, Phys. Rev. B **93**, 104504 (2016).
[52] T. Geisel, R. Ketzmerick, and G. Petschel, New class of level statistics in quantum systems with unbounded diffusion, Phys. Rev. Lett. **66**, 1651 (1991).
[53] K. Machida and M. Fujita, Quantum energy spectra and one-dimensional quasiperiodic systems, Phys. Rev. B **34**, 7367 (1986).
[54] C. L. Bertrand and A. M. García-García, Anomalous Thouless energy and critical statistics on the metallic side of the many-body localization transition, Phys. Rev. B **94**, 144201 (2016).
[55] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A **33**, 1141 (1986).
[56] A. D. Mirlin, Y. V. Fyodorov, A. Mildenberger, and F. Evers, Exact relations between multifractal exponents at the Anderson transition, Phys. Rev. Lett. **97**, 046803 (2006).
[57] R. Dubertrand, I. García-Mata, B. Georgeot, O. Giraud, G. Lemarié, and J. Martin, Two scenarios for quantum multifractality breakdown, Phys. Rev. Lett. **112**, 234101 (2014).
[58] H. Hiramoto and S. Abe, Dynamics of an electron in quasiperiodic systems. II. Harper's model, J. Phys. Soc. Jpn. **57**, 1365 (1988).
[59] R. Ketzmerick, K. Kruse, S. Kraut, and T. Geisel, What determines the spreading of a wave packet?, Phys. Rev. Lett. **79**, 1959 (1997).
[60] M. Larcher, F. Dalfovo, and M. Modugno, Effects of interaction on the diffusion of atomic matter waves in one-dimensional quasiperiodic potentials, Phys. Rev. A **80**, 053606 (2009).
[61] Y. Wang, C. Cheng, X.-J. Liu, and D. Yu, Many-body critical phase: Extended and nonthermal, Phys. Rev. Lett. **126**, 080602 (2021).
[62] T. Xiao, D. Xie, Z. Dong, T. Chen, W. Yi, and B. Yan, Observation of topological phase with critical localization in a quasi-periodic lattice, Sci. Bull. **66**, 2175 (2021).
[63] A. Pal and D. A. Huse, Many-body localization phase transition, Phys. Rev. B **82**, 174411 (2010).
[64] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[65] A. Avila, Global theory of one-frequency Schrödinger operators, Acta. Math. **1**, 215 (2015).
[66] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.176401 for details of the (i) Lyapunov exponent, (ii) larger $\kappa$ case, (iii) finite size scaling, (iv) more perturbation results, (v) localization length, and (vi) experimental realizations, which includes Refs. [65,67,68].
[67] A. Browaeys and T. Lahaye, Many-body physics with individually controlled Rydberg atoms, Nat. Phys. **16**, 132 (2020).
[68] T.-H. Yang, B.-Z. Wang, X.-C. Zhou, and X.-J. Liu, Quantum Hall states for Rydberg arrays with laser-assisted dipole-dipole interactions, Phys. Rev. A **106**, L021101 (2022).
[69] We particularly note that (i) The limit $\epsilon \to \infty$ is not a typo, but that $\gamma_\epsilon$ are actually a constant independent of $\epsilon$, as shown by Avila's global theory and the asymptotic behavior of $\gamma_\epsilon$ in our model. (ii) The second equality is true because in our model, $T_i(\theta) = T_{i+\kappa}(\theta - 2\pi\alpha\kappa)$ and that $\mathcal{T}_\kappa$ is independent of $\theta$ in the limit of $\epsilon \to \infty$ so that in that limit $\mathcal{T}_{m\kappa} = \mathcal{T}_\kappa^m$. See Supplemental Material [66] for a more formal and detailed discussion.
[70] A. Avila, S. Jitomirskaya, and C. A. Marx, Spectral theory of extended Harper's model and a question by Erdős and Szekeres, Inventiones Mathematicae **210**, 283 (2017).
[71] Barry Simon and Thomas Spencer, Trace class perturbations and the absence of absolutely continuous spectra, Commun. Math. Phys., **125**, 113 (1989).
[72] S. Jitomirskaya and C. A. Marx, Analytic quasi-periodic cocycles with singularities and the Lyapunov exponent of extended Harper's model, Commun. Math. Phys. **316**, 237 (2012).
[73] See Ref. [71] or Proposition 7.1 in Ref. [72].
[74] Here incommensurate distributions of zeros (IDZs) refers to that the hopping coefficients approach zero $t_{2j} \to 0$ on the lattice sites $2j$ which are incommensurately distributed over the whole system in the thermodynamic limit. To be more precise, the sequence $\{2j_k\}$ of sites with $t_{2j_k} \to 0$ is an infinite sequence which does not contain any infinite subsequence whose indices increase linearly over the system, or equivalently, it does not contain any infinite subsequence whose indices can match a sublattice of the whole system.
[75] S. Iyer, V. Oganesyan, G. Refael, and D. A. Huse, Many-body localization in a quasiperiodic system, Phys. Rev. B **87**, 134202 (2013).

[76] D. Barredo, S. de Léséleuc, V. Lienhard, T. Lahaye, and A. Browaeys, An atom-by-atom assembler of defect-free arbitrary two-dimensional atomic arrays, Science **354**, 1021 (2016).

[77] M. Endres, H. Bernien, A. Keesling, H. Levine, E. R. Anschuetz, A. Krajenbrink, C. Senko, V. Vuletic, M. Greiner, and M. D. Lukin, Atom-by-atom assembly of defect-free one-dimensional cold atom arrays, Science **354**, 1024 (2016).

[78] S. de Léséleuc, V. Lienhard, P. Scholl, D. Barredo, S. Weber, N. Lang, H. P. Büchlerr, T. Lahaye, and A. Browaeys, Observation of a symmetry-protected topological phase of interacting bosons with Rydberg atoms, Science **365**, 775 (2019).

[79] V. Lienhard, P. Scholl, S. Weber, D. Barredo, S. de Léséleuc, R. Bai, N. Lang, M. Fleischhauer, H. P. Büchler, T. Lahaye, and A. Browaeys, Realization of a density-dependent peierls phase in a synthetic, spin-orbit coupled Rydberg system, Phys. Rev. X **10**, 021031 (2020).

[80] S. Ravets, H. Labuhn, D. Barredo, T. Lahaye, and A. Browaeys, Measurement of the angular dependence of the dipole-dipole interaction between two individual Rydberg atoms at a Förster resonance, Phys. Rev. A **92**, 020701(R) (2015).