Editors' Suggestion

Thermodynamic Bounds on Correlation Times

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We derive a variational expression for the correlation time of physical observables in steady-state diffusive systems. As a consequence of this variational expression, we obtain lower bounds on the correlation time, which provide speed limits on the self-averaging of observables. In equilibrium, the bound takes the form of a trade-off relation between the long- and short-time fluctuations of an observable. Out of equilibrium, the trade-off can be violated, leading to an acceleration of self-averaging. We relate this violation to the steady-state entropy production rate, as well as the geometric structure of the irreversible currents, giving rise to two complementary speed limits. One of these can be formulated as a lower estimate on the entropy production from the measurement of time-symmetric observables. Using an illustrating example, we show the intricate behavior of the correlation time out of equilibrium for different classes of observables and how this can be used to partially infer dissipation even if no time-reversal symmetry breaking can be observed in the trajectories of the observable.

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A characteristic property of noisy systems is that, even in a steady state, where the ensemble probability does not change, there are still dynamics in the system. Characterizing the instantaneous configuration of the system at time t by a collection of degrees of freedom $\boldsymbol{x}(t) = [x_1(t), \ldots, x_d(t)]$, we denote by $p_{\mathrm{st}}(\boldsymbol{x})$ the time-independent steady-state probability of observing a given configuration. Because of the presence of noise, the instantaneous configuration $\boldsymbol{x}(t)$ exhibits time-dependent fluctuations, so does any configuration-dependent observable $z(t) = z(\boldsymbol{x}(t))$, whereas the ensemble average $\langle z \rangle_{\mathrm{st}}$ is time independent. For ergodic systems, the connection between the fluctuating single realizations and the ensemble is that time-averaged observables $\bar{z}_\tau = \int_0^\tau dt z(\boldsymbol{x}(t))/\tau$ converge to their ensemble averages in the long-time limit, $\lim_{\tau\to\infty} \bar{z}_\tau = \langle z \rangle_{\mathrm{st}}$. For a sufficiently long measurement time τ, this self-averaging allows us to deduce ensemble-averaged observables from a single realization.

For any finite time, however, $\bar{z}_\tau$ fluctuates, and we characterize these fluctuations by the variance $\mathrm{Var}(\bar{z}_\tau)$. Ergodicity then implies $\lim_{\tau\to\infty} \mathrm{Var}(\bar{z}_\tau) = 0$. We can characterize the speed of the self-averaging process by defining the correlation time τ^z of the observable as

$$\tau^z = \frac{\int_0^\infty dt \mathrm{Cov}[z(t), z(0)]}{\mathrm{Var}_{\mathrm{st}}(z)}, \tag{1}$$

where Cov denotes the covariance and $\mathrm{Var}_{\mathrm{st}}(z)$ the variance of $z(\boldsymbol{x})$ in the steady state. Intuitively, τ^z measures the typical timescale on which the correlations of $z(\boldsymbol{x}(t))$ decay. For time lags τ longer than τ^z, $z(\boldsymbol{x}(t+\tau))$ is approximately independent of $z(\boldsymbol{x}(t))$, and the time average can be regarded as a sum of independent random variables. We then have

$$\frac{\mathrm{Var}(\bar{z}_\tau)}{2\mathrm{Var}_{\mathrm{st}}(z)} \simeq \frac{\tau^z}{\tau}, \tag{2}$$

in agreement with the central limit theorem.

While Eq. (1) gives a prescription to compute τ^z, an explicit expression is only available in simple cases [1], and the relation to other physical quantities is not readily apparent. In this work, we derive lower bounds on τ^z in and out of equilibrium, which constitute speed limits on the self-averaging of observables. In contrast to existing speed limits [2–9], which describe the transition between different ensemble states in stochastic systems, our speed limits characterize the decay of correlations in a steady state as a consequence of the noisy dynamics. In particular, they highlight the influence of irreversible currents and their geometric structure on correlations and self-averaging out of equilibrium.

Physical setup.—For the sake of concreteness, we will focus on the overdamped Langevin dynamics ($\boldsymbol{x} \in \mathbb{R}^d$),

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}(\boldsymbol{x}(t)) + \boldsymbol{G}\boldsymbol{\xi}(t), \tag{3}$$

with drift vector $\boldsymbol{a}(\boldsymbol{x})$ and full rank matrix $\boldsymbol{G}$, where $\boldsymbol{\xi}(t)$ is a vector of mutually independent Gaussian white noises. We stress that similar bounds can be derived for jump processes; we will address this case in a forthcoming publication. If the drift vector satisfies the potential condition

$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{\nabla}\phi(\boldsymbol{x})$, where $\boldsymbol{B} = \boldsymbol{G}\boldsymbol{G}^{\mathrm{T}}/2$ is the positive definite diffusion matrix, then the steady state of Eq. (3) is the Boltzmann-Gibbs equilibrium $p_{\mathrm{st}}(\boldsymbol{x}) = p_{\mathrm{eq}}(\boldsymbol{x}) = e^{\phi(\boldsymbol{x})}/\int d\boldsymbol{x} e^{\phi(\boldsymbol{x})}$ and the system satisfies detailed balance [10]. For generic $\boldsymbol{a}(\boldsymbol{x})$, however, the steady state is out of equilibrium and exhibits a nonvanishing rate of entropy production

$$\sigma_{\mathrm{st}} = \langle\sigma\rangle_{\mathrm{st}} \quad \text{with} \quad \sigma(\boldsymbol{x}) = \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) \cdot \boldsymbol{B}^{-1}\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}). \tag{4}$$

Here, $\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = \boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{B}\boldsymbol{\nabla} \ln p_{\mathrm{st}}(\boldsymbol{x})$ is called the local mean velocity; it describes the irreversible currents in the system as a consequence of broken detailed balance.

Main results.—For an equilibrium system, we find the variational expression and lower bound

$$\tau^z_{\mathrm{eq}} = \sup_\chi \left[\frac{\frac{\mathrm{Cov}_{\mathrm{eq}}(z,\chi)^2}{\mathrm{Var}_{\mathrm{eq}}(z)}}{D^\chi}\right] \geq \frac{\mathrm{Var}_{\mathrm{eq}}(z)}{D^z}, \tag{5}$$

where the maximum is taken over differentiable functions $\chi(\boldsymbol{x})$ and the inequality is obtained for $\chi(\boldsymbol{x}) = z(\boldsymbol{x})$. D^z quantifies the short-time fluctuations of the displacement $dz = z(\boldsymbol{x}(t+dt)) - z(\boldsymbol{x}(t))$,

$$D^z = \lim_{dt\to 0}\left[\frac{\mathrm{Var}(dz)}{2dt}\right] = \langle \boldsymbol{\nabla} z \cdot \boldsymbol{B}\boldsymbol{\nabla} z\rangle_{\mathrm{eq}}. \tag{6}$$

Since τ^z governs the long-time fluctuations of the time average, Eq. (5) implies a trade-off between short- and long-time fluctuations of the observable. The short-time fluctuations reflect the reversible diffusive motion in the system, which is the only way in which an equilibrium system can explore its configuration space and thereby self-average. We note that the same type of relation also applies to underdamped Langevin dynamics (see Sec. III of the Supplemental Material [11]) and jump processes. Therefore, Eq. (13) constitutes a universal trade-off between diffusion and self-averaging that applies to a wide range of equilibrium processes. Rather than a particular observable, we can also characterize the self-averaging behavior of the system by defining the intrinsic correlation time

$$\tau^* = \sup_z\left[\tau^z\right], \tag{7}$$

that is, by considering the observable with the slowest self-averaging speed. For the latter, we find the identity

$$\tau^*_{\mathrm{eq}} = \sup_\chi\left[\frac{\mathrm{Var}_{\mathrm{eq}}(\chi)}{D^\chi}\right] = \frac{1}{\lambda^1_{\mathrm{eq}}}, \tag{8}$$

and thus the bound Eq. (5) becomes tight for the slowest observable. Crucially, this expression is equivalent to a well-known variational formula (see, e.g., Chap. 6.6.2 in Ref. [10]) for the first nonzero eigenvalue of the generator of the dynamics, λ^1_{eq}. Since the eigenvalue governs the asymptotic approach of the system toward equilibrium, $|p_t(\boldsymbol{x}) - p_{\mathrm{eq}}(\boldsymbol{x})| \sim e^{-\lambda^1_{\mathrm{eq}} t}$, this relation formally establishes that both correlations in equilibrium and the relaxation toward equilibrium are governed by the same timescale.

For a nonequilibrium system, on the other hand, the correlation time is reduced, $\tau^z \leq \tau^z_{\mathrm{eq}}$, corresponding to faster self-averaging [15–18]. Here, τ^z_{eq} is the correlation time in the equilibrium system with the same steady state. Intuitively, nonequilibrium systems can explore their configuration space not only by reversible diffusive motion, but also by irreversible, directed motion in the form of currents, which provide another mechanism for self-averaging. The thermodynamic consequence of the irreversible currents is dissipation characterized by the entropy production rate $\sigma_{\mathrm{st}} > 0$. Indeed, an appropriate nonequilibrium generalization of Eq. (5) can be expressed in terms of entropy production,

$$\tau^z \geq \frac{\mathrm{Var}_{\mathrm{st}}(z)}{D^z + \sigma_{\mathrm{st}}\mathrm{Var}_\sigma(z)}, \tag{9}$$

where $\mathrm{Var}_\sigma(z)$ is the variance of $z(\boldsymbol{x})$ with respect to a probability distribution reweighted according to the local rate of entropy production; see Eq. (16). While the dynamics of the system is accelerated by driving it out of equilibrium, Eq. (9) states that there is a minimal amount of dissipation associated with this acceleration, and we refer to it as the dissipation speed limit.

Moreover, we derive a complementary lower bound on the correlation time,

$$\tau^z \geq \sup_{\chi_\perp}\left[\frac{\frac{\mathrm{Cov}_{\mathrm{st}}(z,\chi_\perp)^2}{\mathrm{Var}_{\mathrm{st}}(z)}}{D^{\chi_\perp}}\right]. \tag{10}$$

While this resembles the equilibrium result Eq. (5), the maximum is restricted to functions whose gradient is orthogonal to the irreversible currents $\boldsymbol{\nabla}\chi_\perp(\boldsymbol{x}) \cdot \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = 0$, and thus, the right-hand side is always smaller than the equilibrium value. In contrast to Eq. (9), this bound is purely geometric as it does not depend on the magnitude of the currents; we refer to it as the geometric speed limit. In particular, this implies that the acceleration of the self-averaging tends to saturate in the strong driving limit. One consequence of Eq. (10) is that, when the observable $z(\boldsymbol{x})$ itself satisfies $\boldsymbol{\nabla} z(\boldsymbol{x}) \cdot \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = 0$, then choosing $\chi_\perp(\boldsymbol{x}) = z(\boldsymbol{x})$, we see that it will obey the equilibrium bound Eq. (5) even out of equilibrium. The geometric intuition behind this is that the observable is constant along the flow lines of the irreversible currents, so the currents do not contribute to changes in the observable and therefore do not accelerate its self-averaging.

Variational formula and equilibrium speed limits.—We now outline how the different speed limits may be derived from a variational formula for the correlation time,

$$\tau^z = \sup_\chi \left[\frac{\frac{\mathrm{Cov}_{\mathrm{st}}(z,\chi)^2}{\mathrm{Var}_{\mathrm{st}}(z)}}{\langle \nabla\chi \cdot \boldsymbol{B}\nabla\chi\rangle_{\mathrm{st}} + \sup_\eta \left[\frac{\langle \chi\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2}{\langle\nabla\eta\cdot\boldsymbol{B}\nabla\eta\rangle_{\mathrm{st}}}\right]} \right], \tag{11}$$

where the maxima are taken with respect to differentiable functions $\chi(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$. While an exact solution is equally hard to obtain, Eq. (11) is immediately useful for deriving bounds, which are not apparent from Eq. (1). The derivation of Eq. (11) is provided in Sec. I of the Supplemental Material [11]. The second term in the denominator is positive and vanishes in equilibrium, where $\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) \equiv 0$. Thus, we immediately conclude that $\tau^z \leq \tau^z_{\mathrm{eq}}$, the latter being the correlation time in the (unique) equilibrium system with the same steady state $p_{\mathrm{eq}}(\boldsymbol{x}) = p_{\mathrm{st}}(\boldsymbol{x})$ and diffusion matrix $\boldsymbol{B}$,

$$\tau^z_{\mathrm{eq}} = \sup_\chi \left[\frac{\frac{\mathrm{Cov}_{\mathrm{eq}}(z,\chi)^2}{\mathrm{Var}_{\mathrm{eq}}(z)}}{\langle \nabla\chi\cdot\boldsymbol{B}\nabla\chi\rangle_{\mathrm{eq}}} \right] \geq \frac{\mathrm{Var}_{\mathrm{eq}}(z)}{\langle\nabla z\cdot\boldsymbol{B}\nabla z\rangle_{\mathrm{eq}}}. \tag{12}$$

The inequality follows by choosing $\chi(\boldsymbol{x}) = z(\boldsymbol{x})$ and noting that $\mathrm{Cov}_{\mathrm{eq}}(z,z) = \mathrm{Var}_{\mathrm{eq}}(z)$. Identifying the term in the denominator with D^z, Eq. (6) (see Sec. II of the Supplemental Material [11]) yields Eq. (5). Using Eq. (7) and taking the maximum over $z(\boldsymbol{x})$ in Eq. (11), we also obtain a variational formula for the intrinsic correlation time

$$\tau^* = \sup_\chi \left[\frac{\mathrm{Var}_{\mathrm{st}}(\chi)}{\langle\nabla\chi\cdot\boldsymbol{B}\nabla\chi\rangle_{\mathrm{st}} + \sup_\eta\left[\frac{\langle\chi\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2}{\langle\nabla\eta\cdot\boldsymbol{B}\nabla\eta\rangle_{\mathrm{st}}}\right]} \right], \tag{13}$$

which immediately gives Eq. (8) in equilibrium.

Nonequilibrium speed limits.—While, in principle, we can obtain a lower bound on τ^z out of equilibrium by any specific choice of $\chi(\boldsymbol{x})$ in Eq. (11), this still involves the maximization over $\eta(\boldsymbol{x})$ and is thus not explicit. However, we can further bound the second term in the denominator of Eq. (11). First, we note that we have

$$\begin{aligned} \langle\chi\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2 &= \langle(\chi-\chi_0)\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2 \\ &\leq \langle(\chi-\chi_0)^2\boldsymbol{\nu}_{\mathrm{st}}\cdot\boldsymbol{B}^{-1}\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}\langle\nabla\eta\cdot\boldsymbol{B}\nabla\eta\rangle_{\mathrm{st}}. \end{aligned} \tag{14}$$

In the first step, we used that, from the steady-state condition of the Fokker-Planck equation, the local mean velocity satisfies $\nabla\cdot[\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x})p_{\mathrm{st}}(\boldsymbol{x})] = 0$, such that $\langle\chi_0\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}} = 0$ for any constant χ_0 after integrating by parts. In the second step, we applied the Cauchy-Schwarz inequality. The second factor precisely cancels the one in the denominator, so that we have the lower bound

$$\tau^z \geq \sup_\chi \left[\frac{\frac{\mathrm{Cov}_{\mathrm{st}}(z,\chi)^2}{\mathrm{Var}_{\mathrm{st}}(z)}}{\langle\nabla\chi\cdot\boldsymbol{B}\nabla\chi\rangle_{\mathrm{st}} + \langle\chi^2\sigma\rangle_{\mathrm{st}} - \frac{\langle\chi\sigma\rangle_{\mathrm{st}}^2}{\sigma_{\mathrm{st}}}} \right], \tag{15}$$

where we maximized with respect to χ_0 and used the definition of the local entropy production rate Eq. (4). Introducing the entropy-rescaled probability density,

$$p_\sigma(\boldsymbol{x}) = \frac{\sigma(\boldsymbol{x})}{\sigma_{\mathrm{st}}} p_{\mathrm{st}}(\boldsymbol{x}), \tag{16}$$

and recalling Eq. (6), this can be written as

$$\tau^z \geq \sup_\chi \left[\frac{\frac{\mathrm{Cov}_{\mathrm{st}}(z,\chi)^2}{\mathrm{Var}_{\mathrm{st}}(z)}}{D^\chi + \sigma_{\mathrm{st}}\mathrm{Var}_\sigma(\chi)} \right], \tag{17}$$

which yields Eq. (9) after choosing $\chi(\boldsymbol{x}) = z(\boldsymbol{x})$.

On the other hand, integrating by parts, we have

$$\langle\chi\nabla\eta\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2 = \langle\eta\nabla\chi\cdot\boldsymbol{\nu}_{\mathrm{st}}\rangle_{\mathrm{st}}^2. \tag{18}$$

For any $\chi_\perp(\boldsymbol{x})$ that satisfies $\nabla\chi_\perp(\boldsymbol{x})\cdot\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = 0$, this term vanishes and we obtain Eq. (10). Thus, whenever there exists a function $\chi(\boldsymbol{x})$ whose gradient is orthogonal to the currents and $\mathrm{Cov}_{\mathrm{st}}(z,\chi) \neq 0$, we obtain a nonzero lower bound whose value is independent of the magnitude of the currents, but only depends on their geometric structure. If the observable itself satisfies $\nabla z(\boldsymbol{x})\cdot\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x})$, we can choose $\chi_\perp(\boldsymbol{x}) = z(\boldsymbol{x})$ in Eq. (10) and have

$$\tau^{z_\perp} \geq \frac{\mathrm{Var}_{\mathrm{st}}(z_\perp)}{D^{z_\perp}}. \tag{19}$$

This implies that observables, whose level lines are parallel to the irreversible currents, obey the equilibrium trade-off between their short- and long-time fluctuations.

Estimation of entropy production.—The fact that Eq. (9) relates the correlation time out of equilibrium to dissipation suggests that it may be possible to estimate the latter by measuring the correlation time. To make this relation explicit, we note that, if $\chi_{\min} \leq \chi(\boldsymbol{x}) \leq \chi_{\max}$ is a bounded function with range $\Delta\chi = \chi_{\max} - \chi_{\min}$, Popoviciu's inequality yields an upper bound on the variance, $\mathrm{Var}_\sigma(\chi) \leq \Delta\chi^2/4$. Plugging this into Eq. (17) and solving for σ_{st} yields

$$\sigma_{\mathrm{st}} \geq \frac{4}{\Delta\chi^2}\left(\frac{2\mathrm{Cov}_{\mathrm{st}}(\chi,z)^2}{\mathrm{Var}(\bar{z}_\tau)} - D^\chi\right). \tag{20}$$

Thus, we can obtain a lower bound on the rate of entropy production by measuring the fluctuations of the time average of an observable and its steady-state correlations with any

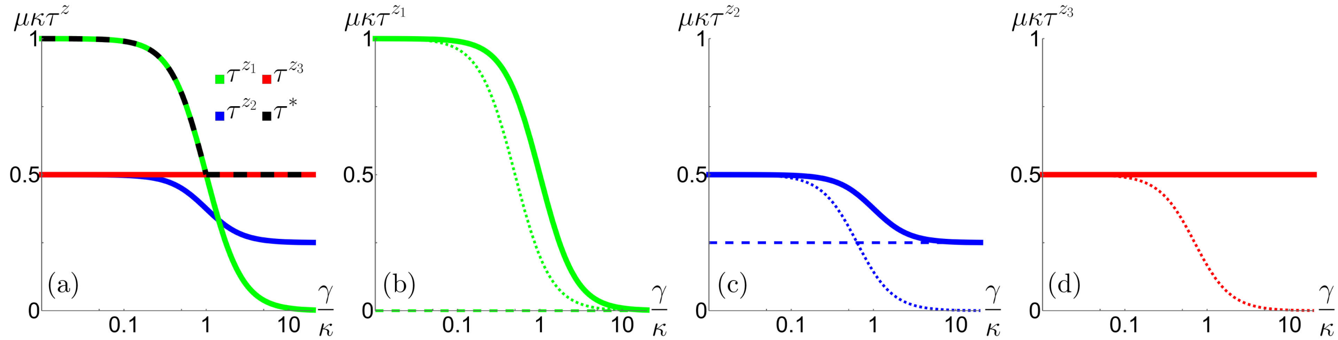

FIG. 1. The correlation times Eq. (22) and corresponding speed limits Eq. (24) for three observables of a driven Brownian particle, as a function of the driving strength. (a) The exact results for the correlation times of the observables $z_1 = x_1$, $z_2 = x_1^2$, and $z_3 = x_1^2 + x_2^2$ (colored lines), as well as the intrinsic (maximal) correlation time τ^* (black dashed line). (b)–(d) The dissipative speed limit [Eq. (9), dotted lines] and the geometric speed limit [Eq. (10), dashed lines] on the correlation times compared with the exact results (solid lines). Note that for z_3, the exact value and the geometric speed limit are identical.

bounded observable. In particular, if $z(\boldsymbol{x})$ itself is bounded, then we have

$$\sigma_{\mathrm{st}} \geq \frac{4}{\Delta z^2}\left(\frac{2\mathrm{Var}_{\mathrm{st}}(z)^2}{\mathrm{Var}(\bar{z}_\tau)} - D^z\right), \tag{21}$$

which allows us to estimate the entropy production by measuring how much the short- and long-time fluctuations violate the equilibrium trade-off Eq. (5). We remark that several bounds relating entropy production to measurable quantities have recently been obtained, most famously the thermodynamic uncertainty relation [19,20] and its many generalizations [21–27]. However, most of these bounds rely on the measurement of some time-antisymmetric observable like a time-integrated current and its fluctuations. A notable exception is Ref. [26], where a bound in terms of time-symmetric observables was obtained, which, however, vanishes in the steady state and only estimate the excess part of the entropy production. To our knowledge, the lower bound Eq. (21) is the first result that only involves time-symmetric quantities in the steady state.

Illustration.—In order to illustrate the above speed limits for a concrete system, we consider a two-dimensional Brownian particle trapped in a parabolic potential $U(\boldsymbol{x}) = \kappa(x_1^2 + x_2^2)/2$ and driven by the nonconservative force $\boldsymbol{F}_{\mathrm{NC}}(\boldsymbol{x}) = \gamma(x_2, -x_1)$. The particle is in contact with a heat bath at temperature T and its mobility is μ. The steady state of this system is given by the Gaussian $p_{\mathrm{st}} = e^{-U(\boldsymbol{x})/T}/\int d\boldsymbol{x}e^{-U(\boldsymbol{x})/T}$, independent of the driving strength γ, while the local mean velocity is given by $\boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = \mu\boldsymbol{F}_{\mathrm{NC}}(\boldsymbol{x})$. Since the forces in this system are linear, all quantities can be computed analytically; see Sec. IVA of the Supplemental Material [11] for the details of the calculation. We consider the observables $z_1(\boldsymbol{x}) = x_1$, $z_2(\boldsymbol{x}) = x_1^2$, and $z_3(\boldsymbol{x}) = x_1^2 + x_2^2$; their correlation times are

$$\tau^{z_1} = \frac{1}{\mu\kappa\left(1+\frac{\gamma^2}{\kappa^2}\right)}, \quad \tau^{z_2} = \frac{2+\frac{\gamma^2}{\kappa^2}}{4\mu\kappa\left(1+\frac{\gamma^2}{\kappa^2}\right)}, \quad \tau^{z_3} = \frac{1}{2\mu\kappa}, \tag{22}$$

which are shown graphically in Fig. 1(a). In equilibrium ($\gamma = 0$), all three observables satisfy the corresponding speed limit Eq. (5) with equality and thus saturate the trade-off between short- and long-time fluctuations. Out of equilibrium, the observables exhibit a markedly different behavior: For $z_1 = x_1$, the correlation time tends to zero, and its self-averaging becomes arbitrarily fast with increasing driving strength. For $z_2 = x_1^2$, the correlation time also decreases when driving the system out of equilibrium; however, it saturates in the limit of strong driving, indicating that stronger driving cannot speed up its self-averaging arbitrarily. The correlation time of $z_3 = x_1^2 + x_2^2$ is not affected at all by the driving, and this particular driving force cannot speed up its self-averaging. By contrast, the intrinsic correlation time is (see Sec. IVB of the Supplemental Material [11] for the calculation)

$$\tau^* = \max(\tau^{z_1}, \tau^{z_3}) = \tau^*_{\mathrm{eq}} \max\left(\frac{1}{1+\frac{\gamma^2}{\kappa^2}}, \frac{1}{2}\right), \tag{23}$$

where $\tau^*_{\mathrm{eq}} = 1/(\mu\kappa)$ is the equilibrium value. Interestingly, the worst-case observable exhibiting the slowest self-averaging depends on the parameters of the system, which leads to a nonsmooth behavior of the intrinsic correlation time.

We now turn to the nonequilibrium speed limits, denoting the lower bound obtained from the dissipation speed limit Eq. (9) by τ^z_{diss} and given by the geometric speed limit Eq. (10) by τ^z_{geom}. For the observables defined above, we have

$$\tau_{\mathrm{diss}}^{z_1} = \frac{1}{\mu\kappa\left(1 + 4\frac{\gamma^2}{\kappa^2}\right)}, \qquad \tau_{\mathrm{geom}}^{z_1} = 0, \tag{24a}$$

$$\tau_{\mathrm{diss}}^{z_2} = \frac{1}{2\mu\kappa\left(1 + \frac{5}{2}\frac{\gamma^2}{\kappa^2}\right)}, \qquad \tau_{\mathrm{geom}}^{z_2} = \frac{1}{2\mu\kappa}, \tag{24b}$$

$$\tau_{\mathrm{diss}}^{z_3} = \frac{1}{2\mu\kappa\left(1 + 2\frac{\gamma^2}{\kappa^2}\right)}, \qquad \tau_{\mathrm{geom}}^{z_3} = \frac{1}{2\mu\kappa}, \tag{24c}$$

which are shown in Figs. 1(b)–1(d). Here we used $\chi_\perp(\boldsymbol{x}) = x_1^2 + x_2^2$ in Eq. (10) to obtain a definite lower bound. Comparing this to the actual values in Eq. (22), we see that the geometric speed limit is tight in the strong driving limit for all observables. For $z_1 = x_1$, the dissipative speed limit captures the qualitative behavior very well, since the correlation time can be reduced arbitrarily much by increasing the driving strength. For $z_2 = x_1^2$, on the other hand, the decrease in the correlation time only occurs for small to moderate driving, whereas for strong driving, we observe saturation to the geometric speed limit. For $z_3 = x_1^2 + x_2^2$, the behavior is described by the geometric speed limit for any driving strength, since $\boldsymbol{\nabla} z_3(\boldsymbol{x}) \cdot \boldsymbol{\nu}_{\mathrm{st}}(\boldsymbol{x}) = 0$ and thus Eq. (19) applies.

Finally, we apply Eq. (20) to obtain an estimate on the dissipation from a measurement of x_1. Since Eq. (20) requires a bounded function, we choose $\chi(\boldsymbol{x}) = x_1$ for $-\Delta/2 \leq x_1 \leq \Delta/2$, $\chi(\boldsymbol{x}) = -\Delta/2$ for $x_1 < -\Delta/2$, and $\chi(\boldsymbol{x}) = \Delta/2$ for $x_1 > \Delta/2$, which corresponds to introducing a cutoff value $\pm\Delta/2$ on x_1. For this choice, the right-hand side of Eq. (20) can be evaluated analytically in terms of Δ; see Eq. (S85) in the Supplemental Material [11]. Numerically maximizing this expression with respect to Δ yields an estimate $\hat{\sigma}$ for the steady-state entropy production, which is shown in Fig. 2. The estimate is positive for any driving strength, allowing us to conclude that the system is out of equilibrium. For strong driving, the estimate tends to $(1/\pi) \approx 0.32$ of the true value and thus reproduces a sizable fraction of the actual dissipation. We stress that this estimate relies only on a measurement of the fluctuations of x_1, while there is no current in the x_1 direction, prohibiting the application of the thermodynamic uncertainty relation if only x_1 can be observed. Moreover, it can be shown (see Sec. IVD of the Supplemental Material [11]) that the joint probability density for x_1, $p(x_1, \tau; x_1', 0)$ is symmetric under exchanging x_1 and x_1', so no dissipation can be inferred from the time-reversal properties of the observable.

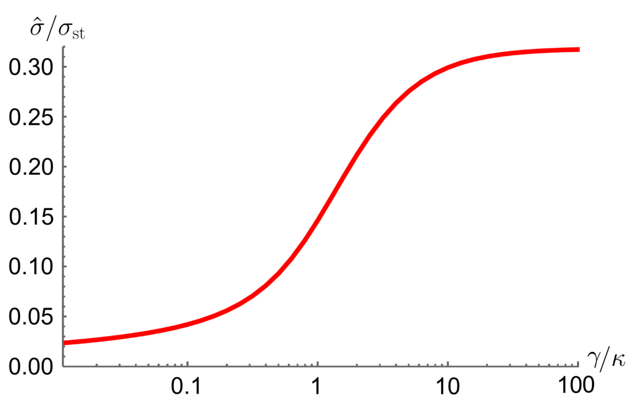

FIG. 2. The ratio of the lower bound $\hat{\sigma}$ on the entropy production rate given by Eq. (20) and the actual value, as a function of the driving strength.

Discussion.—We demonstrated that the self-averaging of observables in the steady state has to obey certain speed limits. For equilibrium systems, the speed limit takes the form of a trade-off between short- and long-time fluctuations, which represents the fact that an equilibrium system can only explore its configuration space via diffusive motion. On the other hand, for nonequilibrium systems, the presence of directed currents generally accelerates the dynamics, allowing for faster self-averaging at the cost of incurring dissipation.

The complimentary speed limits Eqs. (9) and (10) for nonequilibrium systems highlight different aspects of this phenomenon: A given speedup requires a minimal amount of dissipation, enabling us to estimate dissipation from a measurement of the violation of the equilibrium speed limit. Nevertheless, driving the system ever further from equilibrium will not generally result in an arbitrary acceleration of the dynamics. In an equilibrium system, the slowest timescale, often corresponding to a large-scale physical process, limits how fast the system can explore its configuration space. Driving the system out of equilibrium in an appropriate manner can speed up this particular process. However, at some point another timescale associated with another, smaller-scale process unaffected by the driving will become the limiting factor. Increasing the magnitude of the driving will then not result in further acceleration, unless the geometric structure of the driving is changed to also affect smaller scales.

We remark that driving a system out of equilibrium to speed up its relaxation has a concrete application in the form of so-called nonreversible sampling in Monte Carlo simulations [15–18,28–32]. There, a perturbation of a given equilibrium system is constructed to preserve the steady-state distribution while breaking detailed balance, speeding up both the convergence toward the steady state as well as the sampling of the configuration space in the steady state. Our speed limits demonstrate that the effectiveness of this approach is constrained both by the strength of the driving as well as its geometric structure.

From the point of view of thermodynamic inference (see Ref. [33] for a recent perspective), Eq. (21) provides a way of estimating dissipation from the measurement of the fluctuations of a time-symmetric observable, even in cases where no currents or time asymmetry of the transition probability can be used to do so. We speculate that this approach may be useful in models of active matter [34,35], where time asymmetry often manifests only in hidden degrees of freedom and cannot be observed directly from the particles' trajectories, while it may still affect the correlation times of trajectory-dependent observables. In Ref. [36], bounds on the fluctuations of time-symmetric counting observables in terms of the dynamical activity have been obtained; in Ref. [37] it was shown that the entropy production can likewise be bounded by such fluctuations. This suggests that considering time-symmetric observables can be a useful extension of the toolset of thermodynamic inference.

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