Optimized $\mathcal{O}(\alpha_s^2)$ Correction to Exclusive Double-J/ψ Production at B Factories

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The failure of observing the $e^+e^- \to J/\psi J/\psi$ events at B factories to date is often attributed to the significant negative order-α_s correction. In this work we compute the $\mathcal{O}(\alpha_s^2)$ correction to this process for the first time. The magnitude of the next-to-next-to-leading order (NNLO) perturbative correction is substantially negative so that the standard nonrelativistic QCD prediction would suffer from an unphysical, negative cross section. This dilemma may be traced in the fact that the bulk contribution of the fixed-order radiative corrections stems from the perturbative corrections to the J/ψ decay constant. We thus implement an improved nonrelativistic QCD factorization framework, by decomposing the amplitude into the photon-fragmentation piece and the nonfragmentation piece. With the measured J/ψ decay constant as input, which amounts to resumming a specific class of radiative and relativistic corrections to all orders, the fragmentation-induced production rate can be predicted accurately and serves a benchmark prediction. The nonfragmentation type of the amplitude is then computed through NNLO in α_s and at lowest order in velocity. Both the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections in the interference term become positive and exhibit a decent convergence behavior. Our finest prediction is $\sigma(e^+e^- \to J/\psi J/\psi) = 2.13^{+0.30}_{-0.06}$ fb at $\sqrt{s} = 10.58$ GeV. With the projected integrated luminosity of 50 ab^{-1}, the prospect to observe this exclusive process at `Belle 2` experiment appears to be bright.

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Introduction.—In 2006 the `BarBar` Collaboration reported two exclusive processes about production of two neutral vector mesons, and the measured production rates are $\sigma(e^+e^- \to \rho^0\rho^0) = 20.7 \pm 0.7(\mathrm{stat}) \pm 2.7(\mathrm{syst})$ fb and $\sigma(e^+e^- \to \rho^0\phi) = 5.7 \pm 0.5(\mathrm{stat}) \pm 0.8(\mathrm{syst})$ fb, with a cut $|\cos\theta| < 0.8$ imposed [1]. Since the final state has net positive C parity, these exclusive processes must proceed via e^+e^- annihilation into two photons. The suppression caused by extra QED coupling constants can be largely compensated by the significant enhancement brought by the small photon virtuality once the vector mesons are produced through two photon independent fragmentation, hence the production rates can be surprisingly larger than naively expected. Shortly after, Davier *et al.* [2] (see also Bodwin *et al.* [3]) considered these processes in the vector dominance model (VMD). Including the finite width of ρ^0, these authors obtained $\sigma(e^+e^- \to \rho^0\rho^0) = 21.4 \pm 0.7$ and $\sigma(e^+e^- \to \rho^0\phi) = 6.15 \pm 0.22$ fb with $|\cos\theta| < 0.8$, in satisfactory agreement with the `BarBar` measurements [1].

One naturally speculates whether the similar yet much cleaner double J/ψ production process can be observed at B factories or not. In fact, as early as in 2003, the `Belle` experiment had already looked for this channel and not found a clear signal [4]. Instead an upper limit is placed, $\sigma(e^+e^- \to J/\psi J/\psi)\mathcal{B}_{>2} < 9.1$ fb at the 90% confidence level, where $\mathcal{B}_{>2}$ signifies the branching fraction for final states including more than two charged tracks.

On the theoretical side, the $e^+e^- \to J/\psi J/\psi$ process has already been investigated by several different groups over the years. In 2002 Bodwin *et al.* studied this process at lowest order in the nonrelativistic QCD (NRQCD) approach and predicted the cross section to be around 8.7 fb [5], which is even greater than the leading-order (LO) NRQCD prediction for $e^+e^- \to J/\psi\eta_c$ [6–8]. Shortly after, this prediction was updated to 6.65 fb by the same authors [9]. With the aid of the VMD, Davier *et al.* considered the photon fragmentation contribution only and predicted the total cross section to be about 2.38 fb [2]. Besides the photon fragmentation contribution, Bodwin *et al.* further took into account the nonfragmentation contribution within the NRQCD factorization framework, and found a sizable destructive interference

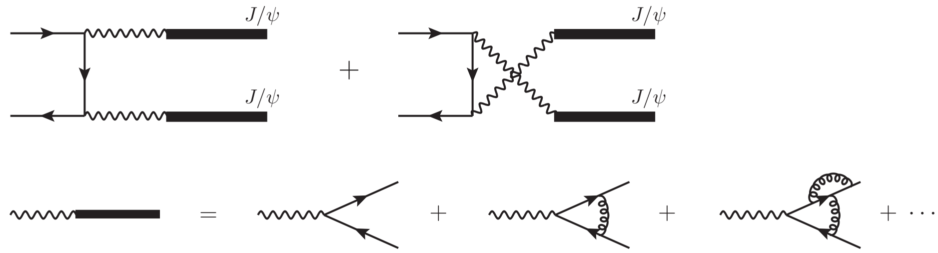

FIG. 1. Illustration of the $e^+e^- \to J/\psi J/\psi$ process through two photon independent fragmentation.

effect, with the cross section predicted to be about 1.69 ± 0.35 fb [3].

Important progress was made by Gong and Wang in 2008 [10], who computed the $\mathcal{O}(\alpha_s)$ correction to this process, yet at the lowest order in velocity. The NLO perturbative correction turns out to be negative and significant. By including the radiative correction, Gong *et al.* found that the LO prediction about 7.4–9.1 fb reduces to −3.4–2.3 fb [10]. Later the combined NLO perturbative and relativistic corrections were investigated by Fan *et al.* [11]. They found the fixed-order NRQCD prediction for the cross section ranges from −12 to −0.43 fb, which is negative and sensitive to the charm quark mass and renormalization scale. On the other hand, following the recipe practiced in [3], splitting the amplitude into the photon-fragmentation and nonfragmentation parts, Fan *et al.* found the predicted cross section boosted to the positive range 1–1.5 fb [11].

The predicted double-J/ψ cross sections at B factories are scattered in a wide range. To provide useful guidance for experimentalists to search for this channel, it is crucial to make the most precise theoretical prediction, which is the chief motivation of this work. Since the $\mathcal{O}(\alpha_s)$ correction is quite important, one naturally wonders what is the impact of $\mathcal{O}(\alpha_s^2)$ correction. It is the goal of this work to investigate the two-loop QCD correction to this double-J/ψ production process, which turns out to be an exceedingly formidable task.

Two photon independent fragmentation.—Since the double J/ψ in the final state has the even C parity, this production process must proceed through e^+e^- annihilation into two virtual photons. The dominant production mechanism is via two photon independent fragmentation into J/ψ, as shown in Fig. 1. According to VMD, the photon-to-J/ψ coupling strength is governed by $ee_cM_{J/\psi}f_{J/\psi}$, where $f_{J/\psi}$ denotes the J/ψ decay constant and is defined by $\langle J/\psi|\bar{c}\gamma^\mu c|0\rangle = -f_{J/\psi}M_{J/\psi}\varepsilon^{*\mu}_{J/\psi}$. The value of the decay constant can be determined from the precisely measured leptonic width of J/ψ, i.e., $\Gamma(J/\psi \to l^+l^-) = 4\pi e_c^2\alpha^2 f_{J/\psi}^2/3M_{J/\psi}$.

The differential unpolarized cross section for $e^+e^- \to J/\psi J/\psi$ through photon fragmentation reads [2]

$$\frac{d\sigma_{\mathrm{fr}}(e^+e^- \to J/\psi J/\psi)}{d\cos\theta} = \left(\frac{ee_cf_{J/\psi}}{M_{J/\psi}}\right)^4 \frac{\pi\alpha^2}{s}\beta\frac{(t^2+u^2)(tu - M_{J/\psi}^4) + 4stuM_{J/\psi}^2}{t^2u^2}, \tag{1}$$

with $\beta = \sqrt{1-4M_{J/\psi}^2/s}$ representing the velocity of the outgoing J/ψ.

Integrating (1) over $\cos\theta$ (one should cover only the hemisphere of the solid angle since two J/ψ are indistinguishable bosons.), one obtains

$$\sigma_{\mathrm{fr}} = \frac{32\pi^3e_c^4\alpha^4f_{J/\psi}^4}{M_{J/\psi}^4}\frac{1}{s}\left[\frac{4+(1-\beta^2)^2}{1+\beta^2}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\beta\right]. \tag{2}$$

Note the fragmentation-initiated total cross section exhibits the asymptotic $1/s$ decrease, which is identical to the scaling behavior of $e^+e^- \to$ hadrons and in sharp contrast with the $1/s^4$ scaling associated with the exclusive process $e^+e^- \to J/\psi + \eta_c(\chi_{c1})$ and the $1/s^3$ scaling affiliated with $e^+e^- \to J/\psi + \chi_{c0,2}$ [6].

Traditional NRQCD factorization.—The mainstream theoretical tool to account for the exclusive charmonium production nowadays is the NRQCD factorization approach [12]. This approach allows one to express the cross section as a double expansion in α_s and v, the typical charm quark velocity inside J/ψ. Concretely speaking, at lowest order in v, the NRQCD prediction for the production rate of $e^+e^- \to J/\psi J/\psi$ can be cast in the following form:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{\beta}{16\pi}\frac{e^8e_c^4}{4}\left(\mathcal{F}^{(0)} + \frac{\alpha_s}{\pi}\mathcal{F}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{F}^{(2)} + \cdots\right)\frac{|\langle\mathcal{O}\rangle_{J/\psi}|^2}{m_c^2}, \tag{3}$$

where $\mathcal{F}^{(i)}$ $(i = 0,\ 1,\ 2)$ represent the short-distance coefficients (SDCs) at various perturbative order, and

$\langle \mathcal{O} \rangle_{J/\psi}$ is the abbreviation of the following NRQCD matrix element: $\langle \mathcal{O} \rangle_{J/\psi} \equiv |\langle J/\psi(\lambda)|\psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\lambda)\chi|0\rangle|^2$.

Improved NRQCD factorization prediction.—At any prescribed order in α_s, the double-J/ψ production from e^+e^- annihilation proceeds through either the photon fragmentation or nonfragmentation channel, where the former always dominates the latter. Following [3], we split the production amplitude into the fragmentation and nonfragmentation pieces:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{\beta}{16\pi}\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}_{\text{fr}} + \mathcal{M}_{\text{nfr}}|^2. \tag{4}$$

One then applies NRQCD factorization to the nonfragmentation part of the amplitude, which is expressed in terms of the charm quark mass and $\langle \mathcal{O} \rangle_{J/\psi}$, instead of $M_{J/\psi}$ and $f_{J/\psi}$. After squaring the amplitude in (4) and summing over spins, we decompose the differential cross section into the fragmentation part, interference part and the nonfragmentation part:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{\beta}{16\pi}\frac{e^8 e_c^4}{4}\left[\mathcal{C}_{\text{fr}} f_{J/\psi}^4 + \mathcal{C}_{\text{int}} f_{J/\psi}^2 \frac{\langle \mathcal{O} \rangle_{J/\psi}}{m_c} + \mathcal{C}_{\text{nfr}}\left(\frac{\langle \mathcal{O} \rangle_{J/\psi}}{m_c}\right)^2\right]. \tag{5}$$

The coefficient affiliated with the fragmentation piece can be read off from (1):

$$\mathcal{C}_{\text{fr}} = \frac{8\left((t^2+u^2)(tu - M_{J/\psi}^4) + 4stuM_{J/\psi}^2\right)}{t^2u^2M_{J/\psi}^4}. \tag{6}$$

The interference and nonfragmentation terms can be tackled in NRQCD factorization approach. At lowest order in v but through α_s^2, the coefficients can be parametrized as

$$\mathcal{C}_{\text{int}} = \mathcal{C}_{\text{int}}^{(0)}\left[1 + \frac{\alpha_s}{\pi}\hat{c}_{\text{int}}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\beta_0}{4}\ln\frac{\mu_R^2}{m_c^2}\hat{c}_{\text{int}}^{(1)} + 2\gamma_{J/\psi}\ln\frac{\mu_\Lambda^2}{m_c^2} + \hat{c}_{\text{int}}^{(2)}\right) + \cdots\right], \tag{7a}$$

$$\mathcal{C}_{\text{nfr}} = \mathcal{C}_{\text{nfr}}^{(0)}\left[1 + \frac{\alpha_s}{\pi}\hat{c}_{\text{nfr}}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\beta_0}{4}\ln\frac{\mu_R^2}{m_c^2}\hat{c}_{\text{nfr}}^{(1)} + 4\gamma_{J/\psi}\ln\frac{\mu_\Lambda^2}{m_c^2} + \hat{c}_{\text{nfr}}^{(2)}\right) + \cdots\right], \tag{7b}$$

where μ_R and μ_Λ refer to renormalization scale and NRQCD factorization scale. $\beta_0 = 11C_A/3 - 2n_f/3$, with $n_f = 4$ signifying the number of active quark flavors. The occurrence of the $\beta_0 \ln\mu_R$ term is dictated by the renormalization group invariance. $\gamma_{J/\psi} = -(\pi^2/12)C_F(2C_F + 3C_A)$ is the two-loop anomalous dimension of the NRQCD vector current [13,14]. The occurrence of the $\gamma_{J/\psi}\ln\mu_\Lambda$ term at two-loop order is demanded by the NRQCD factorization. $\hat{c}^{(i)}$ $(i = 1,\ 2)$ represent the μ-independent order-α_s and order-α_s^2 corrections.

Both $f_{J/\psi}$ and $\langle \mathcal{O} \rangle_{J/\psi}$ enter the improved NRQCD factorization formula (5). $f_{J/\psi}$ is not an entirely nonperturbative object and rather encapsulates some perturbative effect. NRQCD factorization allows one to further factorize the J/ψ decay constant as SDCs multiplied with the NRQCD matrix element $\langle \mathcal{O} \rangle_{J/\psi}$:

$$f_{J/\psi} = \sqrt{\frac{2\langle \mathcal{O} \rangle_{J/\psi}}{M_{J/\psi}}}\left[1 + \mathfrak{f}^{(1)}\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\mathfrak{f}^{(1)}\frac{\beta_0}{4}\ln\frac{\mu_R^2}{m_c^2} + \gamma_{J/\psi}\ln\frac{\mu_\Lambda^2}{m_c^2} + \mathfrak{f}^{(2)}\right) + \cdots\right] + \mathcal{O}(v^2), \tag{8}$$

with $\mathfrak{f}^{(1)} = -2C_F$, $\mathfrak{f}^{(2)} = -43.3288$ [13,14], and $n_f = 4$. The $\mathcal{O}(\alpha_s^3)$ correction [15,16] and $\mathcal{O}(\alpha_s^i v^2)$ $(i = 0,\ 1)$ corrections [17,18] have also been available. We adopt the measured value $f_{J/\psi}$ in (5). This implies that we have resummed infinite towers of perturbative and relativistic corrections to all orders. Actually, one can easily obtain various SDCs in the traditional NRQCD framework (3) from the improved factorization formula (5) by employing (8) and setting $M_{J/\psi} = 2m_c$.

There are two tree-level nonfragmentation diagrams, as depicted in Figs. 2(a) and 2(b). A straightforward calculation yields the tree-level coefficients for the interference and the nonfragmentation terms in (7):

$$\mathcal{C}_{\text{int}}^{(0)} = -\frac{128\left(tu(t^2+u^2) + 20m_c^2stu + 16m_c^4s^2 - 64m_c^6s - 512m_c^8\right)}{3m_c^2tus^3}, \tag{9a}$$

$$\mathcal{C}_{\text{nfr}}^{(0)} = 2048\left(\frac{-12tu(tu - 32m_c^4 + 4m_c^2s) + 5s^2tu}{9s^6} + \frac{16m_c^2(s^3 - 5m_c^2s^2 + 48m_c^4s - 192m_c^6)}{9s^6}\right). \tag{9b}$$

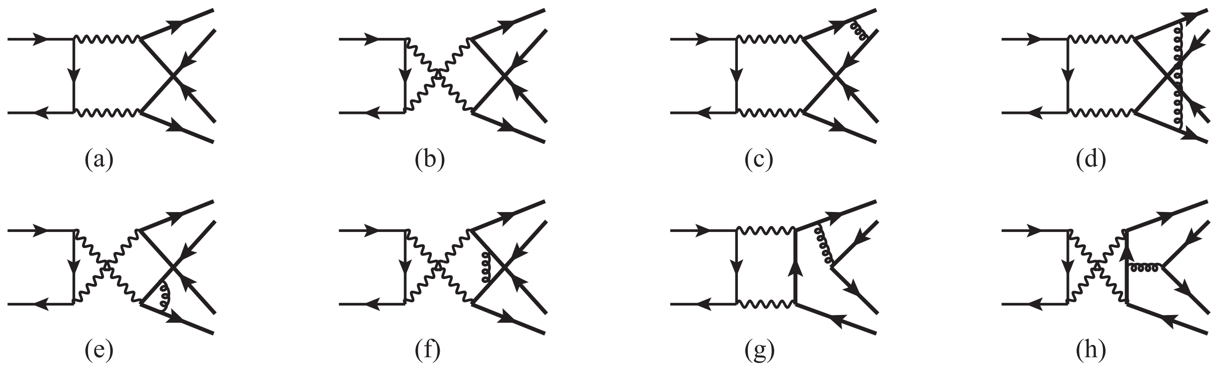

FIG. 2. Nonfragmentation type of tree-level Feynman diagrams [(a) and (b)], together with some sample one-loop nonfragmentation diagrams [(c) through (h)].

Substituting (6) and (9) into (5), integrating over $\cos\theta$ from 0 to 1, we reproduce the fragmentation-initiated integrated cross section (2), and obtain the following interference and nonfragmentation contributions to the integrated cross section:

$$\sigma_{\mathrm{int}}=-\frac{16\pi^3 e_c^4\alpha^4 f_{J/\psi}^2\langle\mathcal{O}\rangle_{J/\psi}}{3m_c^3 s^2}\left[(5-\beta^2)(1-\beta^2)^2\ln\left(\frac{1+\beta}{1-\beta}\right)+22\beta-\frac{40}{3}\beta^3+2\beta^5\right], \tag{10a}$$

$$\sigma_{\mathrm{nfr}}=\frac{2048\pi^3\alpha^4 e_c^4|\langle\mathcal{O}\rangle_{J/\psi}|^2}{45m_c^2 s^3}\beta\left(10-\frac{20}{3}\beta^2+\beta^4\right). \tag{10b}$$

Here the J/ψ velocity β is evaluated by replacing $M_{J/\psi}$ with $2m_c$.

In contrast with the fragmentation part that asymptotically scales as $1/s$, the interference part of the cross section exhibits a $1/s^2$ asymptotic decrease, while the nonfragmentation part exhibits a $1/s^3$ scaling. Adding (2) and (10), setting $f_{J/\psi}\approx\sqrt{\langle\mathcal{O}\rangle_{J/\psi}/m_c}$ and $M_{J/\psi}\approx 2m_c$ everywhere, we reproduce the analytic expression of the tree-level integrated cross section [10].

Higher-order radiative corrections.—We proceed to compute the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections to the unpolarized cross section. We begin with the quark-level amplitude for $e^+e^-\to\gamma^*\gamma^*\to c\bar{c}({}^3S_1^{(1)})+c\bar{c}({}^3S_1^{(1)})$. About 24 one-loop and 506 nonvanishing two-loop diagrams of nonfragmentation type, together with the corresponding amplitudes are generated by `QGraf`/`FeynArts` [19,20]. Some representative nonfragmentation types of one-loop and two-loop diagrams are shown in Figs. 2 and 3. [For simplicity, we have neglected those "light-by-light"-type diagrams exemplified by Fig. 3(h), which generally yield very tiny contributions to higher-order perturbative corrections in various quarkonium production and decay processes [16,21–27].] We use the packages `FeynCalc`/`FormLink` [28,29] to conduct Dirac algebra. At lowest order in v, we neglect the relative momentum in each $c\bar{c}$ pair prior to carrying out the loop integration, which amounts to directly extracting the NRQCD SDCs from the hard loop region [30]. After the integration-by-parts reduction with the aid of `Apart` [31] and `FIRE` [32], we end up with about 2400 two-loop master integrals (MIs). We utilize the package `AMFlow` [33–36] to compute the MIs with high numerical accuracy. Note that the encountered two-loop diagrams bear the genuine $2\to 4$ topology and represent the cutting-edge problem in the area of

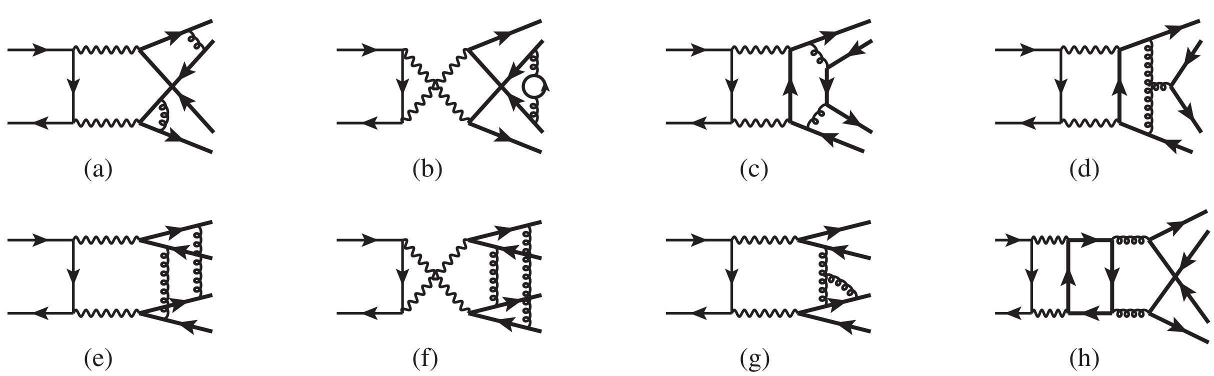

FIG. 3. Some representative two-loop diagrams of nonfragmentation origin for $e^+e^-\to J/\psi J/\psi$.

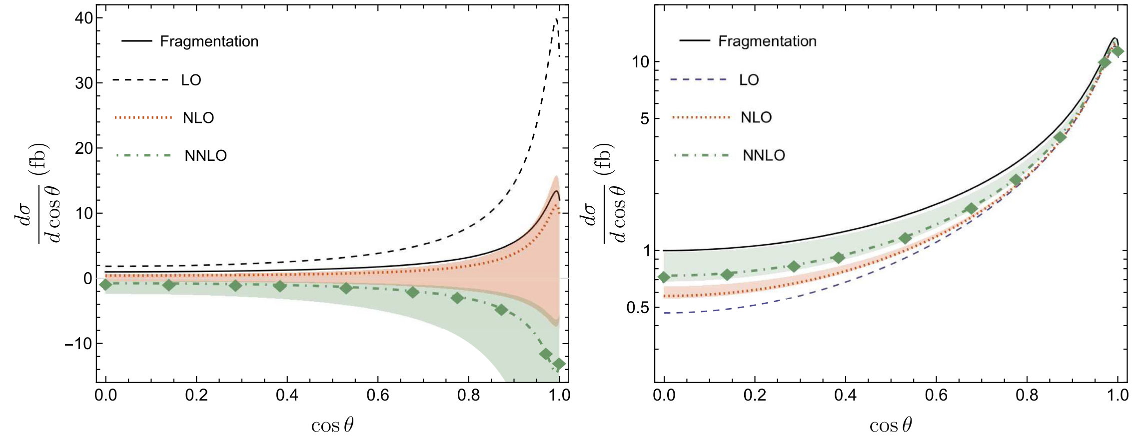

FIG. 4. Differential cross sections for $e^+e^- \to J/\psi J/\psi$ against $\cos\theta$ at various perturbative accuracy from traditional NRQCD factorization (left panel) and improved NRQCD approach (right panel). We have fixed $\mu_\Lambda = 1$ GeV, and taken the central value of μ_R to be $\sqrt{s}/2$. The package RunDec [42] is utilized to compute the running QCD coupling to two-loop accuracy. The error bands of the NLO and NNLO predictions are estimated by sliding μ_R from m_c to $\sqrt{s}$.

multiloop calculation. The IBP reduction and computation of the MIs turn out to be rather time consuming.

Performing the field-strength and mass renormalization, with two-loop expressions of Z_2 and Z_m taken from [37], and renormalizing the strong coupling constant under the $\overline{\mathrm{MS}}$ scheme to one-loop order, we eliminate the UV divergences. Nevertheless, the renormalized two-loop corrections to $\mathcal{C}_{\mathrm{int}}$ and $\mathcal{C}_{\mathrm{nfr}}$ still contain uncancelled single IR poles equal to $\mathcal{C}^{(0)}_{\mathrm{int}}\gamma_{J/\psi}$ and to $2\mathcal{C}^{(0)}_{\mathrm{nfr}}\gamma_{J/\psi}$, respectively. This pattern is exactly what is required by NRQCD factorization for double-J/ψ production at $\mathcal{O}(\alpha_s^2)$, as reflected in (7). These IR poles can be factored into the NRQCD matrix element $\langle\mathcal{O}\rangle_{J/\psi}$ under the $\overline{\mathrm{MS}}$ prescription, which then becomes scale-dependent quantity. Note that the $\gamma_{J/\psi}\ln\mu_\Lambda$ terms in (7) exactly cancel the μ_Λ dependence of the NRQCD matrix element, so that the predicted cross section is independent of μ_Λ. Finally we are able to identify the desired nonlogarithmic piece in the two-loop SDCs, $\hat{c}^{(2)}_{\mathrm{int}}$ and $\hat{c}^{(2)}_{\mathrm{nfr}}$.

Although we adopt the Feynman gauge throughout this work, it is important to stress that our decomposition in (5) explicitly preserves the QED and QCD gauge invariance. QED gauge invariance simply arises from current conservation. Note the fragmentation-type diagrams are essentially the process $e^+e^- \to \gamma^*\gamma^*$ dressed with two independent photon-to-J/ψ fragmentation processes, which encode the gauge-invariant QCD corrections to J/ψ decay constant. Since the full amplitude must be gauge invariant, the nonfragmentation part of the amplitude ought to be gauge invariant. Certainly it will be desirable to explicitly verify the QCD gauge invariance in a general R_ξ gauge. To expedite the future check, we provide Supplemental Material [38] which tabulates the values of various SDCs in each part in both optimized and traditional NRQCD approaches, for ten different values of scattering angles.

In our numerical analysis, we have fixed $\sqrt{s} = 10.58$, $M_{J/\psi} = 3.0969$, and $m_c = 1.5$ GeV. With $\Gamma(J/\psi \to l^+l^-) = 5.56$ keV [39], and the running QED coupling $\alpha(M_{J/\psi}) = 1/132.6$ [40], one obtains $f_{J/\psi} = 403$ MeV. In the phenomenological analysis, we approximate the NRQCD matrix element $\langle\mathcal{O}\rangle_{J/\psi}(\mu_\Lambda = 1\ \mathrm{GeV}) \approx (3/2\pi)R^2_{J/\psi}(0) = 0.387\ \mathrm{GeV}^3$, where the radial Schrödinger wave function at the origin is evaluated from Buchmüller-Tye potential model [41].

In Fig. 4 we plot the angular distribution of J/ψ at various perturbative order within both the improved NRQCD factorization and the traditional NRQCD factorization. For the traditional NRQCD prediction, the LO prediction is considerably greater than the fragmentation contribution. Nevertheless, both $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections significantly decrease the LO prediction. We have confirmed the significant negative $\mathcal{O}(\alpha_s)$ correction first discovered by Gong *et al.* [10]. Remarkably, the next-to-next-to-leading order (NNLO) perturbative correction is also very substantial, which brings the predicted cross section down to unphysical, negative value. The symptom can be attributed to the fact that the bulk contribution of the fixed-order radiative corrections actually originates from the negative and significant perturbative corrections to the J/ψ decay constant (8).

On the other hand, for the improved NRQCD prediction, due to the destructive interference between the tree-level nonfragmentation amplitude [Figs. 2(a) and 2(b)] and fragmentation amplitude, the LO prediction is considerably smaller than the fragmentation cross section. Nevertheless, both $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections become positive, which exhibit decent convergence behavior. In spite of large uncertainty, the finest prediction at NNLO accuracy already

TABLE I. Integrated cross section of $e^+e^- \to J/\psi J/\psi$ at various perturbative accuracy. The uncertainties are estimated by varying μ_R from m_c to $\sqrt{s}$.

σ (fb)	Fragmentation	LO	NLO	NNLO
Optimized NRQCD	2.52	1.85	$1.93^{+0.05}_{-0.01}$	$2.13^{+0.30}_{-0.06}$
Traditional NRQCD		6.12	$1.56^{+0.73}_{-2.95}$	$-2.38^{+1.27}_{-5.35}$

gets quite close to the fragmentation prediction in (1). One also sees that when the outgoing J/ψ is collinear to the electron beam direction, the fragmentation contribution dominates the cross section. As θ deviates from 0, the interference term starts to play some notable role. The nonfragmentation term appears to be insignificant in the entire range of θ.

Finally, in Table I we enumerate our predictions at various perturbative accuracy for the integrated cross section. The NNLO prediction from the improved NRQCD approach is predicted to $2.13^{+0.30}_{-0.06}$ fb. In contrast with the negative cross section predicted by the traditional NRQCD approach, we believe that our prediction based on optimized NRQCD approach is robust and reliable.

To date the `Belle` and the `Belle 2` experiments have accumulated about 1500 fb^{-1} data, so we expect about 3105–3645 exclusive double J/ψ events. Taking into account $\mathcal{B}(J/\psi \to l^+l^-) = 12\%$, about 45–52 four-lepton events from double J/ψ can be produced. Assuming 40% reconstruction efficiency, we expect about 18–21 signal events may be reconstructed. With the designed 50 ab^{-1} integrated luminosity at `Belle 2`, it seems that the observation prospects of exclusive double J/ψ production is promising in the foreseeable future.

Nontrivial examination of NRQCD factorization at two loop.—The validity of NRQCD factorization proves to be rather intriguing at two-loop order by exchange of gluons between double J/ψ. Had these IR poles not been canceled, the NRQCD factorization would break down. Since these IR divergences cannot be affiliated with an individual J/ψ, let alone to be factored into the respective NRQCD matrix element. We pick up a specific nonfragmentation type of two-loop diagram, Fig. 3(e), to illustrate this point. This Abelian two-loop diagram, together with additional 17 similar diagrams, can be obtained by dressing the photon-fragmentation diagram in Fig. 1(a) with two gluon exchanges between two $c\bar{c}$ pairs.

A simplifying situation arises as $\cos\theta = 0$, that the leading IR pole starts at order $1/\epsilon_{\mathrm{IR}}^2$. The origin of this double IR pole stems from the loop regions where both gluons become simultaneously soft. After making eikonal approximation, we find

$$\mathcal{M}^{\text{Fig. 3(e)}}\Big|_{\theta=\frac{\pi}{2}} = \frac{1}{\epsilon_{\mathrm{IR}}^2}\frac{C_F\alpha^2}{2N_c}\frac{(m_c^2+2\mathbf{P}^2)^2}{16\mathbf{P}^2(4m_c^2+\mathbf{P}^2)} \times \left(\ln\frac{1+\beta}{1-\beta} - i\pi\right)^2 \mathcal{M}^{\text{Fig. 1(a)}}_{\mathrm{fr},0}\Big|_{\theta=\frac{\pi}{2}} + \mathcal{O}(1/\epsilon_{\mathrm{IR}}), \tag{11}$$

where $|\mathbf{P}|$ denotes the magnitude of the J/ψ momentum. Summing up 18 diagrams, these double IR poles exactly cancel. The pattern of the cancelation of the single IR poles become much more involved.

When $\cos\theta \neq 0$, Fig. 3(e) exhibits severe IR divergences which start at $1/\epsilon_{\mathrm{IR}}^3$, with complex-valued coefficients. Fortunately, all the IR poles, from $\mathcal{O}(1/\epsilon_{\mathrm{IR}}^3)$ to $\mathcal{O}(1/\epsilon_{\mathrm{IR}})$, exactly cancel upon summing 18 two-gluon exchange diagrams together. The cancellation of IR poles implies that the NRQCD factorization is fulfilled in a highly nontrivial manner.

Summary.—Complementary to the well-studied double charmonium production processes $e^+e^- \to J/\psi + \eta_c(\chi_{cJ})$, a better understanding of double-J/ψ production process can enrich our knowledge about charmonium production mechanisms and test the applicability of the NRQCD factorization approach. An accurate theoretical account for the cross section offers crucial guidance for experimental search of this process.

In this work we calculate the $\mathcal{O}(\alpha_s^2)$ correction to this process for the first time. It is found that the traditional NRQCD approach would result in a substantially negative perturbative correction, and inevitably lead to the unphysical negative cross section. The symptom may be attributed to the observation that the bulk contribution of the fixed-order radiative corrections actually stems from the negative and significant perturbative corrections to the J/ψ decay constant. Motivated by this observation, we implement an improved NRQCD factorization approach, in which the amplitude is split into the photon-fragmentation piece and the nonfragmentation piece. The fragmentation-induced production rate can be predicted unambiguously with the measured J/ψ decay constant as input. The interference part and the nonfragmentation part are then computed through NNLO in α_s. In this optimized scheme, we find that both the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections in the interference part become positive and exhibit a reasonable convergence. The nonfragmentation part turns out to be insignificant numerically. Our most accurate prediction in the optimized NRQCD approach is $\sigma(e^+e^- \to J/\psi J/\psi) = 2.13^{+0.30}_{-0.06}$ fb at $\sqrt{s} = 10.58$ GeV. We believe that this NNLO prediction is much more meaningful and trustworthy than that from the traditional NRQCD approach.

Based on the current 1500 fb^{-1} data accumulated in `Belle` and `Belle 2`, and the projected 50 ab^{-1} full dataset at `Belle 2`, the observation prospect of the exclusive double-J/ψ production process looks very bright.

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