Transport and Entanglement across Integrable Impurities from Generalized Hydrodynamics

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Quantum impurity models (QIMs) are ubiquitous throughout physics. As simplified toy models they provide crucial insights for understanding more complicated strongly correlated systems, while in their own right are accurate descriptions of many experimental platforms. In equilibrium, their physics is well understood and have proven a testing ground for many powerful theoretical tools, both numerical and analytical, in use today. Their nonequilibrium physics is much less studied and understood. However, the recent advancements in nonequilibrium integrable quantum systems through the development of generalized hydrodynamics (GHD) coupled with the fact that many archetypal QIMs are in fact integrable presents an enticing opportunity to enhance our understanding of these systems. We take a step towards this by expanding the framework of GHD to incorporate integrable interacting QIMs. We present a set of Bethe-Boltzmann type equations which incorporate the effects of impurity scattering and discuss the new aspects which include entropy production. These impurity GHD equations are then used to study a bipartioning quench wherein a relevant backscattering impurity is included at the location of the bipartition. The density and current profiles are studied as a function of the impurity strength and expressions for the entanglement entropy and full counting statistics are derived.

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Introduction.—The role of impurities in quantum physics has long been known to be central in our understanding of physical phenomena. Classical scattering from impurities in metals gives a finite lifetime to quasiparticles, taming unphysical divergences in conductivities. Interaction between electrons and magnetic impurities leads to the formation of clouds of many correlated particles resulting in strongly correlated compounds [1]. Pronounced effects in emission and absorption spectra are explained via the orthogonality catastrophe, the thermodynamic vanishing of overlaps between ground states of pure and impure systems [2]. Quantum impurity models (QIMs) are simplified versions of the above scenarios, consisting of a single impurity with few degrees of freedom interacting with a much larger environment, which capture the essential physics. They provide an ideal platform to study strongly correlated phenomena such as dynamical scale generation or asymptotic freedom and as such have become a proving ground for many of the most widely used nonperturbative techniques, both analytical and numerical, of modern theoretical physics. They are also of interest in their own right as quantum impurity systems are now routinely engineered in the laboratory [3–8].

Equilibrium properties of QIMs are quite well understood in large part due to the fact that many of the most important examples are integrable [9,10]. These include the well-known Kondo [11–13], Anderson [14,15], and Kane-Fisher models [16,17], among others [18–29]. However, experimental advances over recent decades have instigated a shift away from studies of equilibrium phenomena to those of far from equilibrium quantum physics and in particular of integrable interacting models [30,31]. At the same time key technical breakthroughs such as the quench action method [32,33] and generalized hydrodynamics [34,35] have facilitated our understanding of these systems. These powerful analytical tools allow one to study the long time behavior and emergent properties of nonequilibrium systems and have been applied to a plethora of scenarios, such as homogeneous [36–54] and inhomogeneous [55–77] quantum quenches. Despite their huge success and widespread use, however, these techniques have yet to be applied to interacting quantum impurity systems. In this Letter we take a step toward rectifying this and enlarge the framework of generalized hydrodynamics (GHD) to incorporate integrable quantum impurities. This is done by modifying the standard Bethe-Boltzmann–type equations of GHD through the inclusion of an exactly determined collision integral describing the integrable scattering with the impurity (1). We make some nontrivial checks on the resulting impurity GHD equations and comment on its new features. Afterward we give an exact solution to these equations for a bipartioning quench in which an interacting backscattering impurity is located at the bipartition. From this we calculate the density and current profiles as well as their full counting statistics and the half system entanglement entropy. This represents a rare example of a nonequilibrium, interacting quantum impurity problem which is analytically tractable.

Integrable impurity models.—Integrable quantum systems generically possess an extensive number of local conserved charges which heavily impact upon the static and dynamic properties of the system. Their spectra consist of a set of stable quasiparticle species, parameterized by a species index j and a rapidity λ and whose properties are described through kinematic data which includes their value under the conserved charges such as their energy $\epsilon_j(\lambda)$ and momentum $p_j(\lambda)$, in addition to the two-particle scattering kernel $T_{jk}(\lambda)$. The state of a system is specified by the types of quasiparticles present and in the thermodynamic limit, this can be done through their distribution in rapidity space, denoted by $\rho_j(\lambda)$. It is also convenient to introduce $\rho^h_j(\lambda)$ the distribution of unoccupied quasiparticles as well as the occupation function $\vartheta_j(\lambda) = \rho_j(\lambda)/\rho^t_j(\lambda), \rho^t_j(\lambda) = \rho_j(\lambda) + \rho^h_j(\lambda)$. These quantities are not unrelated and obey the thermodynamic Bethe Ansatz equations, a set of coupled integral equations arising from the quantization conditions $\rho^t_j(\lambda) = |p'_j(\lambda)|/2\pi - \sum_k \int d\mu T_{jk}(\lambda-\mu)\rho_k(\mu)$, with $'$ denoting $d/d\lambda$. Moreover, in the presence of many excitations the quasiparticle properties become dressed due to the interactions, e.g., $\epsilon_j(\lambda) \to [\epsilon_j(\lambda)]^{\mathrm{Dr}}$. To calculate these dressed quantities it is convenient to introduce a second dressing operation, denoted by $[\cdots]^{\mathrm{dr}}$ which is related to the physical dressing by $[\epsilon_j(\lambda)]^{\mathrm{Dr}} = \int^\lambda d\mu [\epsilon'_j(\mu)]^{\mathrm{dr}}$ and can be computed through the integral equation $[\epsilon'_j(\lambda)]^{\mathrm{dr}} = \epsilon'_j(\lambda) - \sum_k \int d\mu T_{jk}(\lambda-\mu)\vartheta_k(\mu)[\epsilon'_j(\mu)]^{\mathrm{dr}}$.

For an integrable QIM additional information is required to describe the interaction between the bulk quasiparticles and the impurity. This can be encoded via two different bases: (i) the scattering basis or (ii) the diagonal basis. In (i) the bulk quasiparticles are as described above and their interaction with the impurity is given by the impurity S matrix which is in general nondiagonal, giving rise to scattering between different quasiparticle species. We characterize this through its off diagonal components, $R_{ij}(\lambda)$, which are the bare reflection coefficients for a particle i to scatter to a particle j. Evidently this basis does not diagonalize the Hamiltonian but is useful when considering transport properties and has been utilized in earlier studies of nonequilibrium QIMs [78–84]. In (ii) the bulk quasiparticle basis is rotated so as to diagonalize the impurity S matrix and therefore also the Hamiltonian. Their interaction with the impurity is characterized by the impurity phase shifts $\varphi_j(\lambda)$ which the quasiparticles acquire when passing through the impurity [85]. This basis is the natural one for describing equilibrium properties of the system. The two bases are straightforwardly related as are their scattering data, in particular we shall emphasize this by writing the reflection coefficients as $R_{ij}[\varphi(\lambda)]$. In what follows we ignore the shift of the Bethe equations arising from the impurity as it does not contribute to leading order in the system size. The effect of the impurity is considered solely to cause scattering between quasiparticle species.

Generalized hydrodynamics with impurities.—GHD provides the long wavelength description of integrable models in an inhomogeneous far from equilibrium setting. It is obtained by considering the transport of the conserved charges through the system via their continuity equations which are then cast in terms of the quasiparticle distributions. The result is a set of coupled Euler equations for the quasiparticle rapidity distributions which become functions of space and time; $\rho_j(\lambda, x, t)$. They can be viewed semiclassically as describing the ballistic propagation of quasiparticles through the system. A crucial step in this procedure is to express the conserved currents in terms of $\rho_j(\lambda, x, t)$, which was at first conjectured on general grounds and later proven microscopically [88]. In the spirit of [34,35,56] we incorporate the presence of an integrable impurity at the origin through the inclusion of a collision integral term in the GHD equations,

$$\partial_t\rho_j(\lambda, x, t) + \partial_x[v_j(\lambda, x, t)\rho_j(\lambda, x, t)] = \delta(x)\mathcal{I}_j(\lambda, t). \quad (1)$$

Here the left-hand side is the standard GHD equation with $v_j(\lambda, x, t) = [\epsilon'_j(\lambda)]^{\mathrm{dr}}/[p'_j(\lambda)]^{\mathrm{dr}}$ being the dressed quasiparticle velocity. The right-hand side is new and is given by $\mathcal{I}_j = \sum_{k\neq j}\mathcal{I}_{jk}$ with

$$\mathcal{I}_{jk} = |R_{kj}([\varphi]^{\mathrm{Dr}})|^2\rho_k[1-\vartheta_j] - |R_{jk}([\varphi]^{\mathrm{Dr}})|^2\rho_j[1-\vartheta_k]. \quad (2)$$

The first term here represents the scattering of species k into j while the second is for the reverse process. The factors $1-\vartheta_i(\lambda)$ appear as we have assumed that the quasiparticles obey Pauli exclusion, however, other statistics can also be incorporated through appropriate replacements [89]. The terms $|R_{jk}([\varphi]^{\mathrm{Dr}})|^2$ are the reflection amplitudes of the *dressed* quasiparticles and when combined with the other factors present give the total rate of scattering into and out of the species j due to the species k. As a consequence of the integrability of the impurity this preserves the total number of quasiparticles in the system but breaks some of the conservation laws, specifically those for which j and k have different charges. It is important to note that while the GHD equations are for those in the scattering basis it is the phase shift from the diagonal basis that undergoes dressing rather than the reflection coefficients themselves. This is done locally at the impurity site, i.e., using the occupation functions at $x = 0$. Equations (1) and (2) constitute the main result of our Letter. Through them one can study the nonequilibrium dynamics resulting from inhomogeneous quenches such as the bipartioning protocol (see below) in the presence of integrable interacting impurities. They represent a very natural extension of the formalism, indeed such a protocol was the setting for the first appearance of the GHD equations, albeit only a purely reflecting defect was considered [90]. It should be noted, however, that the scattering between quasiparticle species caused by the impurity leads to many nontrivial effects which are absent when the defect is purely reflecting (or equivalently purely transmitting). These include the generation of a strong coupling scale, a feature of interacting QIMs, entropy production and nontrivial charge, and current fluctuations.

Checks.—We can perform some simple analytic checks of our result. First we consider a noninteracting model of one species type, with energy $\epsilon(\lambda)$ and momentum $p(\lambda)$ such that $\mathrm{sgn}[\epsilon'(\lambda)/p'(\lambda)] = \mathrm{sgn}[\lambda]$, coupled to a noninteracting impurity at the origin. The impurity allows for both transmission and reflection, i.e., a flip of the sign of the momentum of an incident particle $p(\lambda) \to -p(\lambda)$ with reflection coefficient $R(\lambda)$. We take the system to be initially decoupled from the impurity and prepared in its ground state with different Fermi levels, $\Lambda_{L,R}$ to the left and right of the origin. The impurity is then suddenly turned on and the system allowed to evolve. This situation has been studied several times in both lattice and continuum systems [91–95] (see also [96,97] for a moving defect). To reproduce those results we treat the quasiparticles with $\lambda > 0$ and $\lambda < 0$ as different species with the impurity causing scattering between the two. As interactions are absent, $T_{jk} = 0$, no dressing occurs and one can straightforwardly solve (1) finding that the total current through the impurity is $J(t) = \int_{\Lambda_L}^{\Lambda_R} d\lambda |p'(\lambda)|[1 - |R(\lambda)|^2]/2\pi$, the usual Landauer-Buttiker result previously obtained.

As a more nontrivial check let us consider an interacting QIM with two quasiparticle species and an impurity which mixes the two. For the bare impurity S matrix we take the generic form

$$S(\lambda) = e^{i\alpha(\lambda)} \begin{pmatrix} \cos\chi(\lambda) & i\sin\chi(\lambda) \\ i\sin\chi(\lambda) & \cos\chi(\lambda) \end{pmatrix} \tag{3}$$

such that the diagonal basis consists of symmetric and antisymmetric combination of the quasiparticles with phase shifts $\alpha(\lambda) \pm \chi(\lambda)$. Suppose now we take the ground state of the system at finite density so that rapidities $\lambda > \Lambda$ are unoccupied. To this we add a single scattering quasiparticle at $\lambda = \lambda^p > \Lambda$. According to (2) this particle is then scattered by the impurity from one species to the other at a rate $\mathcal{I} = |\sin[\chi^{\mathrm{Dr}}(\lambda^p)]|^2$. This is the inverse lifetime of the quasiparticle and as $\lambda^p \to \Lambda$ can be related to the zero temperature resistivity of the system which has been calculated in QIMs such as the Kondo model. Specializing to that case we find agreement with the known exact result calculated using the T-matrix formalism [85,98].

Entropy production.—In the absence of the impurity, the GHD equations preserve entropy; however, diffusive corrections to this have been calculated which allow for the transfer of entropy between scales of the system [99,100]. The collision integral $\mathcal{I}_j$ plays a similar role here and results in the production of entropy even from a zero temperature state. To see this we note that we can derive a GHD equation also for the occupation functions albeit with a modified impurity term,

$$\partial_t \vartheta_j(\lambda, x, t) + v_j(\lambda, x, t)\partial_x \vartheta_j(\lambda, x, t) = \delta(x)\mathcal{I}_j^{\vartheta}(\lambda, x, t) \tag{4}$$

where $[\mathcal{I}_j^{\vartheta}(\lambda, t)]^{\mathrm{dr}}\rho_j^t(\lambda, 0, t) = \mathcal{I}_j(\lambda, t)$. Likewise an Euler-type equation can be derived for ρ_j^h also. Combining these along with the definition of the Yang-Yang entropy [101] we find that the total entropy production rate in the system is [85]

$$\partial_t \mathcal{S}(t) = \sum_j \int d\lambda s_j(\lambda, 0, t)\mathcal{I}_j(\lambda, t) \tag{5}$$

where $s_j(\lambda, x, t) = \log[\rho_j^h(\lambda, x, t)/\rho_j(\lambda, x, t)] + \sum_k \int d\mu T_{jk} \times (\lambda - \mu)\log[1 - \vartheta_k(\mu, x, t)]$. The establishment of a nonequilibrium steady state therefore leads to a linear in time increase in the total entropy of the system.

Kane-Fisher model.—We now look to implement this framework in a specific QIM, the Kane-Fisher model describing a backscattering impurity in a Luttinger liquid. The Hamiltonian is given by

$$H = \int dx \psi^\dagger(x)[-i\sigma^z\partial_x]\psi(x) + U\delta(x)\psi^\dagger(x)\sigma^x\psi(x) + \frac{g}{2}\left([\psi^\dagger(x)\psi(x)]^2 - [\psi^\dagger(x)\sigma^z\psi(x)]^2\right). \tag{6}$$

Here $\psi = (\psi_r, \psi_l)^T$ are two component Weyl fermions with linear dispersion and positive (r) or negative (l) momentum. They interact with each other via a four-fermion interaction of strength g and with an impurity allowing both transmission and reflection at the origin of strength U. In the absence of the impurity, the model has two $U(1)$ charges, the total charge $N = \int dx\psi^\dagger\psi$, and chiral charge $J = \int dx\psi^\dagger\sigma^z\psi$; however, when $U \neq 0$ the latter is no longer conserved and a current is generated by the impurity. The impurity is RG relevant for $g > 0$ and leads to a dynamically generated scale, T_U, and vanishing conductance at zero temperature in equilibrium [102]. The model is integrable and can be solved either through bosonization and mapping onto the boundary sine-Gordon model [16,78] or directly in fermionic form using the coordinate Bethe ansatz [17].

We study a bipartioning quench, where the system is prepared in the ground state of the $U = 0$ system at different chemical potentials to the left and right of the origin and then allowed to evolved according to H at nonzero U. This models the sudden coupling of two disjoint quantum wires through a quantum point contact. In this context the system contains two quasiparticle species labeled $\pm$. Their energy and momenta are $\epsilon_\pm(\lambda) = \pm p(\lambda) = e^\lambda$ while $T_{\pm\pm}(\lambda) = T_{\pm\mp}(\lambda) \equiv T(\lambda)$,

$$T(\lambda) = \int \frac{d\omega}{2\pi} e^{-i\omega\lambda} \frac{\sinh\left(\frac{\pi}{2}(\gamma-1)\omega\right)}{2\sinh\left(\frac{\pi}{2}\gamma\omega\right)\cosh\left(\frac{\pi}{2}\omega\right)}, \tag{7}$$

where $\gamma^{-1} \approx 1 + 2g/\pi$, which we take to be an integer. The quasiparticles scatter from the impurity with an S matrix of the form (3) with $\chi(\lambda) = \pi/2 - \arctan e^{(\lambda-\lambda_U)/\gamma}$ where we have introduced $\lambda_U \approx (1 + 1/\gamma)\log U$ which sets the impurity scale, $T_U \sim e^{\lambda_U}$, and $\alpha(\lambda)$ given in the Supplemental Material [85].

Initially the system is taken to be populated only to the left of the impurity according to $\vartheta_\pm(\lambda, x, 0) = \Theta(-x)\Theta(\Lambda - \lambda)$ with $\Theta(x)$ the Heaviside function and $E_F = e^\Lambda$ being the Fermi level for the quasiparticles. The corresponding rapidity distribution, $\rho_0(\lambda)$, can then be determined analytically through the Wiener-Hopf method [85]. From this initial condition the solution to (1) is given by

$$\begin{aligned} \rho_-(\lambda, x, t) &= \Theta(-x)[\Theta(-x-t) + \Theta(t+x)\mathcal{R}(\lambda)]\rho_0(\lambda) \\ \rho_+(\lambda, x, t) &= [\Theta(-x) + \Theta(x)\Theta(t-x)\mathcal{T}(\lambda)]\rho_0(\lambda) \end{aligned} \tag{8}$$

where we introduced $\mathcal{R}(\lambda) = |R([\chi(\lambda)]^{\mathrm{Dr}})|^2$ dressed reflection amplitude and $\mathcal{T}(\lambda) = 1 - \mathcal{R}(\lambda)$ the dressed transmission amplitude. A similar form holds also for the occupation functions with the replacement $\rho_0(\lambda) \to \Theta(\Lambda - \lambda)$ and $\mathcal{R}(\lambda) \to \mathcal{R}^\vartheta(\lambda)$ such that $[\mathcal{R}^\vartheta(\lambda)]^{\mathrm{dr}} = \mathcal{R}(\lambda)\Theta(\Lambda - \lambda)$ and $\mathcal{T}^\vartheta(\lambda) = 1 - \mathcal{R}^\vartheta(\lambda)$. The dressed phase shift χ^{Dr} can be determined analytically providing a full analytic solution to the problem [85]. From this one finds that for $\Lambda \ll \lambda_U$ the dressing has negligible effect, in essence the natural scale of the system as set by the impurity, T_U is much larger than the one set by the quench, E_F and system the remains close to equilibrium. In the opposite limit, however, the dressed phase shift can be approximated by

$$\chi^{\mathrm{Dr}}(\lambda) \simeq \frac{\pi}{2} + \int_{-\infty}^{\infty} \frac{d\omega}{2i\omega} \frac{\tanh\left(\frac{\pi}{2}\gamma\omega\right)\cosh\left(\frac{\pi}{2}\omega\right)}{\sinh\left[\frac{\pi}{2}(1+\gamma)\omega\right]} e^{-i\omega(\lambda-\lambda_U)}. \tag{9}$$

An important feature here is that for $\lambda_U \ll \lambda$ the phase shift is constant $\chi^{\mathrm{Dr}}(\lambda) \simeq (\pi/2)[(1-\gamma)/(1+\gamma)]$.

The current and charge density are straightforwardly obtained from (8). The former is nonzero only within the light cone surrounding the impurity and takes a Landauer-Buttiker form,

$$J(x, t) = \Theta(t - |x|) \int d\lambda \mathcal{T}(\lambda)\rho_0(\lambda). \tag{10}$$

Using our asymptotic result for the phase shift we see that in the far from equilibrium regime $J(x, t) \simeq \cos^2(\pi/2)[(1-\gamma)/(1+\gamma)]J_0(x, t)$ where $J_0(x, t)$ is the current at $U = 0$.

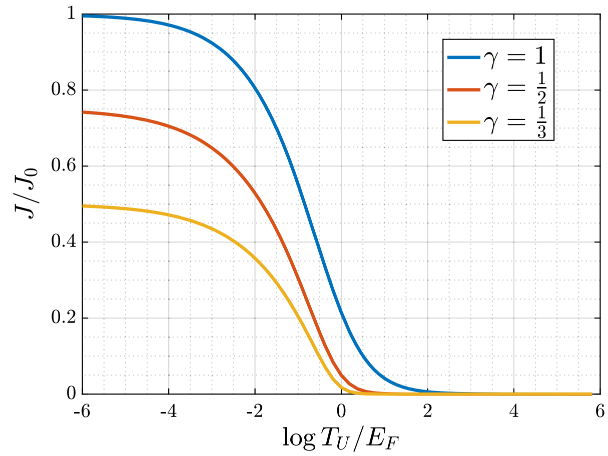

FIG. 1. The current within the light cone about $x = 0$, rescaled by its value without the impurity $J(x, t)/J_0(x, t)$ (10) as a function of $\log T_U/E_F$. The different curves correspond to interaction strength $\gamma = 1, \frac{1}{2}, \frac{1}{3}$. The asymptotic values for large E_F are given by $\cos^2(\pi/2)[(1-\gamma)/(1+\gamma)]$ as explained in the text.

In Fig. 1 we plot the current for different values γ as a function of T_U/E_F normalized by J_0. The density also only deviates from its initial value within the light cone. Within this region we have

$$N(x, t) = \Theta(\pm x) \int d\lambda [1 \mp \mathcal{R}(\lambda)]\rho_0(\lambda). \tag{11}$$

The density therefore exhibits a finite jump across the impurity, i.e., a potential difference of $2\int d\lambda \mathcal{R}(\lambda)\rho_0(\lambda)$. In addition, using the Friedel sum rule and $\alpha^{\mathrm{Dr}}(\lambda)$ we may compute the impurity induced charge deficit at $x = 0$ obtaining to leading order in E_F/T_U, $\delta N_{\mathrm{imp}} = -[2/(1+\gamma)]$ [85]. Thus despite being a theory of macroscopic length scales, we can use GHD to infer microscopic properties.

We may go beyond the expectation values of the current and density and calculate their fluctuations as well. For this we introduce the time integrated current $\mathcal{J}(t) = \int_0^t d\tau J(0^+, \tau)$ as well as its generating function

$$G(t, \beta) = \mathrm{Tr}[\varrho\, e^{\beta\mathcal{J}(t)}] \tag{12}$$

where ϱ is the density matrix of the system. Then upon using the continuity equation $\partial_t N(x, t) + \partial_x J(x, t) = 0$ this will give the fluctuations of the charge across the impurity. This quantity obeys a large deviation principle and has been studied recently in the context of both integrable and nonintegrable models [103–109]. Using recent results for the full counting statistics of integrable models [107–109] we find

$$\log G(t,\beta) = t\int \frac{d\lambda}{2\pi} e^{\lambda} \log\left[\mathcal{R}^{\vartheta}(\lambda) + \mathcal{T}^{\vartheta}(\lambda) e^{y(\lambda,\beta)}\right]$$
$$y(\lambda,\beta) = \beta - \int d\mu T(\lambda-\mu) \log\left[\mathcal{R}^{\vartheta}(\mu) + \mathcal{T}^{\vartheta}(\mu) e^{y(\mu,\beta)}\right]. \tag{13}$$

The function $y(\lambda, \beta)$ is related to the effective charge carried by the quasiparticles, indeed differentiating the second line above we have that $\partial_\beta y(\lambda, \beta) = q^{\mathrm{dr}}$ the dressed quasiparticle charge. These results can be contrasted with those obtained for the related boundary sine-Gordon model [78–80]. In these works the impurity scattering is also treated using a Boltzmann equation, however, the dressing of the scattering amplitudes does not appear. This difference is a result of the physical setup of the problem. Therein the current was studied directly in the nonequilibrium steady state which emerges from an adiabatic turning on of the impurity and the potential difference [110,111]. In that circumstance one can envisage starting in a dilute limit where bare quasiparticles scatter one by one off the impurity, without any dressing, and then slowly increasing the density. In the quench problem considered here this dilute limit cannot be used and we must instead turn to GHD which is a theory of dressed quasiparticles as opposed to bare ones. Nevertheless when $\Lambda \ll \lambda_U$ one obtains the expressions for the current of [78–80].

Lastly we examine the entanglement entropy between the two halves of the system, using $S_R = -\mathrm{Tr}\varrho_R(t) \log \varrho_R(t)$ where ϱ_R is the reduced density matrix of the right half of the system which has been studied previously in noninteracting systems [93,94,112,113]. We do so here by using the quasiparticle picture [114–117] which essentially counts the number of pairs of quasiparticles which are entangled and are shared between the left and right halves of the system. Since pairs are shared within the light cone this will be linear in time and by equating the entanglement entropy with the thermodynamic entropy we obtain

$$S_R = -t\int \frac{d\lambda}{2\pi} e^{\lambda}\left[\mathcal{R}^{\vartheta}(\lambda)\log \mathcal{R}^{\vartheta}(\lambda) + \mathcal{T}^{\vartheta}(\lambda)\log \mathcal{T}^{\vartheta}(\lambda)\right]. \tag{14}$$

This reduces to the known expression in the noninteracting limit [112] and vanishes when the impurity either purely transmits or reflects.

Discussion and conclusions.—In this Letter we have expanded the framework of GHD to include interacting quantum impurity models through the addition of an impurity collision integral which can be determined exactly from integrability. After performing some nontrivial analytic checks on this expression, we presented an exact solution of the impurity GHD equations for a bipartite quench with an interacting impurity. Using this we then derived several results on the current, entanglement entropy and full counting statistics of the model. In addition we also showed how the approach can be used to determine microscopic properties of the impurity like the charge deficit.

While ostensibly a theory of integrable dynamics, GHD facilitates the inclusion of certain mild integrability breaking terms such as external potentials [118], inhomogenous interactions [58], atom losses [119], or extended nonintegrable defects [120], through the use of collision integrals [121]. These naturally limit the applicability of the theory to be shorter than the quasiparticle lifetime, which however may still be quite large. A similar approach can be adopted here through the inclusion of near-integrable impurities with approximate reflection coefficients determined via Fermi's golden rule and compared to numerics [122,123]. Alternatively we can consider including multiple widely separated impurities and determine transport through a system with finite but small impurity concentration.

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