Critical Scaling through Gini Index

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In the systems showing critical behavior, various response functions have a singularity at the critical point. Therefore, as the driving field is tuned toward its critical value, the response functions change drastically, typically diverging with universal critical exponents. In this Letter, we quantify the inequality of response functions with measures traditionally used in economics, namely by constructing a Lorenz curve and calculating the corresponding Gini index. The scaling of such a response function, when written in terms of the Gini index, shows singularity at a point that is at least as universal as the corresponding critical exponent. The critical scaling, therefore, becomes a single parameter fit, which is a considerable simplification from the usual form where the critical point and critical exponents are independent. We also show that another measure of inequality, the Kolkata index, crosses the Gini index at a point just prior to the critical point. Therefore, monitoring these two inequality indices for a system where the critical point is not known can produce a precursory signal for the imminent criticality. This could be useful in many systems, including that in condensed matter, bio- and geophysics to atmospheric physics. The generality and numerical validity of the calculations are shown with the Monte Carlo simulations of the two dimensional Ising model, site percolation on square lattice, and the fiber bundle model of fracture.

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Critical phenomena are observed in an expansive variety of physical systems undergoing equilibrium (fluids, binary mixtures, magnetic systems, superfluidity, superconductivity, etc.), as well as nonequilibrium phase transitions (fracture, active particles, etc.) [1]. When a system approaches a critical point by tuning a driving field F toward its critical value F_c, a suitably defined response function M (e.g., derivatives of free energy) would show a singular variation of the form $M \propto |F - F_c|^{-n}$. While from the universality hypothesis the value of the critical exponent n remains the same within a class of systems, the critical point F_c very much depends on the details of each system, thereby posing one of the major difficulties in estimating the critical exponent values [2].

In this Letter, we present a framework where the critical behavior can be formulated using a measure called the Gini index (g), which quantifies how unequal the response of a system is near the critical point [3]. The Gini index has been used for over a century in quantifying economic inequality. However, for the critical scaling of physical quantities, the Gini index of a response function shows a singularity at a point that is at least as universal as the corresponding critical exponent. Hence, the critical scaling for any unknown function becomes a one-parameter fit. We also formulate a precursory signal for an imminent critical point using a different measure of inequality, the Kolkata index (k) [4].

Because of the singular form of M, small changes in F can result in changes by very unequal amounts in M, depending upon the proximity to the critical point F_c (here we take $F_c > 0$, without loss of generality). Such highly unequal responses are ubiquitously manifested in various physical systems. For example, the drastic changes in the magnetic susceptibility near a ferromagnetic to paramagnetic transition [2], growing avalanche sizes in a stressed quasibrittle material driven toward the failure point [5], widely varying energy releases in earthquake events due to slowly moving tectonic plates [6], occurrences of catastrophic desertification due to small changes in endogenous [7] pressure, are a few instances of such unequal responses of measurable quantities near the corresponding critical points.

Given the often consequential nature of such transitions, along with estimating the critical exponent values, it is also of wide interest to predict the proximity to an imminent critical transition point [8–10]. As the critical point is a nonuniversal quantity, one often has to resort to multi-parameter fitting, Binder cumulant calculations, machine learning based regressions, or other system-specific methods in order to estimate the critical exponent values as well as the proximity to an imminent drastic change in the system, i.e., the critical point [11–18].

Here, we show that any response function $M \propto |F - F_c|^{-n}$ can be written in terms of the corresponding Gini index as $M \propto |g - g_f|^{-n^*}$, where n^* is a function of n and g_f is either a function of n or 1. We also show, through another measure of the inequality in values of M, the so-called Kolkata index (k), that the condition $g = k$ is satisfied for $F = F^* < F_c$ if $n > 1$, with the crossing point value of the two indices approaching $1/2$ from above as $n \to \infty$, thereby the condition acts as a precursor to the approaching criticality. For the ranges of the value of n that usually appears in physical systems, this crossing point value is close to 0.87 (very weakly dependent on n).

The inequality indices g and k (and similar other indices) are defined using the so-called Lorenz function. Lorenz function was introduced in 1905 primarily to quantify wealth inequality in an economy [19]. Traditionally the function $\mathcal{L}(p)$ is defined as the fraction of the total wealth of a society possessed by the poorest p fraction of the population. In the present context, for a monotonically diverging response function, the function $\mathcal{L}(p)$ can be computed within an arbitrary range from $F = A$ to $F = B$ as

$$\mathcal{L}(p, n, A, B) = \frac{\int_A^{A+p(B-A)} M\, dF}{\int_A^B M\, dF}, \tag{1}$$

where $A < B < F_c$. Experimentally and numerically, in the ferromagnets and Ising model, for example, for a series of values of temperature below the critical point, one could compute the Lorenz function with the (unequal) values of susceptibility ($M = \chi$, $F = T$ there) using the above equation. For the other side of criticality, the limits need to be appropriately reversed. By definition $\mathcal{L}(p = 0) = 0$ and $\mathcal{L}(p = 1) = 1$. Within the range $0 < p < 1$, $\mathcal{L}(p)$ is continuous, monotonically growing, and with a positive curvature, if any. In the extreme limit if M is independent of F, i.e., the responses are always equal no matter the driving field, then $\mathcal{L}(p) = p$, which is called the equality line. The departure of $\mathcal{L}(p)$ from this equality line, therefore, is a measure of the inequality in M.

To put a value to this inequality, i.e., to define an inequality index or coefficient, one needs to look at what is called the summary statistics of $\mathcal{L}(p, n, A, B)$, i.e., the p dependence needs to be removed. This can be done by either integrating $\mathcal{L}(p, n, A, B)$ over the full range of p, or by evaluating it at a particular value of p. The Gini index, defined as

$$g(n, A, B) = 1 - 2\int_0^1 \mathcal{L}(p, n, A, B)dp, \tag{2}$$

is an exercise of the former, while the Kolkata index, defined as the fixed point $1 - k = \mathcal{L}(k, n, A, B)$, is that of the latter. The interpretation for g is that it is the area between the equality line and the Lorenz curve divided by the area under the equality line (necessarily 1/2). Therefore, it varies between $g = 0$ (complete equality) to $g = 1$ (just one value is nonzero). The k index has the interpretation that $1 - k$ fraction of the largest values accounts for the k fraction of the total value. It is a generalization of Pareto's law [20].

We will first look at the properties of the Lorenz function and particularly the Gini index, when measured near the critical point of a system. To quantify proximity to the critical point, let us write $A = aF_c$ and $B = bF_c$. Then from Eq. (1), using the power-law variation of M, we get ($n \neq 1$, $n \neq 2$)

$$\mathcal{L}(p, n, a, b) = \frac{(1-a)^{1-n} - [1 - a - p(b-a)]^{1-n}}{(1-a)^{1-n} - (1-b)^{1-n}}. \tag{3}$$

It is then straightforward to evaluate the Gini index

$$g(n,a,b) = 1 - \frac{2}{(1-a)^{1-n} - (1-b)^{1-n}} \times \left[(1-a)^{1-n} + \frac{(1-b)^{2-n} - (1-a)^{2-n}}{(2-n)(b-a)}\right], \tag{4}$$

while the Kolkata index needs to be numerically evaluated from

$$1 - k(n,a,b) = \frac{(1-a)^{1-n} - [1 - a - k(n,a,b)(b-a)]^{1-n}}{(1-a)^{1-n} - (1-b)^{1-n}}. \tag{5}$$

We will use the notation $g(b = 1, n) = g_f$ and $k(b = 1, n) = k_f$ and keep $n > 0$. In the following we consider the cases $n < 1$, $n > 1$, and $n > 2$ separately, with the corresponding consequences in the scaling form of M. Note that the Gini index can be calculated from the qth order derivative of M with respect to F, which is also a diverging function at F_c with a different exponent. Let us, therefore, fix the notation that $\Delta g_{(\phi,q)}$ denotes the critical interval, when g (and g_f) are calculated using the qth order derivative of some response function having the original (i.e., in terms of $|F - F_c|$) exponent ϕ. So, for example, $\Delta g_{(\alpha,1)}$ would mean that it is calculated for the first derivative of specific heat and so on. The same is true for the rescaled exponents, i.e., $\gamma_{(\beta,1)}$ would mean the susceptibility exponent appearing in the power of $\Delta g_{(\beta,1)}$, where g is calculated using the first derivative of the order parameter (a diverging quantity at F_c).

Case I ($0 < n < 1$).—Clearly, $g(b = 1, n) = g_f = n/(2 - n)$, which is independent of a. It also follows from Eq. (4) that for $b \to 1$, keeping up to the leading order term in $(1 - b)^{1-n}$, we have [the details of the calculations are given in the Supplemental Material (SM) [21]]

$$g(n, a, b) \approx \frac{n}{2-n} - 2(1-b)^{1-n}(1-a)^{n-1}, \tag{6}$$

which means $|g - g_f| = \Delta g_{(n,0)} \propto (1 - b)^{1-n}(1 - a)^{n-1}$. Since $1 - b \propto F_c - F$ and a is constant,

$$M \propto \Delta g_{(n,0)}^{-n/(1-n)}, \tag{7}$$

with $g_f = n/(2 - n)$ and $0 < n < 1$. See Fig. 1(a) for comparisons with numerical evaluations for some typical values of n.

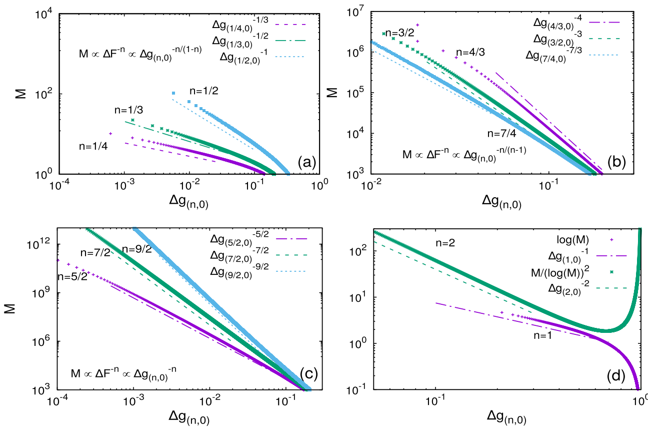

FIG. 1. The scaling behavior of a response function in terms of the Gini index is shown. As $M \propto \Delta F^{-n}$, the divergence with respect to the Gini index is $M \propto \Delta g_{(n,0)}^{-n^*}$, where the divergence exponent $n^* = n/(1-n)$ for $0 < n < 1$ [(a) showing some typical examples], $n^* = n/(n-1)$ for $1 < n < 2$ [(b) showing some typical examples], $n^* = n$ for $n > 2$ [(c) showing some typical examples], and (d) shows the particular cases of $n = 1$ and $n = 2$.

Case II ($1 < n < 2$).—In the limit $b \to 1$, Eq. (4) gives $g_f = 1$. It also follows from Eq. (4) that up to the leading order $|g - g_f| = \Delta g_{(n,0)} \propto (1-a)^{1-n}(1-b)^{n-1}$, which as before leads to

$$M \propto \Delta g_{(n,0)}^{-n/(n-1)}, \tag{8}$$

with $g_f = 1$ and $1 < n < 2$. See Fig. 1(b) for comparisons with numerical evaluations for some typical values of n. See also Fig. 2(a) of the SM for the manifestation of this scaling in two dimensional Ising model.

Case III ($n > 2$).—Here, also $g_f = 1$. Then up to the leading order, $|g - g_f| = \Delta g_{(n,0)} \propto (1-b)/(1-a)$. This implies

$$M \propto \Delta g_{(n,0)}^{-n}, \tag{9}$$

with $g_f = 1$ and $n > 2$. See Fig. 1(c) for comparisons with numerical evaluations for some typical values of n. See also Fig. 2(b) of the SM for the manifestation of this scaling in the site percolation.

Case IV ($n = 1$ *and* $n = 2$).—For $n = 1$, up to the leading order $g \approx 1 + (2/b)[1/\ln(1-b)]$ and with $g_f \to 1$ for $n \to 1$, $\ln(M) \sim \Delta g_{n,0}^{-1}$. Similarly, for $n = 2$, $g \approx 1 + (2/b)[(1-b)\ln(1-b)]$. With $g_f \to 1$ for $n \to 2$, we have $M/[\ln(M)]^2 \sim \Delta g_{(n,0)}^{-2}$. Numerically these are verified in Fig. 1(d).

In the above cases, we have written a generic response function near any critical point in such a way that the critical exponent (n^*) and the critical point (g_f) are equally universal. We have calculated g from one side of the critical point for the diverging response function, but it is extendable to the other side of the critical point (see Fig. 1 in SM). Also, it follows that the corresponding coefficients of such a diverging function are expected to be the same on both sides.

Note that, for practical purposes, when the proximity to the critical point is *a priori* not known, for a series of values of the driving parameter F, one can calculate a series of values of g [from Eq. (4)] for a response function. An estimate of the critical point could be made beforehand by noting the maximum of g (see Sec. IV in SM, specifically Fig. 5). These g values can then be fitted using Eqs. (7), (8), or (9), which is then a single parameter fit, since g_f is solely dependent on n, the exponent value. This is a considerable simplification from the usual situation where the critical point (F_c) and the critical exponent (n) are independent.

It is useful to revisit the implications of the critical scaling, using g on (a) the finite size scaling, to show what is expected for a simulation study with finite system sizes, and (b) precursor to critical point, for a practical application of the inequality measures in a variety of systems.

Finite size scaling.—A characteristic feature of second order phase transition is the divergence of a correlation length ξ at the critical point $F_c(\infty)$ (in the infinite system size limit), where F_c can be critical temperature (T_c) in Ising model, percolation threshold (p_c) in percolation, or critical applied stress (σ_c) in the fiber bundle model (FBM) of fracture, etc. In a finite system, however, near $F_c(L)$ the role of ξ is taken over by the linear system size L:

$$|F - F_c(\infty)| \propto \xi^{-\frac{1}{\nu}} \longrightarrow |F_c(L) - F_c(\infty)| \propto L^{-\frac{1}{\nu}}. \tag{10}$$

Recalling that $|g - g_f| \propto c(n)|F - F_c|^{\theta}$, where $c(n)$ is only dependent on n and $\theta = 1 - n$ for $0 < n < 1$, $\theta = n - 1$ for $1 < n < 2$ and $\theta = 1$ for $n > 2$, at the critical point of a finite system (of linear size L) we would have

$$|g_f(L) - g_f(\infty)| \propto L^{-\frac{\theta}{\nu}}, \tag{11}$$

with the values of θ depending on n as mentioned above. This is numerically verified for the Ising model on square lattice (see Fig. 6 in SM).

Precursor to critical point.—The closed form of the Kolkata index k is not possible for arbitrary n [see Eq. (5)]. However, its numerical evaluation shows the remarkable property that for $n > 1$, k becomes equal to g at two points: one is the trivial point where $g_f = k_f = 1$ at the critical point, but the other point (say, $g^* = k^*$ at $F = F^*$) is necessarily below the critical point and usually very close to it (see Fig. 7 in SM). Therefore, for any system approaching a critical point (from either side, if possible), monitoring g and k for a sufficiently strongly diverging response function ($n > 1$) would indicate an imminent critical point when the two quantities become equal and have a value smaller than 1 (see Fig. 2).

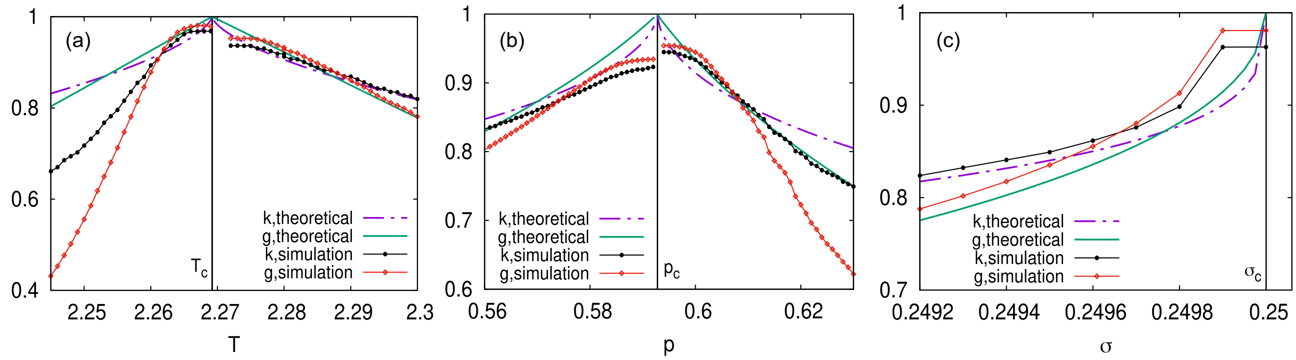

FIG. 2. The precursory signals from the crossing points of Gini (g) and Kolkata (k) indices. (a) The values of g and k are measured for χ^2 in the two dimensional Ising model from either side of the critical point (by increasing and decreasing temperature from below and above the critical point, respectively). The crossing happens close to the critical point. In simulation, g and k do not reach 1 due to finite size effect. (b) Here, the same is done for the second moment of cluster sizes for the site percolation in two dimensions. (c) Here, the cube of the avalanche sizes are taken for the fiber bundle model [$S^3 \propto (\sigma_c - \sigma)^{-3/2}$]. The crossing can only be shown here in the precritical regime, since there is no stable configuration of the model for $\sigma > \sigma_c$, the catastrophic failure point. In all cases the analytical estimates are also shown, which do not match very well since in the simulations the power-law variation is only valid very close to the critical point. But the crossing point values for g and k are almost independent of the associated exponent value.

Having a reliable precursory signal to an imminent critical point is a crucial issue in many physical systems, including fracture, environmental catastrophe, market crash, etc. In the case of fracture [5], this issue have been addressed in several different ways, including using inequality indices [23–25].

Here, we take three paradigmatic examples, the Ising model, and the site percolation problem on square lattices and the fiber bundle model of fracture and show that $g^* = k^*$ at $F = F^* < F_c$ (where $F = T$ in Ising model, $F = p$ in percolation, and $F = \sigma$ in FBM) is a reliable precursor to critical point on both sides of criticality for the first two models and for one side in the case of FBM (since there is no stable state on the other side of criticality in this case).

First we consider the dynamics of the fiber bundle model for fracture, which has been viewed as a critical phenomena for several decades. It is a threshold activated cellular automata type model that reproduces many features of fracture dynamics (see Ref. [26] for a review), including the intermittent scale-free avalanche dynamics in disordered quasibrittle materials. With N elements (fibers) carrying a load W, the mean field version of the model is analytically tractable. For a mild restriction on the failure threshold (load beyond which a fiber breaks and redistributes its load to the remaining fibers) probability distributions of the individual fibers, the fraction of surviving fibers $U(\sigma) = N(\sigma)/N$ for an applied load per fiber $\sigma = W/N$ has the form $U(\sigma) = U(\sigma_c) + D(\sigma_c - \sigma)^{1/2}$, where D is a constant that depends on the distribution function and σ_c is the critical load beyond which the system collapses [21]. One can then consider the response function

$$S(\sigma) = \left|\frac{dU}{d\sigma}\right| \propto (\sigma_c - \sigma)^{-1/2}, \qquad (12)$$

which has the physical interpretation of the avalanche size (if a constant amount of load $d\sigma$ is added to the system every time it comes to a stable state). For detecting the precursory signal from the Gini and Kolkata indices (i.e., to make them cross), we need a function that diverges with an exponent higher than 1. We consider the function $S^3(\sigma)$, which will diverge with an exponent $3/2$. A higher power would still work, but will give a precursory signal earlier, eventually leading to the trivial limit where precursor is set as soon as $F > 0$ (see Fig. 7 in SM). We numerically evaluate $S^3(\sigma)$ from the simulation data. Then we calculated the inequality indices $g_{(3/2,0)}(a, b, n = 3/2)$ and $k_{(3/2,0)}(a, b, n = 3/2)$ and found that they cross at a point prior to the critical point [see Fig. 2(c)]. The crossing point, therefore can serve as an indicator to imminent critical point (catastrophic breakdown in this case) irrespective of the threshold distribution function.

Note that in an SOC state, the system is always very close to the critical point. Its response statistics are generally scale-free. It is analytically known for the FBM that the avalanche size distribution exponent value is the same for both the (mean field) self-organized criticality (SOC) case [27] and for the avalanches occurring only very close to the (tuned) critical point [28]. Note that the crossing point value $g^* = k^* \approx 0.87$, which is almost independent of the divergence exponent, is what was numerically observed in simulations [29] of SOC models (including FBM) and the real data of many systems assumed to be in the SOC state [25,30]. This near-universal observation can now be argued from the above to be a consequence of the measurements of inequality indices [from Eqs. (4) and (5)] of the corresponding response functions very close to the critical point.

The generality of this precursory signal can be seen by applying it for the two dimensional Ising model and site percolation on square lattice. For the Ising model, the susceptibility (χ) diverges with an exponent $\gamma = 7/4$ [1]. While g and k are expected to cross for this, the crossing point is expected to be very close to the critical point (see Fig. 7 in the SM). So, we take χ^2 instead (diverging with an exponent $7/2$), for which the crossing points for g and k can be seen [Fig. 2(a)] from both sides of the critical point. Similarly, for the site percolation on square lattice, the second moment of the cluster size distribution diverges with an exponent $43/18$ [31]. Here also, the crossing of g and k could be seen prior to the critical point on both sides of the critical point [Fig. 2(b)].

In conclusion, inequality measures of diverging response functions near a critical point enable a superuniversal representation of such functions [see Eqs. (7), (8), and (9)] that are free from the nonuniversal, model specific critical point. It also allows for a precursory signal of an approaching criticality, which is crucial in many systems. The analytical results are verified through numerical simulations of the two dimensional Ising model, site percolation on square lattice, and the fiber bundle model of fracture, but these are applicable to any equilibrium or nonequilibrium critical phenomenon.

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[1] S.-k. Ma, *Modern Theory of Critical Phenomena* (Taylor & Francis, New York, 2001), 10.4324/9780429498886.

[2] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (CRC Press, New York, 1992), 10.1201/9780429493492.

[3] C. Gini, Measurement of inequality of incomes, Econ. J. **31**, 124 (1921).

[4] A. Ghosh, N. Chattopadhyay, and B. K. Chakrabarti, Inequality in societies, academic institutions and science journals: Gini and k-indices, Physica (Amsterdam) **410A**, 30 (2014).

[5] S. Biswas, P. Ray, and B. K. Chakrabarti, *Statistical Physics of Fracture, Breakdown and Earthquake* (Wiley-VCH, Weinheim, 2015).

[6] H. Kawamura, T. Hatano, N. Kato, S. Biswas, and B. K. Chakrabarti, Statistical physics of fracture, friction, and earthquakes, Rev. Mod. Phys. **84**, 839 (2012).

[7] A. E. Noble, T. S. Rosenstock, P. H. Brown, J. Machta, and A. Hastings, Spatial patterns of tree yield explained by endogenous forces through a correspondence between the Ising model and ecology, Proc. Natl. Acad. Sci. U.S.A. **115**, 1825 (2018).

[8] J. M. Drake and B. D. Griffen, Early warning signals of extinction in deteriorating environments, Nature (London) **467**, 456 (2010).

[9] S. R. Carpenter, J. J. Cole, M. L. Pace, R. Batt, W. A. Brock, T. Cline, J. Coloso, J. R. Hodgson, J. F. Kitchell, D. A. Seekell, L. Smithand, and B. Weidel, Early warnings of regime shifts: A whole-ecosystem experiment, Science **332**, 1079 (2011).

[10] M. Scheffer, S. R. Carpenter, T. M. Lenton, J. Bascompte, W. Brock, V. Dakos, J. van de Koppel, I. A. van de Leemput, S. A. Levin, E. H. van Nes, M. Pascualand, and J. Vandermeer, Anticipating critical transitions, Science **338**, 344 (2012).

[11] E. van Nieuwenburg, Y.-H. Liu, and S. Huber, Learning phase transitions by confusion, Nat. Phys. **13**, 435 (2017).

[12] J. Carrasquilla and R. G. Melko, Machine learning phases of matter, Nat. Phys. **13**, 431 (2017).

[13] R. A. Vargas-Hernández, J. Sous, M. Berciu, and R. V. Krems, Extrapolating quantum observables with machine learning: Inferring multiple phase transitions from properties of a single phase, Phys. Rev. Lett. **121**, 255702 (2018).

[14] N. Maskara, M. Buchhold, M. Endres, and E. van Nieuwenburg, Learning algorithm reflecting universal scaling behavior near phase transitions, Phys. Rev. Res. **4**, L022032 (2022).

[15] M. Yang, T. Karmakar, and M. Parrinello, Liquid-liquid critical point in phosphorous, Phys. Rev. Lett. **127**, 080603 (2021).

[16] J. C. Xavier and F. C. Alcaraz, Precise determination of quantum critical points by violation of the entropic area law, Phys. Rev. B **84**, 094410 (2011).

[17] T. F. J. Bögels and R. Caracas, Critical point and supercritical regime of MgO, Phys. Rev. B **105**, 064105 (2022).

[18] K. Binder, Critical properties from Monte Carlo coarse graining and renormalization, Phys. Rev. Lett. **47**, 693 (1981).

[19] M. O. Lorenz, Methods of measuring the concentration of wealth, Publ. Am. Stat. Assoc. **9**, 209 (1905).

[20] V. Pareto and A. N. Page, *Translation of 'Manuale di Economia Politica' (Manual of Political Economy)* (A.M. Kelley Publishing, New York, 1971).

[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.157101 for Sec. IIIB, one could also use something called a branching ratio as the order parameter, as defined in [22].

[22] Y. Moreno, J. B. Gómez, and A. F. Pacheco, Fracture and second-order phase transitions, Phys. Rev. Lett. **85**, 2865 (2000).

[23] S. Biswas and B. K. Chakrabarti, Social inequality analysis of fiber bundle model statistics and prediction of materials failure, Phys. Rev. E **104**, 044308 (2021).

[24] Diksha and S. Biswas, Prediction of imminent failure using supervised learning in a fiber bundle model, Phys. Rev. E **106**, 025003 (2022).

[25] A. Ghosh, S. Biswas, and B. K. Chakrabarti, Success of social inequality measures in predicting critical or failure points in some models of physical systems, Front. Phys. **10**, 990278 (2022).

[26] S. Pradhan, A. Hansen, and B. K. Chakrabarti, Failure processes in elastic fiber bundles, Rev. Mod. Phys. **82**, 499 (2010).

[27] S. Biswas and L. Goehring, Interface propagation in fiber bundles: Local, mean-field and intermediate range-dependent statistics, New J. Phys. **18**, 103048 (2016).

[28] S. Pradhan, A. Hansen, and P. C. Hemmer, Crossover behavior in burst avalanches: Signature of imminent failure, Phys. Rev. Lett. **95**, 125501 (2005).

[29] S. S. Manna, S. Biswas, and B. K. Chakrabarti, Near universal values of social inequality indices in self-organized critical models, Physica (Amsterdam) **596A**, 127121 (2022).

[30] S. Banerjee, S. Biswas, B. K. Chakrabarti, S. Challagundla, A. Ghosh, S. Guntaka, H. Koganti, A. Kondapalli, R. Maiti, M. Mitra, and D. R. Ram, Evolutionary dynamics of social inequality and coincidence of Gini and Kolkata indices under unrestricted competition, Int. J. Mod. Phys. C **34**, 2350048 (2023).

[31] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 2003).