Absence of Mobility Edge in Short-Range Uncorrelated Disordered Model: Coexistence of Localized and Extended States

Adway Kumar Das* and Anandamohan Ghosh†
Indian Institute of Science Education and Research Kolkata, Mohanpur, 741246 India

Ivan M. Khaymovich‡
Nordita, Stockholm University and KTH Royal Institute of Technology Hannes Alfvéns väg 12, SE-106 91 Stockholm, Sweden and Institute for Physics of Microstructures, Russian Academy of Sciences, 603950 Nizhny Novgorod, GSP-105, Russia

(Received 11 May 2023; accepted 26 August 2023; published 18 October 2023)

Unlike the well-known Mott's argument that extended and localized states should not coexist at the same energy in a generic random potential, we formulate the main principles and provide an example of a nearest-neighbor tight-binding disordered model which carries both localized and extended states without forming the mobility edge. Unexpectedly, this example appears to be given by a well-studied β ensemble with independently distributed random diagonal potential and inhomogeneous kinetic hopping terms. In order to analytically tackle the problem, we locally map the above model to the 1D Anderson model with matrix-size- and position-dependent hopping and confirm the coexistence of localized and extended states, which is shown to be robust to the perturbations of both potential and kinetic terms due to the separation of the above states in space. In addition, the mapping shows that the extended states are nonergodic and allows one to analytically estimate their fractal dimensions.

DOI: 10.1103/PhysRevLett.131.166401

A mobility edge (ME) [1], separating localized and extended states in disordered systems, has been established and studied for decades. Known to be present in various semiconductors, amorphous media, and even in disordered liquid metals, ME has become a hallmark of the Anderson [2] and many-body [3] localization transitions. It is commonly believed that in any short-range model, with random uncorrelated entries, just below the localization transition ME separates the localized and extended states in the energy spectrum. Therefore, eigenstates with different localization properties cannot coexist at the same energy for the same system parameter values. The argument behind this, given by Nevill F. Mott [1], is straightforward: if extended and localized states coexist at the same energy, any perturbation of the disorder potential immediately hybridizes them, making both extended. The main result of this Letter is the formulation of the main ingredients, necessary to realize both extended or localized states to emerge at a given energy, present in any realization, and provide a corresponding example of one-dimensional (1D) disordered short-range model, demonstrating this. Hence, disorder averaging forbids the ME formation.

The first of these ingredients is that the system should avoid level degeneracy or attraction, i.e., it should possess some (residual) level repulsion. Indeed, any resonance in the energy levels, corresponding to localized and extended states, should be suppressed in order to observe their coexistence without hybridization. Among short-range uncorrelated models, the natural ensemble for tunable and controllable level repulsion is the so-called β ensemble, represented by real symmetric tridiagonal matrices, with independent random elements [4]. Such an ensemble is parametrized by the Dyson's index β and has the same joint probability distribution of eigenvalues like in the well-known Gaussian random-matrix ensembles [5], but for any real β along with $\beta = 1, 2, 4$. The limit $\beta \to 0$ yields uncorrelated eigenvalues as observed in integrable systems [6], whereas $\beta \geq 1$ produces correlated spectra as in chaotic systems [7].

Second, to suppress the overlap of the localized and extended states, the latter ones should be nonergodic, with a support set smaller than the entire Hilbert space. It is known that the presence of disorder may break ergodicity in some quantum systems, while they remain nonintegrable, thus, delocalized [8–14]. Such a nonergodic extended (NEE) phase is associated with a nontrivial scaling of the eigenstate fluctuations [15] and can be captured by various random matrix models [16–26], hierarchical graphs [27], and many-body disordered systems [3,12,28]. Quite surprisingly, it has been recently shown that the above mentioned β ensemble also hosts the NEE phase over a finite parameter range [29].

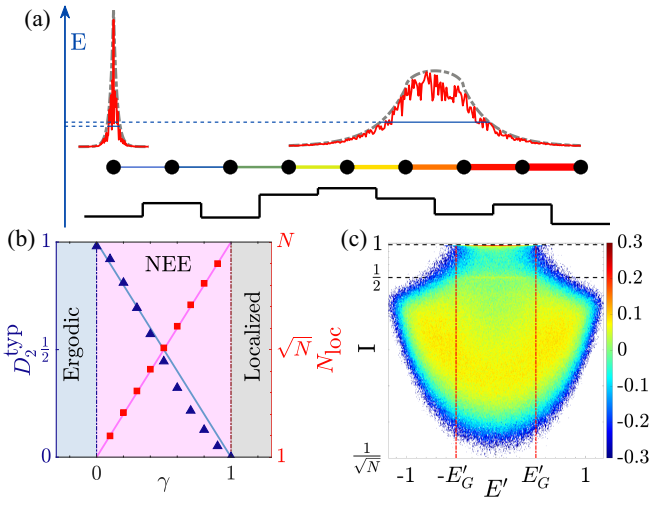

FIG. 1. (a) Schematic of β ensemble, given by 1D lattice in Eq. (1). The hopping increases along the lattice (thickness, color of links) and yields localized (left) and extended (right) states, coexisting at the same energy, but living in spatially different system parts. (b) Phase diagram of the β ensemble, with three distinct phases along with typical fractal dimension in the NEE phase, $D_2^{\rm typ} \approx 1-\gamma$, and the number $N_{\rm loc} \sim N^{\gamma}$ of strongly localized states, $|\Psi_{\rm loc}\rangle$. (c) Joint density $\mathrm{P}(I, E')$ of IPR, I and the energy, $E' = E/\epsilon_\beta$, rescaled to the bandwidth ϵ_β for $\gamma = 0.7$. The colorbar indicates the values of joint density in $\log_N$ scale, and $(-E'_{\rm G}, E'_{\rm G})$ is the rescaled energy band for coexistent states. Numerical results are for $N = 8192$ and 128 realizations.

Last, but not least, the NEE states should be separated in space from the localized ones. The construction of the β ensemble [4] introduces inhomogeneity, where the distributions of hopping matrix elements y_n significantly depend on the lattice coordinate n; see Fig. 1(a) and Eq. (2). Consequently the eigenstates of β ensemble become spatially separated.

In this Letter, we show that by fulfilling all of the above three crucial ingredients, β ensemble provides an ideal platform for realizing coexistent localized and extended eigenstates. We numerically confirm that $\mathcal{O}(\beta^{-1})$ localized states coexist along with the extended states within the middle of the spectral band without forming any ME. In addition, the NEE phase of β ensemble is shown to exhibit anomalies in the spectral statistics [29]: nearby eigenvalues remain uncorrelated, while two distant eigenvalues, separated by $\Delta E > (N\beta)^{-\frac{1}{2}}$, can be correlated. Here and further, N is the system size. Such a feature is in sharp distinction from the NEE states observed in a paradigmatic Rosenzweig-Porter ensemble (RPE) [16,19,30]. This last aspect unveils the origin of the absence of the ME. We analytically explain it by locally mapping β ensemble to a 1D Anderson model with N-dependent hopping strength. This mapping demonstrates that β ensemble separates into nearly independent blocks, where localized and (nonergodic) extended eigenstates appear to be located in spatially separated blocks, but share nearly the same spectral energies.

The β-ensemble is composed of the matrices H, with the following nonzero elements [4]:

$$H_{n,n} = x_n, \quad H_{n,n+1} = H_{n+1,n} = y_n$$
$$x_n \sim \mathcal{N}(0,1), \quad \sqrt{2}y_n \sim \chi_{n\beta}, \tag{1}$$

where $\mathcal{N}(0,1)$ is the normal distribution and χ_k is the chi distribution with a degree of freedom k. H represents a 1D lattice with an open boundary, where a particle can randomly hop to the nearest neighbors under disordered on-site potentials [Fig. 1(a)]. The relative strengths of on site potentials $\{x_n\}$ and the hopping amplitudes $\{y_n\}$ make it convenient to reparametrize β as $\beta = N^{-\gamma}$ [29,31], leading to the typical behavior of hopping amplitudes, as shown in the Supplemental Material [32],

$$y_n^{\rm typ} \sim \begin{cases} \exp\left(-\frac{N^\gamma}{n}\right), & n < N^\gamma \\ \sqrt{\frac{n}{N^\gamma}}, & n > N^\gamma \end{cases}. \tag{2}$$

Thus, on average y_n increases across the lattice and presents a highly inhomogeneous system. As we will show, it is this inhomogeneity which makes the β ensemble host three distinct phases: ergodic ($\gamma \le 0$), NEE ($0 < \gamma < 1$), and localized ($\gamma \ge 1$), reported earlier [29]. In the localized phase, $\gamma > 1$, all the levels are uncorrelated and Poisson-distributed as the eigenstates are localized with a finite support in the thermodynamic limit ($N \to \infty$); see the left state in Fig. 1(a). Contrarily in the ergodic phase, all energies are correlated irrespective of their distance, and the bulk eigenstates are extended over the entire Hilbert space. NEE phase in β ensemble appears at $0 < \gamma < 1$ due to its inhomogeneous hopping terms; otherwise phase transition is absent in generic 1D systems with uncorrelated short-range hopping [35,36]. Inhomogeneity of the β ensemble is formed by the transformation from the Gaussian ensemble [4] and shows a particular case of the Krylov-basis representation of any generic Hamiltonian [37]. The β ensemble can be realized in ultracold atoms with the tunable location of optical tweezers [38–42] or in photonic systems of optical waveguides [43–47].

As known from [29], the NEE phase, characterized by the scaling with the system size N of the inverse participation ratio (IPR), $I_j = \sum_{n=1}^{N} |\Psi_j(n)|^4$ of the eigenstate at the energy E_j having nth component $\Psi_j(n)$, shows the typical fractal dimension $D_2^{\rm typ} \approx 1-\gamma$, extracted from the spectral average $\langle \log I_j \rangle \simeq -D_2^{\rm typ} \log N$ [Fig. 1(b)]. Hence most of the eigenstates occupy an extensive number, but a vanishing fraction of the Hilbert space in the NEE phase. However, here we see the first unexpected feature that the density of IPR shows a peak around $I = 1$ indicating the presence of strongly localized states, $|\Psi_{\rm loc}\rangle$, along with a finite fraction of extended states with $I \ll 1$ [32]. We consider a small tolerance value $\delta I \ll 1$ and identify $|\Psi_{\rm loc}\rangle$

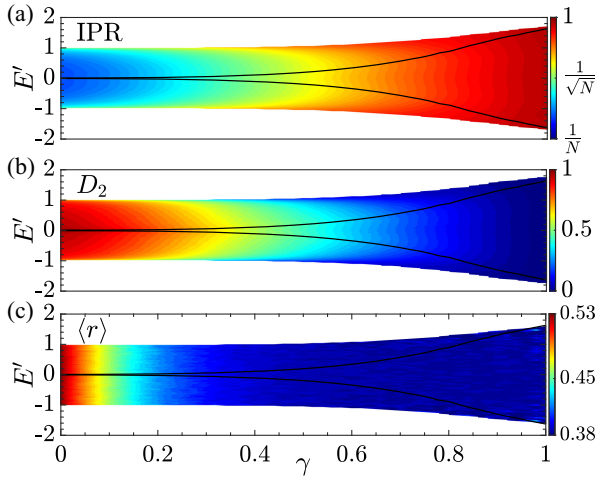

FIG. 2. (a) IPR, (b) fractal dimension D_2, (c) mean level-spacing ratio r for $N = 8192$ in the γ-E' plane. Solid black lines indicate $(-E'_G, E'_G)$, the energy band of localized states.

as a state with $I > 1 - \delta I$. In Fig. 1(b), we show the ensemble-averaged number of strongly localized states, $N_{\rm loc} \propto N^{\gamma}$ coexisting with the finite fraction of NEE states.

Looking at the joint density of IPR and energy, Fig. 1(c), we unveil the spectral structure of $|\Psi_{\rm loc}\rangle$. In the NEE phase of the β ensemble, $|\Psi_{\rm loc}\rangle$ appears only within an energy window $(-E_G, E_G)$, centered around midspectrum ($E = 0$). The ensemble average $\langle E_G \rangle \propto N^{\alpha_G}$ is nearly N independent, with $\alpha_G \ll 1$, irrespective of $\delta I \ll 1$ [32]. Importantly, Fig. 2(c) indicates that within $(-E_G, E_G)$, IPR takes a wide range of values from $\mathcal{O}(1)$ to N^{-D_2}, convincingly demonstrating the coexistence of localized and extended states.

Does this coexistence form a ME? ME has been observed in the Lévy ensemble [20,48], quasiperiodic lattice [11,46,47,49–55], 3D Anderson model [56], quantum random energy model [57], and many-body localization [3]. In order to search for ME in the β ensemble, first, we compute the energy-dependent IPR,

$$I(E) = \frac{1}{N\rho(E)} \sum_{j=1}^{N} I_j \delta(E - E_j), \tag{3}$$

where $\rho(E)$ is the global density of states (DOS). For a given energy $I(E) \to 0$ [$\simeq \mathcal{O}(1)$] for extended [localized] states. Hence, an existence of a ME would have implied $I(E)$ exhibiting an energy-dependent crossover from 0 to $\mathcal{O}(1)$ within $(-E_G, E_G)$. Figure 2(a), showing IPR in the γ-E' plane of the energy $E' = E/\epsilon_\beta$, rescaled by the global energy bandwidth $\epsilon_\beta = 2\sqrt{\langle E^2 \rangle} = \sqrt{4 + 2N^{1-\gamma}}$, provides *no evidence* of ME at any value of γ.

However, as IPR in the β ensemble is fat-tail distributed and may not be a self-averaging quantity [29], we extract the energy-dependent fractal dimension $D_2(E)$ from the system-size scaling of median(I) within small windows across the energy spectrum. Figure 2(b) shows that $D_2(E)$ is energy independent in $(-E_G, E_G)$ irrespective of γ. This convincingly shows that the ME is absent in the NEE phase of the β ensemble despite the coexistence of $\mathcal{O}(N^{\gamma})$ localized states around $E = 0$.

In addition to the spectral homogeneity of the eigenstate properties, in Fig. 2(c) we show it in energy-level correlations across the spectrum. The ensemble- and spectral-averaged level-spacing ratio $\langle r \rangle$ [33,58] exhibits criticality only around $\gamma = 0$, implying that the neighboring eigenvalues are typically uncorrelated in the NEE phase [29] in the thermodynamic limit, having some residual level repulsion $\beta = N^{-\gamma}$ at finite sizes. To uncover the energy-resolved short-range spacing correlations, we compute

$$r(E) = \frac{1}{N\rho(E)} \sum_{n=1}^{N} r_n \delta(E - E_n)$$
$$r_n = \min\left(\tilde{r}_n, \frac{1}{\tilde{r}_n}\right), \quad \tilde{r}_n = \frac{E_{n+1} - E_n}{E_n - E_{n-1}}. \tag{4}$$

The ensemble-averaged r in the γ-E' plane, Fig. 2(c), also shows no energy-dependent crossover from Poisson, $\langle r \rangle \approx 0.38$, to Wigner-Dyson, $\langle r \rangle \approx 0.53$, statistics [33,58], for all γ. Thus, short-range energy correlations are also homogeneous over the bulk energy spectrum.

Unlike short-range spectral statistics, the long-range two-level correlations can be characterized by the power spectrum $P(\omega)$ of the fluctuations of the nth unfolded energy level $\mathcal{E}_n$ [59] around its mean position, n vs frequency ω, Fourier dual to n. For Poisson (Wigner-Dyson) statistics, the power spectrum of the fluctuations $\delta_n \equiv \mathcal{E}_n - n$ decays as ω^{-2} (ω^{-1}) [34,60,61]. In the β ensemble, $P(\omega)$ is known to show a crossover in the frequency domain [29] (see Fig. 3):

$$P(\omega) \propto \begin{cases} \omega^{-1}, & \omega < \omega_c \\ \omega^{-2}, & \omega > \omega_c \end{cases}, \qquad \omega_c = \frac{\pi}{N^{\gamma}}. \tag{5}$$

The critical frequency, ω_c, corresponds to the unfolded energy scale N^{γ} such that two unfolded energy levels $\mathcal{E}_{1,2}$ are correlated if $|\mathcal{E}_1 - \mathcal{E}_2| > N^{\gamma}$, and uncorrelated otherwise. Therefore, in an energy window $[\mathcal{E} - (\Delta\mathcal{E}/2), \mathcal{E} + (\Delta\mathcal{E}/2)]$, $P_{\mathcal{E}}(\omega) \sim \omega^{-2}$ shows only Poisson behavior for $\Delta\mathcal{E} < N^{\gamma}$ irrespective of $\mathcal{E}$. Such a long-range correlation is unusual and complimentary to the energy correlations typically observed in various models like RPE [16,62,63], deformed Poisson ensemble [21], or driven Aubry-André models [10]. Usually the eigenstates hybridize below the Thouless energy, $\mathcal{E}_{\rm Thouless}$ [64], while distant eigenvalues, separated by $\Delta\mathcal{E} > \mathcal{E}_{\rm Thouless}$, remain uncorrelated [23,65]. The spectral homogeneity of unusual $P_{\mathcal{E}}(\omega)$ of the β ensemble is related to the ME absence and can be understood from the following analytical consideration.

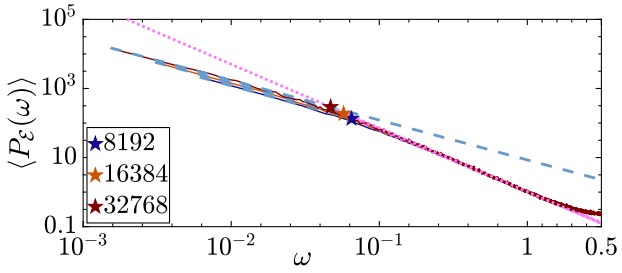

FIG. 3. Ensemble-averaged power spectrum vs dimensionless frequency for $\gamma = 0.4$ in a randomly chosen energy window with width $\Delta\mathcal{E} = (N/8)$. Dashed (dotted) lines show ω^{-1} (ω^{-2}) fits, and the star denotes the critical frequency.

The above numerical results unambiguously show that the coexistence of localized and extended states fails to form any ME. All this can be analytically understood from a spatially local mapping of the β ensemble to an Anderson model. Indeed, in the β ensemble, the hopping amplitudes over the first $\mathcal{O}(N^\gamma)$ sites are much smaller in magnitude in comparison to the typical on-site potentials $x_n^{\rm typ} \sim \mathcal{O}(1)$. Hence the sites $1, 2, \ldots, n \lesssim N^\gamma$ can be considered to be effectively disconnected from the rest of the lattice and hosts $|\Psi_{\rm loc}\rangle$. Corresponding DOS follows the normal distribution, while both short- and long-range energy correlations are given by the Poisson statistics [32].

The structure of the eigenstates at the remaining sites can be understood as follows. The hopping amplitudes for lattice sites $n > N^\gamma$, given by Eq. (2), are self-averaging and homogeneous with small relative fluctuations $|y_{n+\delta n} - y_n| \ll y_n$ for $\delta n \ll n$. So, we can formally separate the lattice into the spatial blocks $n \in \Delta_l$, where hopping is approximately the same and

$$\Delta_l \equiv [N^{\gamma+\zeta_l}, cN^{\gamma+\zeta_l}]. \tag{6}$$

Here $\zeta_{l+1} - \zeta_l \equiv \delta\zeta = (\log c/\log N)$, $c \sim \mathcal{O}(1)$, and $N^{\gamma+\zeta_{l_{\rm max}}} = N$. The number of sites in Δ_l is $N_l \sim N^{\gamma+\zeta_l}$, and the hopping amplitudes are asymptotically constant and given by $y_l^{\rm typ} \approx \sqrt{(N^{\gamma+\zeta_l}/N^\gamma)} = N^{(\zeta_l/2)}$. Thus, the β ensemble within Δ_l is equivalent to the 1D Anderson model, with uncorrelated diagonal disorder $\mathcal{O}(1)$ and homogeneous N-dependent hopping $N^{\zeta_l/2}$. As the localization length in the 1D Anderson model is given by $\xi_l \sim \langle y_n^2\rangle/\langle x_n^2\rangle = N^{\zeta_l}$ [66,67] within each Δ_l, the eigenstates exponentially decay, $|\Psi(j)| \sim \exp(-|j - j_{\rm loc}|/\xi_l)$, and localization centers $j_{\rm loc}$ randomly distributed in Δ_l. Furthermore, as each sub-block of length $\xi_l \ll N_l$ has N^{ζ_l} eigenstates, each block Δ_l can accommodate $N_l/\xi_l \approx N^\gamma$ sub-blocks. The eigenstates within each sub-block hybridize, but not across sub-blocks. This local mapping implies that each sub-block has Gaussian DOS with the bandwidth [68]

$$\Delta E_l \simeq \sqrt{\frac{1}{N^{\zeta_l}} \sum_{i_l \in \Delta_l} N^{\zeta_l}} = N^{\frac{\zeta_l}{2}} \tag{7}$$

and the mean level spacing $\delta_l = \Delta E_l/N^{\zeta_l} = N^{-\zeta_l/2}$. The above mapping explains all the NEE-phase properties of the β ensemble as shown below.

First, $\mathcal{O}(N^\gamma)$ states, confined in the first spatial block Δ_0 are all localized with $\xi_0 \sim N^{\delta\zeta} \sim \mathcal{O}(1)$. Equation (7) also justifies that such δ-localized states have an energy bandwidth $E_{\rm G} \approx \mathcal{O}(1)$, not scaling with N [32].

Second, the eigenstates in the largest block are the least localized with a localization length $\xi_{l_{\rm max}} \sim N^{1-\gamma}$. This block contains a finite fraction of all sites, $N_{l_{\rm max}} \sim \mathcal{O}(N)$ and then defines the typical fractal dimension $D_2^{\rm typ} = 1 - \gamma$ in the NEE phase of the β ensemble [29]. As both the number of eigenstates, with localization length $1 \le \xi_l \le N^{1-\gamma}$, and the bandwidth, Eq. (7), increase with l, the distribution of any localization measure exhibits a fat tail [32]. Thus, within $(-E_{\rm G}, E_{\rm G})$, where bands from all l overlap, all the localization lengths are possible. This structure of spatial-separated states with different ξ_l explains the ME absence and the coexistence of localized and extended states in the β ensemble.

Third, the above mapping explains the anomalous long-range energy correlations in the NEE phase of the β ensemble. The eigenvalues from all blocks constitute the global DOS; hence the bandwidth ϵ_β is given by that of the largest block at $l_{\rm max} \simeq (1-\gamma)/\delta\zeta$ with a bandwidth $N^{(1-\gamma)/2}$. Thus, global mean level spacing is given by $\delta \sim N^{(1-\gamma)/2}/N = N^{-(1+\gamma)/2}$. Contrarily, the smallest level spacing, locally in a sub-block is $\delta_{\rm min} = \delta_{l_{\rm max}} \sim N^{-(1-\gamma)/2}$. As $\delta_{\rm min} > \delta$, neighboring eigenvalues come from different sub-blocks and are, thus, uncorrelated, while the correlated ones have at least the energy difference $\delta_{\rm min}$. The unfolding procedure rescales $\delta \to 1$, setting a critical dimensionless energy $\delta_{\rm min}/\delta = N^\gamma$, in agreement with numerics. Any two unfolded energy levels $\mathcal{E}_{1,2}$ are uncorrelated if $|\mathcal{E}_1 - \mathcal{E}_2| \lesssim N^\gamma$ being from different sub-blocks, while for $|\mathcal{E}_1 - \mathcal{E}_2| \gtrsim N^\gamma$ (actual level spacing $\gtrsim N^{(\gamma-1)/2}$), they may belong to the same sub-block and be correlated. This explains the origin of the anomalous behavior in the power spectrum of the energy fluctuations in the β ensemble [Eq. (5)].

Finally, we numerically confirm that the eigenstates of the β ensemble in the NEE phase are exponentially decaying using the metric defined in [32,69]. Therefore we can order the eigenstates according to their localization centers instead of energy. To understand the correlation among such spatially ordered eigenstates, we look at the covariance matrix

$$M_{ij} = \sum_{k=1}^{N} |\Psi_i(k)\Psi_j(k)| = \begin{cases} 1, & i = j \\ \frac{2}{\pi}, & \text{ergodic} \\ \to 0, & \text{localized} \end{cases}. \tag{8}$$

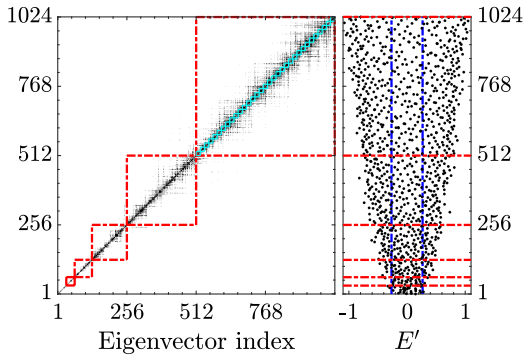

FIG. 4. Left: threshold-filtered covariance matrix $\tilde{M}$, Eq. (8), for $N = 1024$, $\gamma = 0.5$, and $\delta M = 0.5$ The red dashed lines denote spatial blocks of the form Δ_l [Eq. (6)] with $\delta\zeta = 0.1$ and $l = 0, 1, \ldots, 5$, while the cyan lines show N^{γ} spatial sub-blocks in the last block. Right: energies of spatially ordered eigenstates. The blue dashed vertical lines denote $(-E'_{\mathrm{G}}, E'_{\mathrm{G}})$, i.e., energy bound of localized states.

The covariance matrix gives a rather complete idea about the Hilbert space structure. By plotting the threshold-filtered covariance matrix $\tilde{M}_{ij} = \Theta(M_{ij} - \delta M)$, with the Heaviside step function $\Theta(x)$ and a threshold $\delta M < (2/\pi)$, we unveil the eigenstate spatial correlation structure. $\tilde{M}_{ij} = 1$ implies that the ith and jth states have a high degree of overlap, i.e., they are hybridizing and vice versa. In the ergodic phase $\tilde{M}$ is a dense matrix, while it is sparse in the localized regime. In the β ensemble (see Fig. 4), $\tilde{M}$ shows a banded structure with the spatial band, increasing with the indices i, j, confirming the analytical picture of the block Δ_l. We have also shown the energy levels of the spatially ordered eigenstates. This further shows that in the β ensemble the localized and the NEE states can appear at nearly same energy eventually leading to coexistence in the thermodynamic limit.

To sum up, in this Letter, we provide the set of main principles on how to avoid the mobility-edge emergence in short-range disordered models and illustrate them in a well-known example of the β ensemble. With various spectral and localization measures, we uncover the structure of the coexistence of localized and extended states in such a model and confirm these results analytically by the spatially local mapping to the 1D Anderson model with system-size-dependent hopping. The general principles, formulated in this work and verified on the β ensemble, allow one to realize the coexistence of the localized and extended states in the same energy interval without fine-tuning which is robust against perturbations and disorder realizations. Such systems can be used for quantum memory and fault-tolerant quantum calculations, where the localized states, decoupled from the extended modes of the bath, are free from decoherence. As an outlook, it would be of particular interest to find many-body realizations of the above concepts.

A. K. D. is supported by an INSPIRE Fellowship, DST, India. I. M. K. acknowledges the support of the Russian Science Foundation, Grant No. 21-12-00409.

*akd19rs062@iiserkol.ac.in
†anandamohan@iiserkol.ac.in
‡ivan.khaymovich@gmail.com

[1] N. Mott, Electrons in disordered structures, Adv. Phys. **16**, 49 (1967).
[2] P. W. Anderson, Absence of diffusion in certain random lattices, Phys. Rev. **109**, 1492 (1958).
[3] D. J. Luitz, N. Laflorencie, and F. Alet, Many-body localization edge in the random-field Heisenberg chain, Phys. Rev. B **91**, 081103(R) (2015).
[4] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys. (N.Y.) **43**, 5830 (2002).
[5] M. Mehta, *Random Matrices*, Pure and Applied Mathematics (Elsevier Science, New York, 2004), https://books.google.co.in/books?id=Fe6uoQEACAAJ.
[6] M. Berry and M. Tabor, Level clustering in the regular spectrum, Proc. R. Soc. A **356**, 375 (1977).
[7] O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, Phys. Rev. Lett. **52**, 1 (1984).
[8] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, Anderson localization on the Bethe lattice: Nonergodicity of extended states, Phys. Rev. Lett. **113**, 046806 (2014).
[9] I. García-Mata, O. Giraud, B. Georgeot, J. Martin, R. Dubertrand, and G. Lemarié, Scaling theory of the Anderson transition in random graphs: Ergodicity and universality, Phys. Rev. Lett. **118**, 166801 (2017).
[10] S. Ray, A. Ghosh, and S. Sinha, Drive-induced delocalization in the Aubry-André model, Phys. Rev. E **97**, 010101(R) (2018).
[11] Y. Wang, L. Zhang, W. Sun, T. F. J. Poon, and X.-J. Liu, Quantum phase with coexisting localized, extended, and critical zones, Phys. Rev. B **106**, L140203 (2022).
[12] M. Pino, V. E. Kravtsov, B. L. Altshuler, and L. B. Ioffe, Multifractal metal in a disordered Josephson junctions array, Phys. Rev. B **96**, 214205 (2017).
[13] M. Pino, L. B. Ioffe, and B. L. Altshuler, Nonergodic metallic and insulating phases of Josephson junction chains, Proc. Natl. Acad. Sci. U.S.A. **113**, 536 (2016).
[14] A. K. Das and A. Ghosh, Transport in deformed centrosymmetric networks, Phys. Rev. E **106**, 064112 (2022).
[15] Y. V. Fyodorov and O. Giraud, High values of disorder-generated multifractals and logarithmically correlated processes, Chaos, Solitons Fractals **74**, 15 (2015).
[16] V. Kravtsov, I. Khaymovich, E. Cuevas, and M. Amini, A random matrix model with localization and ergodic transitions, New J. Phys. **17**, 122002 (2015).
[17] P. A. Nosov, I. M. Khaymovich, and V. E. Kravtsov, Correlation-induced localization, Phys. Rev. B **99**, 104203 (2019).
[18] P. A. Nosov and I. M. Khaymovich, Robustness of delocalization to the inclusion of soft constraints in long-range random models, Phys. Rev. B **99**, 224208 (2019).

[19] I. M. Khaymovich, V. E. Kravtsov, B. L. Altshuler, and L. B. Ioffe, Fragile extended phases in the log-normal Rosenzweig-Porter model, Phys. Rev. Res. **2**, 043346 (2020).

[20] P. Cizeau and J. P. Bouchaud, Theory of Lévy matrices, Phys. Rev. E **50**, 1810 (1994).

[21] A. K. Das and A. Ghosh, Chaos due to symmetry-breaking in deformed Poisson ensemble, J. Stat. Mech. (2022) 063101.

[22] V. E. Kravtsov, I. M. Khaymovich, B. L. Altshuler, and L. B. Ioffe, Localization transition on the random regular graph as an unstable tricritical point in a log-normal Rosenzweig-Porter random matrix ensemble, arXiv:2002.02979.

[23] I. M. Khaymovich and V. E. Kravtsov, Dynamical phases in a "multifractal" Rosenzweig-Porter model, SciPost Phys. **11**, 45 (2021).

[24] A. G. Kutlin and I. M. Khaymovich, Emergent fractal phase in energy stratified random models, SciPost Phys. **11**, 101 (2021).

[25] W. Tang and I. M. Khaymovich, Non-ergodic delocalized phase with Poisson level statistics, Quantum **6**, 733 (2022).

[26] V. R. Motamarri, A. S. Gorsky, and I. M. Khaymovich, Localization and fractality in disordered Russian doll model, SciPost Phys. **13**, 117 (2022).

[27] K. S. Tikhonov and A. D. Mirlin, Fractality of wave functions on a Cayley tree: Difference between tree and locally treelike graph without boundary, Phys. Rev. B **94**, 184203 (2016).

[28] D. J. Luitz, I. M. Khaymovich, and Y. B. Lev, Multifractality and its role in anomalous transport in the disordered XXZ spin-chain, SciPost Phys. Core **2**, 006 (2020).

[29] A. K. Das and A. Ghosh, Nonergodic extended states in the β ensemble, Phys. Rev. E **105**, 054121 (2022).

[30] N. Rosenzweig and C. E. Porter, Repulsion of energy levels in complex atomic spectra, Phys. Rev. **120**, 1698 (1960).

[31] T. K. Duy and F. Nakano, Gaussian beta ensembles at high temperature: Eigenvalue fluctuations and bulk statistics, J. Stat. Phys. **173**, 295 (2018).

[32] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.166401 for details, which includes Refs. [33,34].

[33] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Distribution of the ratio of consecutive level spacings in random matrix ensembles, Phys. Rev. Lett. **110**, 084101 (2013).

[34] E. Faleiro, J. M. G. Gómez, R. A. Molina, L. Muñoz, A. Relaño, and J. Retamosa, Theoretical derivation of $1/f$ noise in quantum chaos, Phys. Rev. Lett. **93**, 244101 (2004).

[35] J. A. Cuesta and A. Sánchez, General non-existence theorem for phase transitions in one-dimensional systems with short range interactions, and physical examples of such transitions, J. Stat. Phys. **115**, 869 (2004).

[36] M. Albert and P. Leboeuf, Localization by bichromatic potentials versus Anderson localization, Phys. Rev. A **81**, 013614 (2010).

[37] V. Balasubramanian, J. M. Magan, and Q. Wu, A tale of two Hungarians: Tridiagonalizing random matrices, arXiv:2208.08452.

[38] J. Beugnon, C. Tuchendler, H. Marion, A. Gaëtan, Y. Miroshnychenko, Y. R. Sortais, A. M. Lance, M. P. Jones, G. Messin, A. Browaeys *et al.*, Two-dimensional transport and transfer of a single atomic qubit in optical tweezers, Nat. Phys. **3**, 696 (2007).

[39] M. A. Norcia, A. W. Young, and A. M. Kaufman, Microscopic control and detection of ultracold strontium in optical-tweezer arrays, Phys. Rev. X **8**, 041054 (2018).

[40] A. Cooper, J. P. Covey, I. S. Madjarov, S. G. Porsev, M. S. Safronova, and M. Endres, Alkaline-earth atoms in optical tweezers, Phys. Rev. X **8**, 041055 (2018).

[41] G. Pagano, F. Scazza, and M. Foss-Feig, Fast and scalable quantum information processing with two-electron atoms in optical tweezer arrays, Adv. Quantum Technol. **2**, 1800067 (2019).

[42] A. M. Kaufman and K.-K. Ni, Quantum science with optical tweezer arrays of ultracold atoms and molecules, Nat. Phys. **17**, 1324 (2021).

[43] A. W. Elshaari, W. Pernice, K. Srinivasan, O. Benson, and V. Zwiller, Hybrid integrated quantum photonic circuits, Nat. Photonics **14**, 285 (2020).

[44] J. Chang, J. Gao, I. Esmaeil Zadeh, A. W. Elshaari, and V. Zwiller, Nanowire-based integrated photonics for quantum information and quantum sensing, Nanophotonics **12**, 339 (2023).

[45] G. Moody, V. J. Sorger, D. J. Blumenthal, P. W. Juodawlkis, W. Loh, C. Sorace-Agaskar, A. E. Jones, K. C. Balram, J. C. Matthews, A. Laing *et al.*, 2022 roadmap on integrated quantum photonics, J. Phys.: Photonics **4**, 012501 (2022).

[46] J. Gao, I. M. Khaymovich, X.-W. Wang, Z.-S. Xu, A. Iovan, G. Krishna, A. V. Balatsky, V. Zwiller, and A. W. Elshaari, Experimental probe of multi-mobility edges in quasiperiodic mosaic lattices, arXiv:2306.10829.

[47] J. Gao, I. M. Khaymovich, A. Iovan, X.-W. Wang, G. Krishna, Z.-S. Xu, E. Tortumlu, A. V. Balatsky, V. Zwiller, and A. W. Elshaari, Coexistence of extended and localized states in finite-sized mosaic Wannier-Stark lattices, Phys. Rev. B (to be published).

[48] F. L. Metz, I. Neri, and D. Bollé, Localization transition in symmetric random matrices, Phys. Rev. E **82**, 031135 (2010).

[49] S. Gopalakrishnan, Self-dual quasiperiodic systems with power-law hopping, Phys. Rev. B **96**, 054202 (2017).

[50] X. Deng, S. Ray, S. Sinha, G. V. Shlyapnikov, and L. Santos, One-dimensional quasicrystals with power-law hopping, Phys. Rev. Lett. **123**, 025301 (2019).

[51] Y. Wang, X. Xia, L. Zhang, H. Yao, S. Chen, J. You, Q. Zhou, and X.-J. Liu, One-dimensional quasiperiodic mosaic lattice with exact mobility edges, Phys. Rev. Lett. **125**, 196604 (2020).

[52] S. Das Sarma, S. He, and X. C. Xie, Mobility edge in a model one-dimensional potential, Phys. Rev. Lett. **61**, 2144 (1988).

[53] J. Biddle and S. Das Sarma, Predicted mobility edges in one-dimensional incommensurate optical lattices: An exactly solvable model of Anderson localization, Phys. Rev. Lett. **104**, 070601 (2010).

[54] S. Ganeshan, J. H. Pixley, and S. Das Sarma, Nearest neighbor tight binding models with an exact mobility edge in one dimension, Phys. Rev. Lett. **114**, 146601 (2015).

[55] A. Ahmed, A. Ramachandran, I. M. Khaymovich, and A. Sharma, Flat band based multifractality in the all-band-flat diamond chain, Phys. Rev. B **106**, 205119 (2022).

[56] B. Bulka, B. Kramer, and A. MacKinnon, Mobility edge in the three dimensional Anderson model, Z. Phys. B Condens. Matter **60**, 13 (1985).

[57] C. R. Laumann, A. Pal, and A. Scardicchio, Many-body mobility edge in a mean-field quantum spin glass, Phys. Rev. Lett. **113**, 200405 (2014).
[58] V. Oganesyan and D. A. Huse, Localization of interacting fermions at high temperature, Phys. Rev. B **75**, 155111 (2007).
[59] T. Guhr, A. Müller–Groeling, and H. A. Weidenmüller, Random-matrix theories in quantum physics: Common concepts, Phys. Rep. **299**, 189 (1998).
[60] R. Riser, V. A. Osipov, and E. Kanzieper, Power spectrum of long eigenlevel sequences in quantum chaotic systems, Phys. Rev. Lett. **118**, 204101 (2017).
[61] A. Relaño, L. Muñoz, J. Retamosa, E. Faleiro, and R. A. Molina, Power-spectrum characterization of the continuous Gaussian ensemble, Phys. Rev. E **77**, 031103 (2008).
[62] R. Berkovits, Super-Poissonian behavior of the Rosenzweig-Porter model in the nonergodic extended regime, Phys. Rev. B **102**, 165140 (2020).
[63] A. K. Das and A. Ghosh, Eigenvalue statistics for generalized symmetric and Hermitian matrices, J. Phys. A **52**, 395001 (2019).
[64] B. Altshuler and V. Kravtsov, Random cantor sets and minibands in local spectrum of quantum systems, Ann. Phys. (Amsterdam) **456**, 169300 (2023).
[65] G. D. Tomasi, M. Amini, S. Bera, I. M. Khaymovich, and V. E. Kravtsov, Survival probability in generalized Rosenzweig-Porter random matrix ensemble, SciPost Phys. **6**, 14 (2019).
[66] F. Izrailev, S. Ruffo, and L. Tessieri, Classical representation of the one-dimensional Anderson model, J. Phys. A **31**, 5263 (1998).
[67] L. Sanchez-Palencia, D. Clément, P. Lugan, P. Bouyer, G. V. Shlyapnikov, and A. Aspect, Anderson localization of expanding Bose-Einstein condensates in random potentials, Phys. Rev. Lett. **98**, 210401 (2007).
[68] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices, Combinatorica **1**, 233 (1981).
[69] T. Prosen and M. Robnik, Energy level statistics and localization in sparsed banded random matrix ensemble, J. Phys. A **26**, 1105 (1993).