Stochastic Operator Variance: An Observable to Diagnose Noise and Scrambling

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Noise is ubiquitous in nature, so it is essential to characterize its effects. Considering a fluctuating Hamiltonian, we introduce an observable, the stochastic operator variance (SOV), which measures the spread of different stochastic trajectories in the space of operators. The SOV obeys an uncertainty relation and allows us to find the initial state that minimizes the spread of these trajectories. We show that the dynamics of the SOV is intimately linked to that of out-of-time-order correlators, which define the quantum Lyapunov exponent λ. Our findings are illustrated analytically and numerically in a stochastic Lipkin-Meshkov-Glick Hamiltonian undergoing energy dephasing.

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Any realistic quantum system inevitably experiences some fluctuations induced by its surrounding environment. Current quantum technologies are limited by the action of this noise, motivating the pragmatic focus on noisy-intermediate-scale quantum (NISQ) devices [1,2]. In any experimental setting, tunable parameters such as Hamiltonian coupling constants may exhibit fluctuations due to interactions with the environment [3–5]. In this context, the dynamics of an ensemble of noisy realizations can be described in terms of the noise-averaged density matrix, which evolves according to a master equation describing nonunitary evolution [3,4,6,7]. Alternatively, noise can be utilized as a resource for the quantum simulation of open systems [4]. The study of fluctuations in noisy quantum systems is also connected to free probability [8–10].

The quest for understanding noise in chaotic systems has recently led to a flurry of activities exploring the signatures of quantum chaos when the dynamics is no longer unitary [6,11–18]. Out-of-time-order correlators (OTOCs) offer an important diagnostic tool, which was initially proposed in the theory of superconductivity [19]. Their use experienced renewed interest in defining a quantum analog of the Lyapunov exponent [20,21], which measures the exponential sensitivity to the initial conditions in chaotic systems and is universally bounded by the system's temperature [22]. The existence of a positive Lyapunov exponent classically is a necessary but not sufficient condition for the system to be chaotic—e.g., [23,24]. Similarly, the exponential growth of the OTOC is not a sufficient signature for quantum chaos but rather indicates *scrambling* [25,26]. OTOCs have been studied experimentally [27–32] and in open systems where their evolution is changed by dissipation [21,33–37].

In this Letter, we consider the dynamics generated by a stochastic Hamiltonian and go beyond an average description of the state by introducing the stochastic operator variance (SOV), an observable that characterizes the spread of stochastic trajectories of any operator. This notion is directly relevant to experiments, particularly in NISQ devices subject to various sources of noise. We compute the evolution of the SOV and show that it obeys a generalized uncertainty relation. It allows us to identify the initial state that minimizes the deviation from Lindblad dynamics at long times. Surprisingly, we also find that the SOV evolution relates to that of an OTOC, which connects fluctuations of the system with scrambling. This is pictorially represented in Fig. 1.

We illustrate our results in a stochastic generalization of the Lipkin-Meshkov-Glick (sLMG) model. The LMG model [38] describes an Ising spin chain with infinite

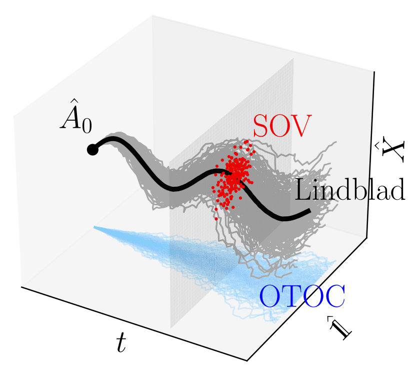

FIG. 1. The SOV-OTOC connection. Illustration of the stochastic operator variance and its connection to the out-of-time-order correlator. An operator $\hat{A}$ evolves through different realizations (gray) of a stochastic Hamiltonian, as illustrated by its projections over the identity $\hat{\mathbb{1}}$ and another operator $\hat{X}$. The noise-averaged evolution (black) follows Lindblad dissipative dynamics. The SOV $\Delta\hat{A}_t^2$ characterizes the deviation of different trajectories (red). Its projection over the identity (blue) is related to the evolution of the OTOC.

range interactions and exhibits scrambling from an unstable fixed point [25,39,40]. It can be realized experimentally with trapped ions [41] and is amenable to dynamical control techniques such as shortcuts to adiabaticity [42]. Considering fluctuations in the energy scale, we compute the SOV in this model, show its connection to the OTOC, and use it to characterize the Lyapunov exponent in the classical limit.

The stochastic operator variance.—Let us consider a system evolving under a Hermitian Hamiltonian $\hat{H}_0$ and subject to classical noise ξ_t modulating the coupling constant of a Hermitian operator $\hat{L}$, i.e.,

$$\hat{H}_t = \hat{H}_0 + \sqrt{2\gamma}\xi_t \hat{L}, \quad (1)$$

where γ measures the coupling strength between the system and noise—we set $\hbar = 1$. The stochastic process is taken as real Gaussian white noise, that is, $\langle \xi_t \rangle = 0$ and $\langle \xi_t \xi_{t'} \rangle = \delta(t - t')$. We will represent the stochastic averages by $\langle \bullet \rangle$; quantum expectation values will be written explicitly taking the trace over the state density matrix, $\mathrm{Tr}(\bullet\rho)$. We introduce the Wiener process $dW_t \equiv \xi_t dt$, which is convenient to deal with the formal treatment of stochastic differential equations (SDEs). It obeys Itō's rules $dW_t^2 = dt$ with vanishing higher-order terms, $dt\, dW_t = dW_t^{k+1} = dt^k = 0\ \forall\ k > 1$ [43]. Over a time increment dt, the evolution of a Hermitian operator $\hat{A}$ in the Heisenberg picture is $\hat{A}_{t-dt} = \hat{U}_{dt}^\dagger \hat{A}_t \hat{U}_{dt}$ [44], since operators in the Heisenberg picture evolve backward in time [45,46]. The associated propagator reads [44]

$$\hat{U}_{dt} = e^{-i\hat{H}_0 dt - i\sqrt{2\gamma} dW_{t-dt} \hat{L}}. \quad (2)$$

Expanding the propagator according to those rules and introducing the backward differential $d\hat{A}_t = \hat{A}_{t-dt} - \hat{A}_t$ [45,46] yields the SDE for the evolution of $\hat{A}_t$ under the stochastic Hamiltonian (1),

$$d\hat{A}_t = \mathcal{L}^\dagger[\hat{A}_t]dt + i\sqrt{2\gamma}[\hat{L}, \hat{A}_t]dW_{t-dt}, \quad (3)$$

where $\mathcal{L}^\dagger[\hat{A}_t] = i[\hat{H}_0, \hat{A}_t] - \gamma[\hat{L}, [\hat{L}, \hat{A}_t]]$ is the adjoint Lindbladian. Upon averaging (3), all the linear terms in dW_{t-dt} vanish [43] and we find that the noise-averaged operator evolves with an adjoint Lindblad equation $d_t\langle \hat{A}_t \rangle = \mathcal{L}^\dagger[\langle \hat{A}_t \rangle]$. This corresponds to the standard evolution of an observable in an open quantum system with a Hermitian jump operator $\hat{L}$ [47]. The formalism described so far has been introduced in [3] and used in [4] to engineer long-range and many-body interactions. Here, we focus on the stochastic variance of an observable.

In order to find the variance, the second stochastic moment $\langle \hat{A}_t^2 \rangle$ is needed. Considering the reference operator to be $\hat{A}^2$ instead of $\hat{A}$, one finds that its evolution follows the Lindblad equation, $d_t\langle \hat{A}_t^2 \rangle = \mathcal{L}^\dagger[\langle \hat{A}_t^2 \rangle]$. Recall that the average is over realizations of the noise and that $\langle \hat{A}_t^2 \rangle$ is still an operator acting on the Hilbert space. Subtracting $d_t\langle \hat{A}_t \rangle^2 = \langle \hat{A}_t \rangle \mathcal{L}^\dagger[\langle \hat{A}_t \rangle] + \mathcal{L}^\dagger[\langle \hat{A}_t \rangle]\langle \hat{A}_t \rangle = \mathcal{L}^\dagger[\langle \hat{A}_t \rangle^2] + 2\gamma[\hat{L}, \langle \hat{A}_t \rangle]^2$ from both sides [44], we find the evolution of the SOV, defined as $\Delta\hat{A}_t^2 = \langle \hat{A}_t^2 \rangle - \langle \hat{A}_t \rangle^2$, to be given by

$$\frac{d(\Delta\hat{A}_t^2)}{dt} = \mathcal{L}^\dagger[\Delta\hat{A}_t^2] - 2\gamma[\hat{L}, \langle \hat{A}_t \rangle]^2. \quad (4)$$

The SOV $\Delta\hat{A}_t^2$ is an observable that characterizes the deviation of any (stochastic) operator $\hat{A}_t$ from the noise-averaged operator in a stochastic evolution governed by the Hamiltonian (1)—see Fig. 1 for a scheme and [44] for a quantitative illustration. Although its equation of motion depends on out-of-time-order terms like $\hat{L}\langle \hat{A}_t \rangle \hat{L}\langle \hat{A}_t \rangle$, it can easily be computed from the evolution of $\langle \hat{A}_t \rangle$ and $\langle \hat{A}_t^2 \rangle$. Indeed, the SOV evolves as

$$\Delta\hat{A}_t^2 \equiv \langle \hat{A}_t^2 \rangle - \langle \hat{A}_t \rangle^2 = e^{\mathcal{L}^\dagger t}[\hat{A}^2] - (e^{\mathcal{L}^\dagger t}[\hat{A}])^2. \quad (5)$$

The average $\langle \hat{A}_t \rangle$ follows the standard Lindblad dissipative dynamics from which the individual trajectories deviate as dictated by the variance $\Delta\hat{A}_t^2$. Since the SOV is Hermitian, it is an observable. It differs from the quantum variance of an operator $\hat{A}$ over a state ρ_t, commonly defined as $\mathrm{Var}(\hat{A}, \rho_t) = \mathrm{Tr}(\hat{A}^2\rho_t) - \mathrm{Tr}(\hat{A}\rho_t)^2$. For an initially pure state, this variance reads $\mathrm{Var}(\hat{A}_t, \psi_0) = \langle \psi_0 | e^{\mathcal{L}^\dagger t}[\hat{A}^2] | \psi_0 \rangle - \langle \psi_0 | e^{\mathcal{L}^\dagger t}[\hat{A}] | \psi_0 \rangle^2$. Such quantum variance cannot be obtained from the expectation value of an observable since it is nonlinear in the state $|\psi_0\rangle$. The difference between the SOV (5) evaluated on a pure state and the quantum variance reads $\langle \psi_0 | \Delta\hat{A}_t^2 | \psi_0 \rangle - \mathrm{Var}(\hat{A}_t, \psi_0) = \langle \psi_0 | e^{\mathcal{L}^\dagger t}[\hat{A}] \hat{\mathbb{Q}} e^{\mathcal{L}^\dagger t}[\hat{A}] | \psi_0 \rangle$, where $\hat{\mathbb{Q}} = \hat{\mathbb{1}} - |\psi_0\rangle\langle\psi_0|$ is a projection operator on the complementary subspace of $|\psi_0\rangle\langle\psi_0|$. Therefore, the SOV contains also contributions from the projector over the complementary subspace, similarly to the term governing the recombination of decay products in unstable systems [48].

A related notion of variance—without the stochastic interpretation—has been introduced in the theory of positive definite matrices [49], which provides us with tools to formally characterize the SOV. First, the SOV is positive semidefinite, $\Delta\hat{A}_t^2 \geq 0$. This can be shown from Kadison's inequality [50] which ensures that $e^{\mathcal{L}^\dagger t}[\hat{A}^2] \geq (e^{\mathcal{L}^\dagger t}[\hat{A}])^2$ for a positive and unital map, $e^{\mathcal{L}^\dagger t}[\hat{\mathbb{1}}] = \hat{\mathbb{1}}$. Second, our formalism leads to a generalization of the Robertson-Schrödinger uncertainty principle [51,52]. Indeed, considering two operators $\hat{A}$, $\hat{B}$ and their SOVs and an initial state ρ_0, we show in [44] that

$$\mathrm{Tr}(\Delta\hat{A}_t^2\rho_0)\mathrm{Tr}(\Delta\hat{B}_t^2\rho_0) \geq |\mathrm{Tr}(\Delta\widehat{AB}_t\rho_0)|^2, \\ \geq \frac{1}{4}\left(D_+^2(\hat{A},\hat{B}) - D_-^2(\hat{A},\hat{B})\right), \quad (6)$$

where $\Delta\widehat{AB}_t = e^{\mathcal{L}^\dagger t}[\hat{A}\,\hat{B}] - e^{\mathcal{L}^\dagger t}[\hat{A}]e^{\mathcal{L}^\dagger t}[\hat{B}]$ is the stochastic operator covariance of $\hat{A}$ and $\hat{B}$. The quantity $D_\eta(\hat{A},\hat{B}) \equiv \mathrm{Tr}(e^{\mathcal{L}^\dagger t}([\hat{A},\hat{B}]_\eta)\rho_0 - [e^{\mathcal{L}^\dagger t}(\hat{A}), e^{\mathcal{L}^\dagger t}(\hat{B})]_\eta)\rho_0)$ measures the difference between evolving the commutator ($\eta = -$) and anticommutator ($\eta = +$) as a whole or each separately [44]. Note that D_+ is purely real and D_- is purely imaginary, so $D_+^2 - D_-^2 \geq 0$. For $\hat{A} = \hat{B}$, the uncertainty relation is saturated and Eq. (6) becomes an equality.

Bipartite interpretation.—Many quantum properties, such as entanglement or scrambling, are best understood in a bipartite system [53–56]. The SOV can actually be interpreted analogously: consider a doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$, with an operator $\hat{A}$ living in each of the copies of the Hilbert space, denoted as $\mathcal{H}_1$ and $\mathcal{H}_2$. Using the swap operator $\mathbb{S}$ [55,56], which introduces an interaction between the Hilbert spaces and is defined as $\mathbb{S}(|i\rangle_1|j\rangle_2) = |j\rangle_1|i\rangle_2$, the product between operators can be interpreted as an operation over a doubled Hilbert space, namely, $\hat{X}\hat{Y} = \mathrm{Tr}_{\mathcal{H}_2}((\hat{X}\otimes\hat{Y})\mathbb{S})$—see details in [44]. Thus, the first term in the SOV (5) can be interpreted as first letting the operators $\hat{A}$ interact to form $\hat{A}^2$ in the bipartite system and then letting that operator evolve under the dissipative evolution of a single bath, $e^{\mathcal{L}^\dagger t}[\hat{A}^2]$. By contrast, the second term corresponds to letting each of the uncoupled systems evolve with their own bath and then making them interact at time t. As expected, the difference between these terms is positive since $\Delta\hat{A}_t^2 \geq 0$, and the first protocol always suffers less decoherence.

The SOV-OTOC connection.—Remarkably, the expectation value of (4) over the completely mixed state, $\rho = \hat{\mathbb{1}}/N$, gives a dissipative version of the OTOC, namely,

$$\frac{1}{N}\frac{d\,\mathrm{Tr}(\Delta\hat{A}_t^2)}{dt} = -\frac{2\gamma}{N}\mathrm{Tr}([\hat{L},\langle\hat{A}_t\rangle]^2). \quad (7)$$

OTOCs are typically defined from two operators as $C_t = -\mathrm{Tr}([\hat{B}_0,\hat{A}_t]^2)/N$, and measure the exponential sensitivity on initial conditions in quantum chaotic systems [20]. Indeed, in a quantum system with scrambling, one expects $C_t \sim \epsilon e^{\lambda_\mathrm{Q} t}$ in the time window $t_s \ll t \ll t_\mathrm{E}$ between the saturation time of two-point functions, $t_s \sim 1/\lambda_\mathrm{Q}$, and that of the OTOC, known as the Ehrenfest time, $t_\mathrm{E} \sim \ln(\hbar^{-1})/\lambda_\mathrm{Q}$ [22]. The main difference in our setting is that the evolved operator follows dissipative dynamics, $e^{\mathcal{L}^\dagger t}[\hat{A}]$, instead of unitary evolution, $e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}$. The connection between this OTOC and the SOV is pictorially shown in Fig. 1. It can be used to compute the Lyapunov exponent through

$$C_t = \frac{1}{2\gamma N}\frac{d\mathrm{Tr}(\Delta\hat{A}_t^2)}{dt} \sim \epsilon e^{\lambda_\mathrm{Q} t}, \quad (8)$$

where the exponential behavior holds only in systems with scrambling over the appropriate period $t_s \ll t \ll t_\mathrm{E}$. The SOV-OTOC connection is complementary to the optimal-path approach to study chaos in continuously monitored systems [57]. Note that the SOV $\Delta\hat{A}_t^2$ is an observable constructed from the knowledge of the evolution under different noise realizations. The classical limit of the above equation is similar, the only difference being that ΔA_t^2 becomes a function of time and the trace an average over a region of phase space [58].

The short-time decay of the OTOC, $C_t \sim C_0 e^{-t/\tau_D}$, is characterized by the dissipation time $\tau_D = (2\gamma\mathrm{Tr}([\hat{L},[\hat{L},\hat{A}]]^2)/(C_0 N))^{-1}$ and the initial value $C_0 = \mathrm{Tr}([\hat{L},\hat{A}]^2)/N$. Interestingly, this dissipation time is related to the Hilbert-Schmidt norm of the dissipator acting on the initial operator [44]. It is analogous to the decoherence rate found in [6] but now obtained for operators in the Heisenberg picture. When $[\hat{H}_0,\hat{L}] = 0$, the operators share a common eigenbasis, and using $\hat{L}|n\rangle = l_n|n\rangle$, we can write the dissipative OTOC (7) as

$$C_t = \sum_{m,n}(l_m - l_n)^2 e^{-2\gamma(l_m-l_n)^2 t}|A_{nm}|^2, \quad (9)$$

where $A_{nm} = \langle n|\hat{A}|m\rangle$. From this expression, the two exponentially decaying regimes reported in [34] appear from the largest and smallest eigenvalue differences governing the short- and long-time dynamics, respectively.

Note that, for simplicity, we have focused on the infinite temperature OTOC. However, the connection to OTOC (7) can be generalized to unregularized thermal OTOC [59] by tracing (4) over a thermal state $\rho_\beta = e^{-\beta\hat{H}}/\mathrm{Tr}(e^{-\beta\hat{H}})$. In this case, the SOV-OTOC relation also involves an additional expectation value of the Lindbladian $\mathrm{Tr}(\mathcal{L}[\Delta\hat{A}_t^2]\rho_\beta)$, which is nonzero in general. Fidelity OTOCs [60] can be obtained by taking the jump operator to be a projector over a pure state. Regularized OTOCs [22] can also be obtained by modifying the jump operator as $\hat{L} \to \rho_\beta^{1/4}\hat{L}\rho_\beta^{1/4}$ and tracing over the completely mixed state.

We next illustrate our findings in the Lipkin-Meshkov-Glick (LMG) model subject to energy dephasing and characterize its Lyapunov exponent using the SOV-OTOC connection.

Stochastic Lipkin-Meshkov-Glick (sLMG) model.—The LMG model describes the collective behavior of N identical two-level systems fully connected to each other with the same coupling strength [38]. Its quantum Hamiltonian reads

$$\hat{H}_\mathrm{LMG} = \Omega\hat{S}_z - \frac{2}{N}\hat{S}_x^2, \quad (10)$$

where Ω is the frequency in units of the coupling strength, and $\hat{S}_j$ are the general spin operators of dimension $2S+1$. Since the total spin $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ commutes with the spin operators, $[\hat{H}_{\rm LMG}, \hat{\mathbf{S}}^2] = 0$, the total angular momentum is conserved; we stay in the sector $S = N/2$. Because of time-translational symmetry, energy is conserved, and since there is only 1 degree of freedom, this model is integrable. If this continuous symmetry is broken by periodic kicks in $\hat{S}_x^2$, the model turns into the known kicked top [11].

Here, we break time-translational symmetry by adding noise in the energy scale and consider

$$\hat{H}_t = \hat{H}_{\rm LMG}(1 + \sqrt{2\gamma}\xi_t). \tag{11}$$

This leads to dephasing in the energy eigenbasis at the ensemble level. The evolution of the SOV instantaneous (ordered) eigenvalues satisfying $\Delta\hat{A}_t^2|v_k(t)\rangle = \Lambda_k(t)|v_k(t)\rangle$ for this model, is shown in Fig. 2(a). In analogy with the theory of quantum transport [61,62], we find *diffusive modes* in which $\Lambda_k(t) \sim t$ and *superdiffusive modes* in which $\Lambda_k(t) \sim t^{3/2}$. In the more general case $[\hat{H}_0, \hat{L}] \neq 0$, one also finds ballistic modes $\Lambda_k(t) \sim t^2$, as we detail in [44]. Based on the eigenvalues, one can find a state which minimizes the SOV at long times

$$|\Psi\rangle = \lim_{t\to\infty} |v_0(t)\rangle, \tag{12}$$

which corresponds to the steady state of the eigenvector with the smallest deviation, and is thus the state minimally affected by the noise. The expectation value over this state is shown in Fig. 2(a) (black dash-dotted) for the sLMG model, and we verify that it minimizes the spread at long times. Figure 2(b) shows the evolution of the dissipative OTOC, computed from the SOV using Eq. (8). We observe several exponential decays, as apparent from (9) and in agreement with [34]. This illustrates how the SOV can be used to obtain the dissipative OTOC.

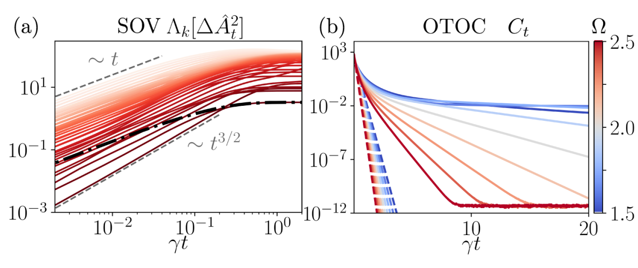

FIG. 2. Evolution of (a) the SOV eigenvalues, and (b) the OTOC C_t for the quantum sLMG model. The operator $\hat{A} = (\hat{S}_x + \hat{S}_y + \hat{S}_z)/\sqrt{3}$ evolves under the stochastic Hamiltonian (11) with $\gamma = 2$, $\Omega = 1$, and $S = 20$. (a) Eigenvalues of the SOV as a function of time (solid red) and expectation value of the SOV for the state which minimizes the deviation at long times, $\langle\Psi|\Delta\hat{A}_t^2|\Psi\rangle$ (black dash-dotted). (b) Dissipative OTOC obtained from the SOV-OTOC relation (7) (solid line) and short-time expansion (dashed line) for different values of Ω (color bar) across the phase transition—at $\Omega_c = 2$.

The classical limit of the Hamiltonian (10) is obtained by taking its expectation value over SU(2) coherent states $|\zeta\rangle$ [44,63] in the thermodynamic limit, $N\to\infty$. We introduce the canonical variables Q and P as $\zeta = [(Q - iP)/\sqrt{4-(Q^2+P^2)}]$ [39,40], which yields $H_{\rm LMG} = \lim_{S\to\infty}\langle\zeta|\hat{H}_{\rm LMG}|\zeta\rangle/S = (\Omega/2)P^2 + [(\Omega/2) - 1]Q^2 + \frac{1}{4}(Q^2P^2 + Q^4)$, where the terms of $\mathcal{O}(1/N)$ are neglected [44]. This model is integrable and exhibits an unstable fixed point at the origin, $Q^* = P^* = 0$, for $0 < \Omega < 2$. Since scrambling originates from an unstable point, it is already present in the semiclassical limit [25,39,40].

Here, we consider the classical equivalent of (11), namely, $H_t = H_{\rm LMG}(1 + \sqrt{2\gamma}\xi_t)$. The evolution of the noise-averaged observable displays the classical analog of energy dephasing, namely, $\partial_t\langle A_t\rangle = -\{H_{\rm LMG}, \langle A_t\rangle\}_{\rm P} + 2\gamma\{H_{\rm LMG}, \{H_{\rm LMG}, \langle A_t\rangle\}_{\rm P}\}_{\rm P}$, where $\{f, g\}_{\rm P}$ denotes the Poisson bracket of f and g. We characterize the Lyapunov exponent using three complementary methods: Analytically, the approach proposed by van Kampen [64,65] yields the Lyapunov for the sLMG as [44] $\lambda^{(1)} = \sqrt{2\Omega - \Omega^2} - \gamma(2\Omega - \Omega^2)$. (ii) Second, numerically. The standard, classical definition of the average Lyapunov exponent gives $\lambda^{(2)} = \langle \lim_{t\to\infty} \ln(\ell_t/\ell_0)/t\rangle$, where $\ell_t^2 = (Q_t - Q_t')^2 + (P_t - P_t')^2$ is the distance between two initially close trajectories evolving with the same realization of the noise, which is found solving the stochastic Hamilton's equations with a stochastic Runge-Kutta method [44,66]. (iii) Finally, our formalism gives the Lyapunov from Eq. (7), by taking as observable the position

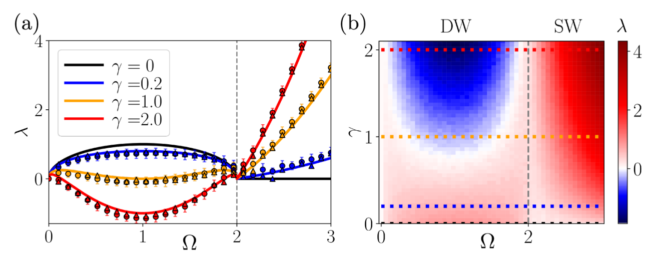

FIG. 3. Lyapunov exponent of the classical sLMG model at the saddle point $Q^* = P^* = 0$ as function of Ω (a) for different values of the noise strength γ and (b) over the phase diagram. (a) λ as computed (i) analytically using van Kampen's method $\lambda^{(1)}$ (solid lines), (ii) from the standard definition $\lambda^{(2)}$ (circles with error bar), and (iii) from the SOV-OTOC connection $\lambda^{(3)}$ (triangles). The known results for the LMG correspond to $\gamma = 0$ (black). (b) Phase diagram. The color scale represents the Lyapunov exponent $\lambda^{(2)}$ as a function of the model parameter Ω and the noise strength γ. A positive value of λ (red) implies exponential divergence of close initial conditions, while a negative value (blue) indicates exponential convergence. The dotted horizontal lines represent the values of γ sampled in (a). The vertical dashed gray line represents the transition between the double well ($\Omega < 2$) and single well ($\Omega \geq 2$) phases.

$A_t = Q_t$, the jump operator being the Hamiltonian itself, $L = H_0$. Namely, $\lambda^{(3)} = \lim_{t\to\infty}(1/2t)\ln(d_t\Delta Q_t^2/\epsilon)$.

Figure 3(a) shows the Lyapunov exponent λ obtained from the above methods as a function of Ω for different noise strengths. We verify that the three methods are in good agreement up to numerical errors. For $\gamma = 0$ (black line), we recover the behavior of LMG with $\lambda > 0$ in the double well (DW) phase and $\lambda = 0$ in the single well (SW) phase. Introducing a weak stochastic perturbation with γ, the Lyapunov exponent becomes smaller in the DW phase—trajectories diverge more slowly—while the SW phase acquires a positive λ. Increasing the strength of the noise causes the Lyapunov exponent in the DW phase to decrease and even reach negative values, where trajectories converge exponentially, while the value of λ in the SW phase increases. This rich behavior is summarized in the phase diagram presented in Fig. 3(b). The latter further shows that, under the application of strong noise, the origin $(Q^*, P^*) = (0, 0)$ is stable in the DW phase—trajectories converge exponentially to it—while it is unstable in the SW phase—trajectories diverge exponentially from it. Therefore the sLMG in the DW phase shows a noise-induced transition to stability [67], analogous to the stabilization seen for periodic driving [68].

In summary, we have introduced the stochastic operator variance and shown it is a valuable tool to study quantum systems driven by noise. We have shown that this observable obeys an uncertainty relation (6), and that it can be used to identify the state (12) which minimizes deviation from the Lindblad dissipative dynamics. We have provided a bipartite interpretation of the SOV. In addition, our results unveil a SOV-OTOC connection (8), which provides an operational protocol harnessing noise as a resource to probe OTOC and extract the Lyapunov exponent in noisy quantum chaotic systems. To illustrate our results, we introduced a stochastic generalization of LMG model and characterized its behavior in the quantum and classical realms. Our results provide the means to elucidate the fate of quantum chaos in noisy systems and benchmark NISQ devices.

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