Editors' Suggestion

# Universal Lower Bound on Topological Entanglement Entropy

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Entanglement entropies of two-dimensional gapped ground states are expected to satisfy an area law, with a constant correction term known as the topological entanglement entropy (TEE). In many models, the TEE takes a universal value that characterizes the underlying topological phase. However, the TEE is not truly universal: it can differ even for two states related by constant-depth circuits, which are necessarily in the same phase. The difference between the TEE and the value predicted by the anyon theory is often called the "spurious" topological entanglement entropy. We show that this spurious contribution is always nonnegative, thus the value predicted by the anyon theory provides a universal lower bound. This observation also leads to a definition of TEE that is invariant under constant-depth quantum circuits.

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Ground states of 2D gapped Hamiltonians are believed to satisfy an area law: the entanglement entropy of a region cannot increase faster than its perimeter. In many examples, the entropy of the reduced density matrix on a disk $A$ takes the form

$$S(\sigma_A) = \alpha|\partial A| - \gamma + \cdots, \tag{1}$$

where $\alpha|\partial A|$ is the leading "area law" term proportional to the boundary length, $\gamma$ is a constant term, and the ellipsis represents terms that vanish for large regions.

The constant term $\gamma$, under natural assumptions, was argued to be universal, i.e., the same for all gapped ground states in a given phase [1,2]. In particular, $\gamma$ takes a form determined solely by the underlying anyon theory of the phase, $\gamma = \log \mathcal{D}$, where $\mathcal{D} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension of the anyons and $d_a$ is the quantum dimension of the anyon $a$. Given its connection to anyons, the constant $\gamma$ has been termed the "topological entanglement entropy" (TEE). The TEE can be computed in both nonsolvable [3,4] and solvable models [5–9], and it is often used as a smoking gun signature of topological order or to distinguish two phases.

A common way to extract the TEE is to use a judicious linear combination of entropies of adjacent regions. We focus on the definition in Ref. [2], where the TEE $\gamma$ is defined using the conditional mutual information $I(A:C|B)_\sigma \coloneqq S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_B) - S(\sigma_{ABC})$ of regions $A$, $B$, and $C$ forming an annulus as in Fig. 1(a),

$$I(A:C|B)_\sigma \equiv 2\gamma, \tag{2}$$

where $\sigma$ is a ground state.

While the TEE is a useful diagnostic of topological order, it was soon observed [10] that it is not a genuine invariant of the topological phase, unlike, e.g., [11,12]. Two ground states are in the same phase if they are connected by a constant-depth circuit consisting of local gates. However, $\gamma$ as defined by Eq. (2) can change under such a circuit. In fact, a shallow circuit acting on a product state may achieve a nonzero value of $I(A:C|B)$ for arbitrarily large regions [13–15]. Deviations of $\gamma$ from the purportedly universal value $\log \mathcal{D}$ have been called spurious contributions or the spurious TEE. States with spurious TEE exist in both trivial and nontrivial topological phases. These examples often arise from symmetry-protected topological phases [14–18], but perhaps not always [19].

Our main result partially restores the universality of the TEE by showing the spurious contribution is always nonnegative. Thus, $\log \mathcal{D}$ provides a universal lower bound for the TEE $\gamma$,

$$\gamma \geq \log \mathcal{D}. \tag{3}$$

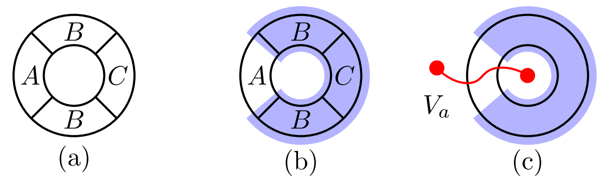


FIG. 1. (a) The partition used to calculate the TEE. (b) Support of the unitary $U$ considered. (c) String operator $V_a$ creates an anyon $a$ in the interior of the annulus and its antiparticle in the exterior.

 

This observation leads directly to a definition of TEE on infinite systems that is invariant under constant-depth circuits, by minimizing the ordinary TEE over such circuits. More specifically, for a state $\rho$ defined on the infinite 2D plane, the following quantity

$$\gamma_{\min} = \lim_{R\to\infty} \min_{U} \frac{1}{2} I(A_R:\ C_R|B_R)_{U\rho U^\dagger}, \tag{4}$$

yields $\log \mathcal{D}$ (for a class of states elaborated below), where the regions $A_R$, $B_R$, $C_R$ have a radius and thickness of order $R$, and the minimum is taken over circuits $U$ of depth $d < cR$ for some fixed constant $c \in (0,1)$.

Our result in Eq. (3) helps restore the TEE as a rigorous diagnostic to distinguish topological phases, albeit with limitations. For instance, if a state has $I(A:C|B) = \log 2$ for some large regions, it may still be in the trivial phase (where $2\log\mathcal{D} = 0$), and indeed such examples exist. However, it *cannot* be in the same phase as the toric code, which has $2\log\mathcal{D} = 2\log 2$; the latter would require a negative spurious TEE, which we rule out.

*Setup.*—We now explain our main result more precisely. We consider a special class of bosonic quantum many-body states, defined on infinite two-dimensional lattices, which we refer to as "reference states," closely related to the states considered in Ref. [20].

*Definition 1.*—A state $\sigma$ is a reference state if (i) the TEE calculated as $\gamma_0 = \frac{1}{2} I(A:C|B)_\sigma$ is the same for any choice of regions topologically equivalent to Fig. 1(a), and (ii) the mutual information between two subsystems is zero for any two nonadjacent subsystems.

Our main technical result is the following inequality, which holds for any reference state $\sigma$ and for any circuit $U$ whose depth is small compared to the radius and thickness of the annulus $ABC$:

$$I(A:C|B)_{U\sigma U^\dagger} \geq I(A:C|B)_\sigma \equiv 2\gamma_0. \tag{5}$$

In other words, we show that constant-depth circuits can never decrease the TEE, when acting on a reference state.

To understand the implications of this result, note that the set of reference states includes all ground states of string-net [7–9] and quantum double models [6] and, more generally, any state satisfying the entanglement bootstrap axioms [20]. For all of these examples, the rhs of Eq. (5) is known to equal $2\log\mathcal{D}$ [1,2,20]. Therefore, (5) implies the claimed lower bound (3) for any state obtained by a constant-depth circuit acting on a string-net, quantum double, or entanglement bootstrap state. [We discuss a generalization of (3) to general 2D gapped ground states in the Supplemental Material [21], Appendix G.] Similarly, we deduce (4) with $\gamma_{\min} = \log\mathcal{D}$ for any state $\rho$ given by a finite-depth circuit $V$ applied to a reference state, where the minimum is achieved by the circuit $U = V^{-1}$.

While we work in the plane for concreteness, our proof also applies to the TEE defined on any disklike region embedded in an arbitrary manifold.

*Example: Toric code.*—To explain the key idea behind our proof, it is instructive to first focus on a concrete reference state $\sigma$, namely, the toric code ground state [6] on a plane. (Our argument here will rely on special properties of the toric code state, but later we will generalize the proof to all states satisfying Definition 1.) For this state, $I(A:C|B)_\sigma = 2\log 2$ so that $\gamma_0 = \log 2$ [5]. If we now apply a constant-depth quantum circuit $U$, defining $\tilde{\sigma} = U\sigma U^\dagger$, in general, $I(A:C|B)_{\tilde{\sigma}} \neq 2\log 2$. Nevertheless, we will show that, for a sufficiently large annulus $ABC$, we still have the lower bound

$$I(A:C|B)_{\tilde{\sigma}} \geq 2\log 2. \tag{6}$$

We first prove the bound (6) for a special class of constant-depth circuits $U$, namely, those that are supported within a constant distance of $BC$ [Fig. 1(b)]. Later, we will extend this result to general constant-depth circuits.

Our basic strategy is to construct a state $\tilde{\lambda}$ that is "locally indistinguishable" from $\tilde{\sigma}$. More precisely, we will construct a state $\tilde{\lambda}$ that is indistinguishable from $\tilde{\sigma}$ over $AB$ and $BC$; that is, $\tilde{\lambda}_{AB} = \tilde{\sigma}_{AB}$ and $\tilde{\lambda}_{BC} = \tilde{\sigma}_{BC}$. We can then express $I(A:C|B)_{\tilde{\sigma}}$ in terms of $I(A:C|B)_{\tilde{\lambda}}$ using the identity

$$I(A:C|B)_{\tilde{\sigma}} = I(A:C|B)_{\tilde{\lambda}} + S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC}). \tag{7}$$

By the strong subadditivity of the entropy (SSA) [27], $I(A:C|B)_{\tilde{\lambda}} \geq 0$, so

$$I(A:C|B)_{\tilde{\sigma}} \geq S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC}). \tag{8}$$

We will obtain the desired lower bound (6) from a judicious choice of $\tilde{\lambda}$.

The easiest way to construct an appropriate $\tilde{\lambda}$ is to first find a state $\lambda$ that is locally indistinguishable from the toric code ground state $\sigma$. More precisely, we need a $\lambda$ that is indistinguishable from $\sigma$ over the past light cone of $AB$ and $BC$ (with respect to $U$). Once we find such a $\lambda$, we can then set $\tilde{\lambda} = U\lambda U^\dagger$.

We construct such a $\lambda$ using a probabilistic mixture of toric code *excited* states. (Later, we use a more general approach.) For each anyon type $a \in \mathcal{C} = \{1, e, m, \epsilon\}$, we define a corresponding excited state $\rho^{(a)}$ by $\rho^{(a)} = V_a \sigma V_a^\dagger$, where $V_a$ is a unitary (open) string operator that places an anyon excitation $a$ in the interior of the annulus and its antiparticle in the exterior [Fig. 1(c)]. We then define $\lambda = \sum_a p_a \rho^{(a)}$ for some probability distribution $\{p_a:\ a \in \mathcal{C}\}$. Note that $\lambda$ has the requisite indistinguishability property as long as the end points of the string operators $V_a$ (where the anyons are created) are far enough away from the annulus to lie outside the past light cones of $AB$ and $BC$.

To proceed, we must evaluate the entropy difference $S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC})$. Here it is convenient to choose the path of the string operators $V_a$ so that they avoid the region of support of the constant-depth circuit $U$ (which by assumption is supported near $BC$). Then $V_a$ commutes with $U$ so $\tilde{\lambda}$ can be written as a probabilistic mixture of the form

$$\tilde{\lambda} = \sum_a p_a \tilde{\rho}^{(a)}, \quad \tilde{\rho}^{(a)} = V_a \tilde{\sigma} V_a^\dagger. \tag{9}$$

Crucially, the $\tilde{\rho}^{(a)}$ states have two simplifying properties: (i) different $\tilde{\rho}^{(a)}_{ABC}$ are orthogonal, and (ii) $S(\tilde{\rho}^{(a)}_{ABC}) = S(\tilde{\sigma}_{ABC})$. Intuitively, property (i) follows from the fact that each $\tilde{\rho}^{(a)}_{ABC}$ belongs to a different anyon sector on the annulus. More formally, (i) follows from the existence of a collection of (closed) string operators supported within $ABC$ that take on different eigenvalues in each state $\tilde{\rho}^{(a)}_{ABC}$. (These string operators are simply the closed versions of $UV_aU^\dagger$; they can be drawn within $ABC$ whenever $ABC$ is wider than twice the circuit depth of $U$). Meanwhile, property (ii) follows from the fact that the $V_a$ are products of single-site unitaries; in particular, each $V_a$ can be written as a product of a unitary acting entirely within $ABC$ and a unitary acting entirely outside $ABC$, neither of which changes the entanglement entropy of $ABC$.

Given properties (i) and (ii) of $\tilde{\rho}^{(a)}$, the entropy difference can be computed as

$$S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC}) = H(\{p_a\}), \tag{10}$$

where $H(\{p_a\}) = -\sum_a p_a \log(p_a)$ is the Shannon entropy of the probability distribution $\{p_a\}$. Substituting (10) into (8), we obtain

$$I(A:C|B)_{\tilde{\sigma}} \geq H(\{p_a\}). \tag{11}$$

To get the best bound, we choose the probability distribution that maximizes $H(\{p_a\})$, namely, the uniform $p_a = \frac{1}{4}$. Then $H(\{p_a\}) = 2\log 2$, yielding the desired bound (6).

To complete the argument, we extend the bound (11) to general constant-depth circuits $U$. First, recall the entanglement entropy of a subsystem is invariant under unitaries acting exclusively within the subsystem or its complement. Thus, we can make the replacement

$$I(A:C|B)_{U\sigma U^\dagger} = I(A:C|B)_{U'\sigma U'^\dagger}, \tag{12}$$

where $U'$ is a constant-depth quantum circuit that acts trivially deep in the interior of $A$ and also trivially far outside $ABC$ [Fig. 3(a)]. Here, we are using the fact that $U$ is a constant-depth quantum circuit, and therefore we can "cancel out" its action in a subsystem $S$ by multiplying by an appropriate unitary supported in the light cone of $S$ (Fig. 2).

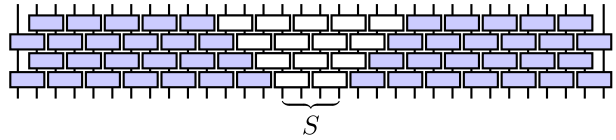


FIG. 2. For any constant-depth circuit $U$, for any subsystem $S$, we can obtain a circuit $U'$ of same depth acting trivially on $S$, by removing from $U$ the "light cone" (white gates) of $S$.

By SSA, $I(A:C|B)$ cannot increase when $A$ shrinks. Therefore,

$$I(A:C|B)_{U'\sigma U'^\dagger} \geq I(A':\ C|B)_{U'\sigma U'^\dagger}, \tag{13}$$

where $A' \subset A$ is shown in Fig. 3(b). Finally, applying the same reasoning as in (12), we can replace

$$I(A':\ C|B)_{U'\sigma U'^\dagger} = I(A':C|B)_{U''\sigma U''^\dagger}, \tag{14}$$

where $U''$ is a constant-depth circuit acting on the region shown in Fig. 3(c). Combining (12)–(14), we deduce that

$$I(A:C|B)_{U\sigma U^\dagger} \geq I(A':\ C|B)_{U''\sigma U''^\dagger}. \tag{15}$$

The lower bound (15) is useful because it allows us to leverage our results from the first part of the proof. In particular, $U''$ is precisely the kind of constant-depth quantum circuit that we analyzed above, so $I(A':\ C|B)_{U''\sigma U''^\dagger} \geq 2\log 2$ for any sufficiently large annulus $A'BC$. Substituting this inequality into (15), we obtain the desired bound (6).

*General case.*—Our proof for the toric code proceeded in three steps. First, we derived a lower bound (8) for $I(A:C|B)_{\tilde{\sigma}}$ in terms of the entropy difference $S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC})$, where $\tilde{\lambda}$ is any state that is indistinguishable from $\tilde{\sigma}$ over $AB$ and $BC$. Second, we constructed an appropriate $\tilde{\lambda}$ and computed the desired entropy difference (10) in the special case where $U$ is a constant-depth circuit supported within a constant distance of $BC$ [Fig. 1(b)]. Combining these two results, we obtained the desired lower bound, but only for this special class of circuits $U$. In the third and final step, we extended this bound to arbitrary constant-depth circuits $U$ using the inequality (15).

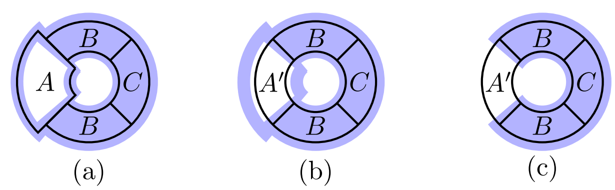


FIG. 3. (a) We first remove gates from $U$ on a "hole" within $A$; we call the new circuit $U'$. (b) We then deform $A$ to $A' \subset A$ such that the boundary of the annulus $A'BC$ is only partially covered. (c) We then further remove some of the gates in the vicinity of $A'$, obtaining $U''$.

Conveniently, the first and third steps of our proof immediately generalize to any reference state $\sigma$ since they do not use any properties of the toric code. On the other hand, in the second step, we used the specific structure of the toric code string operators [28], so we need a different argument for this step in the general case. In particular, instead of defining $\tilde{\lambda}$ in terms of a mixture of excited states, we will now define it in terms of the "maximum-entropy state." Consider a larger annulus $Y = ABC \cup \mathrm{Supp}(U)$, with $U$ again as in Fig. 1(b). Define a density matrix $\lambda$ to be the maximum-entropy state consistent with the reduced density matrices of $\sigma$ over the past light cones of $AB$ and $BC$. We then define $\tilde{\lambda} = U\lambda U^\dagger$.

By construction, $\tilde{\lambda}$ is indistinguishable from $\tilde{\sigma}$ over $AB$ and $BC$ and therefore the lower bound (8) still holds. The only remaining question is the value of the entropy difference on the right-hand side. We claim that

$$S(\tilde{\lambda}_{ABC}) - S(\tilde{\sigma}_{ABC}) = S(\lambda_Y) - S(\sigma_Y) \tag{16}$$

and, in turn,

$$S(\lambda_Y) - S(\sigma_Y) = 2\gamma_0, \tag{17}$$

provided that $A$, $B$, and $C$ are sufficiently large compared to the circuit depth. See Eq. (E1) in the Supplemental Material [21] for a self-contained derivation of Eq. (17) starting from Definition 1. We are finished once we prove Eq. (16).

We now present the proof of (16). Our main tool is the following lemma about entropy differences under reversible channels, proven in the Supplemental Material [21].

*Lemma 1.*—Let $\rho$ and $\rho'$ be density matrices over $PQ$ such that $\rho'_Q = \rho_Q$ and $\rho'_P = \rho_P$. Let $\mathcal{R}$, $\mathcal{T}$ be a pair of quantum channels $\mathcal{R}: Q \to \hat{Q}$ and $\mathcal{T}: \hat{Q} \to Q$. If

$$\mathcal{T}\circ\mathcal{R}(\rho_{PQ}) = \rho_{PQ},$$
$$\mathcal{T}\circ\mathcal{R}(\rho'_{PQ}) = \rho'_{PQ}, \tag{18}$$

then

$$S(\rho_{PQ}) - S(\rho'_{PQ}) = S[\mathcal{R}(\rho_{PQ})] - S[\mathcal{R}(\rho'_{PQ})]. \tag{19}$$

To apply Lemma 1 to our setup, we let $\rho = \bar{\lambda}$ and $\rho' = \bar{\sigma}$, where $\bar{\lambda} = \bar{U}\lambda\bar{U}^\dagger$ and $\bar{\sigma} = \bar{U}\sigma\bar{U}^\dagger$ and where $\bar{U}$ is a unitary obtained by removing the gates in $U$ that are deep in the interior of the annulus $ABC$ [Figs. 4(a) and 4(b)]. We then let $P$, $Q$ be a partition of the annulus $ABC$ of the form shown in Fig. 4, such that $P \subset A$ is sufficiently far away from the support of $\bar{U}$ (see Fig. 9 in the Supplemental Material [21] and Ref. [30]). By construction, $\bar{\lambda}$ and $\bar{\sigma}$ are indistinguishable on $P$. In the Supplemental Material [21], Appendix E, we show that the two states are indistinguishable on $Q$ as well [Eq. (E2)], thus fulfilling the premise of Lemma 1.

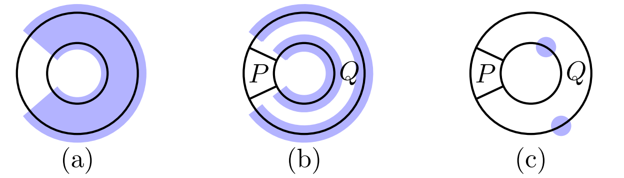


FIG. 4. Using the procedure in Fig. 2, we remove the gates in $U$ that act deep in the interior of the annulus without changing the entropy; starting from $U$, whose support is depicted as the blue region in (a), we obtain a unitary $\bar{U}$, whose support is shown in (b). Then the support of $\bar{U}$ becomes a union of two disks, which is topologically equivalent to (c) after regrouping sites.

Below we will construct quantum channels $\mathcal{R}: Q \to \hat{Q}$ and $\mathcal{T}: \hat{Q} \to Q$, with $\hat{Q} \equiv Q \cup \bar{u}$, where $\bar{u}$ is the support of $\bar{U}$. These will obey (18) with $\mathcal{R}(\bar{\lambda}_{PQ}) = \lambda_{PQ\cup\bar{u}}$ and $\mathcal{R}(\bar{\sigma}_{PQ}) = \sigma_{PQ\cup\bar{u}}$. Because $PQ \cup \bar{u} = Y$ and also $S(\bar{\lambda}_{PQ}) = S(\tilde{\lambda}_{PQ})$ and $S(\bar{\sigma}_{PQ}) = S(\tilde{\sigma}_{PQ})$, once we construct these channels, we can immediately deduce Eq. (16) from Lemma 1. This will then complete our proof of the bound (5), as explained earlier.

Now let us discuss our construction of $\mathcal{R}$ and $\mathcal{T}$. These maps are constructed from compositions of $\bar{U}$, partial trace, and the Petz map [31].

To clearly render the construction of $\mathcal{R}$ and $\mathcal{T}$, we depict $\bar{u}$ as two disks of smaller sizes, as in Fig. 4(c). Loosely speaking, $\mathcal{R}$ removes the circuits in the disks and $\mathcal{T}$ restores them.

The map $\mathcal{R}$ is constructed by applying a partial trace followed by a Petz map, best described by Fig. 5. In the first step, we trace out the region $Q \cap \bar{u}$. This step effectively removes the circuit $\bar{U}$, mapping $\bar{\lambda}_{PQ}$ to $\lambda_{PQ\setminus\bar{u}}$ and $\bar{\sigma}_{PQ}$ to $\sigma_{PQ\setminus\bar{u}}$. In the second step, we apply the Petz map $\Phi^\sigma_{v\to v\bar{u}}$. We show in the Supplemental Material [21], Eq. (E3), that this step extends $\lambda_{PQ\setminus\bar{u}}$ to $\lambda_{PQ\cup\bar{u}}$ and $\sigma_{PQ\setminus\bar{u}}$ to $\sigma_{PQ\cup\bar{u}}$,

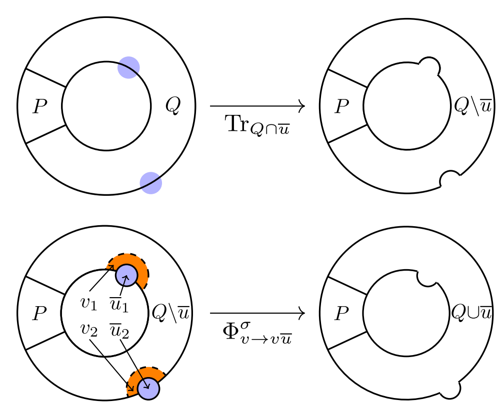


FIG. 5. The construction of $\mathcal{R}$. The first step is the partial trace $\mathrm{Tr}_{\bar{u}}$ over the support of $\bar{U}$, and the second step applies the Petz map $\Phi^\sigma_{v\to v\bar{u}}$, where $v = v_1 v_2$ and $\bar{u} = \bar{u}_1\bar{u}_2$. The blue subsystem is $\bar{u}$, the support of $\bar{U}$.

$$\lambda_{PQ\cup\bar{u}} = \Phi^{\sigma}_{v\to v\bar{u}}(\lambda_{PQ\setminus\bar{u}}), \quad \sigma_{PQ\cup\bar{u}} = \Phi^{\sigma}_{v\to v\bar{u}}(\sigma_{PQ\setminus\bar{u}}). \quad (20)$$

Combining the two steps, we see that $\mathcal{R}$ maps $\bar{\lambda}_{PQ}$ to $\lambda_{PQ\cup\bar{u}}$ and $\bar{\sigma}_{PQ}$ to $\sigma_{PQ\cup\bar{u}}$, as required. As for the map $\mathcal{T}$, this can be constructed by simply applying $\bar{U}$ and tracing out $\bar{u}\setminus Q$. Clearly, these operations map $\lambda_{PQ\cup\bar{u}}$ to $\bar{\lambda}_{PQ}$ and $\sigma_{PQ\cup\bar{u}}$ to $\bar{\sigma}_{PQ}$, as required.

*Discussion.*—The lower bound (5) can be generalized to the case that $\sigma$ contains an anyon in the interior of the annulus, because the proof only required the invariance of the TEE under deformations of the annulus. The lower bound then becomes $\gamma_0 = \log(\mathcal{D}/d_a)$ [20] for anyon $a$ in the interior. We expect similar lower bounds can be derived in a variety of setups, including systems with defects or higher-dimensional systems.

Although we have only proven the lower bound (3) for states obtained by constant-depth circuits acting on reference states, we expect that (3) holds more generally. In fact, we argue heuristically in the Supplemental Material [21], Appendix G, that (3) holds for any 2D gapped bosonic ground state. The key idea is to use the fact that reference states can already realize all "doubled" 2D topological phases obtained by stacking a bosonic topological phase onto its time-reversed partner [2,8,9].

Our Definition 1 and bound (5) actually apply beyond area law states. For instance, coupling an area law reference state to a hot surface (modeled by an identity density matrix) for a short period of time cannot decrease the TEE of the joint system. We speculate that similar arguments may apply to the 3D toric code at finite temperature [32].

An interesting open question is whether our bounds apply to the TEE defined using the alternative partition in Ref. [1]. It would also be interesting to know whether similar results hold for a TEE defined via Rényi entropies, which are easier to measure in quantum simulators [33].

A final question is to understand how generically our bound (3) is *saturated.* How much fine-tuning is required to obtain a spurious TEE that does not decay with distance? Despite hints in this direction [14,15,18,34], the general question remains open.

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[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.166601 for development of some lemmas related to quantum Markov chains for the purpose of deriving Eq. (17), which includes Refs. [22–26].

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