Testing Heisenberg-Type Measurement Uncertainty Relations of Three Observables

Ya-Li Mao,[1] Hu Chen,[1,2] Chang Niu,[3] Zheng-Da Li,[1,2,*] Sixia Yu,[3,†] and Jingyun Fan[1,2,4,‡]

[1]*Shenzhen Institute for Quantum Science and Engineering and Department of Physics, Southern University of Science and Technology, Shenzhen, 518055, China*

[2]*Guangdong Provincial Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen, 518055, China*

[3]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[4]*Center for Advanced Light Source, Southern University of Science and Technology, Shenzhen, 518055, China*

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Heisenberg-type measurement uncertainty relations (MURs) of two quantum observables are essential for contemporary research in quantum foundations and quantum information science. Going beyond, here we report the first experimental study of MUR of three quantum observables. We establish rigorously MURs for triplets of unbiased qubit observables as combined approximation errors lower bounded by an incompatibility measure, inspired by the proposal of Busch *et al.* [Phys. Rev. A **89**, 012129 (2014)]. We develop a convex programming protocol to numerically find the exact value of the incompatibility measure and the optimal measurements. We propose a novel implementation of the optimal joint measurements and present several experimental demonstrations with a single-photon qubit. We stress that our method is universally applicable to the study of many qubit observables. Besides, we theoretically show that MURs for joint measurement can be attained by sequential measurements in two of our explored cases. We anticipate that this work may stimulate broad interests associated with Heisenberg's uncertainty principle in the case of multiple observables, enriching our understanding of quantum mechanics and inspiring innovative applications in quantum information science.

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Introduction.—Heisenberg's uncertainty principle [1] is one of the most distinctive features in which quantum mechanics differs from classical theories. The extensive exploration of the uncertainty principle about a pair of quantum observables has revealed two types of uncertainty relations, namely, the preparation uncertainty relations (PURs, also known as the Heisenberg-Robertson uncertainty relation) [2,3] and the measurement uncertainty relations (MURs) [4–31]. While the PURs prohibit us from preparing quantum states with definite values for incompatible observables, the MURs capture the essence of measurement incompatibility [32,33], namely, quantum observables may be immeasurable in a single apparatus. These uncertainty relations deepen our understanding of quantum mechanics and crucially underlie quantum measurements and quantum information science [33–42]. Hence it is of high interest to explore uncertainty relations of multiple (≥ 3) quantum observables from both fundamental and practical perspectives. There has been significant progress in the study of PUR for multiple quantum observables [38,43–54]. In contrast, the study of MUR for multiple quantum observables has remained largely unexplored. A central problem is how to establish MUR for multiple observables and to find and experimentally realize the optimal measurement. Furthermore, it has been shown that joint measurement can always be implemented by sequential measurements for two observables [55], thus, the MUR for joint measurement applies also to the sequential scenario. However, the relation between joint measurement and sequential measurement schemes for multiple observables is another open problem.

Heisenberg discussed in his original gedanken experiment of microscopes [1], when a particle encounters both a position and a momentum measurement, the accuracy of an approximate position measurement is related to the disturbance of the particle's momentum measurement. This uncovers a deep nature of quantum mechanics: some quantum measurements disturb each other. Translating Heisenberg's intuition into the modern quantum formalism has led to the derivations of different measurement uncertainty relations by viewing Heisenberg's gedanken experiment at different angles, with each having its own merits [33,40]. As noted by Busch, Lahti, and Werner (BLW), Heisenberg's original description of the gedanken experiment of microscopes in a manner of sequential measurement is covered by the joint measurement scheme. By approximating two incompatible quantum observables (A, B) via the implementation of two compatible quantum observables (C, D) and adopting the measures of error and disturbance to quantify the differences between two probability distributions obtained in separate runs of measurements, BLW arrived at a state-independent uncertainty relation for position-momentum measurement in the gedanken experiment of microscope and its variant for a pair of qubit observables. The BLW uncertainty relation presents the characterization of the overall performance of measuring device and is of unrestricted applicabilitys [13,14]. On the other hand, it turns out that joint measurement of two observables can always be implemented sequentially [55].

In this Letter we report a significant advancement regarding the central problem by presenting the Heisenberg-type measurement uncertainty relation of three qubit observables with attainable lower bounds and its experimental demonstration. Besides, we also report a positive progress in the exploration of the other open problem. Specifically, we first establish the MUR for a triplet of unbiased qubit observables by approximating three incompatible quantum observables via the implementation of three compatible quantum measurements and provide a lower bound for the quantum incompatibility measure with exact condition of attainability, following the approach of BLW [14]. Second, we find the exact value of incompatibility measure and the corresponding optimal measurement via a convex programming protocol. Third, we design a novel implementation of optimal measurement, with which we showcase several experimental demonstrations saturating the MURs. We stress that this is the first experimental test of MUR of multiple observables with an attainable lower bound, which is directly relevant to the fundamental limit of quantum precision measurement [23,26]. Lastly, We discuss that the triplet MUR for joint measurement can also be attained by sequential measurements in two special cases explored in our experiments.

Triplet measurement uncertainty relation.—We consider three ideal qubit observables $\mathcal{M}=\{M_i=\vec{m}_i\cdot\vec{\sigma}\}_{i=1}^3$, where $\vec{\sigma}=\{\sigma_X,\sigma_Y,\sigma_Z\}$ are Pauli matrices and $\vec{m}_i$ are unit Bloch vectors. If the qubit system is prepared in the state $\rho_s=(1+\vec{r}_s\cdot\vec{\sigma})/2$, the distributions of the measurement outcomes are given by $P(\pm|M_i)=[(1\pm\vec{m}_i\cdot\vec{r}_s)/2]$. The most general measurement of a qubit observable with two outcomes is described by the positive-operator-valued measures (POVM) $\Omega=\{\Omega^\pm\}$ with $\Omega^\pm=\{[1\pm(x+\vec{\omega}\cdot\vec{\sigma})]/2\}$, which is normalized, $\Omega^++\Omega^-=1$, and nonnegative, $\Omega^\pm\geq 0$, as long as $|x|+|\vec{\omega}|\leq 1$, where $\vec{\omega}$ is the Bloch vector and $|x|$ stands for the biasedness. The triplet $\mathcal{M}$ are unbiased and are incompatible. A set of general qubit observables are compatible, or jointly measurable, if there exists a parent POVM $\mathcal{R}_p$ with multiple outcomes such that each observable in the given set arises as a marginal measurement or equivalently from a postmeasurement processing [56].

We approximate $\{M_i\}$ via the implementation of compatible POVMs $\{N_i\}$, respectively. It was shown in an earlier work that the necessary and sufficient conditions for three unbiased qubit observables to be compatible, i.e., with vanishing biasedness x, is given by [51]

$$\sum_{k=0}^{3}|\vec{p}_k-\vec{p}_f|\leq 4, \tag{1}$$

where $\vec{p}_k=\sum_{j=1}^3\gamma_{jk}\vec{m}_j$ with $\gamma_{jk}=(-1)^{k\lfloor j/2\rfloor+j\lfloor k/2\rfloor}$ and $\vec{p}_f$ is the Fermat-Torricelli (FT) point of $\{\vec{p}_k\}_{k=0}^3$, i.e., the vector that minimizes the left-hand side in the above inequality.

A measurement N_1 as an approximation to M_1 alters the state of the system so that a subsequent measurement N_2 approximates M_2 with limited accuracy if M_1 and M_2 are incompatible. Similarly, a subsequent measurement N_3 approximates M_3 with limited accuracy if M_2 and M_3 are incompatible. This scheme is a special case of joint measurement and the measure of disturbance is an instance of an approximation error [14]. Defining the combined approximation errors as $\Delta_\rho=\sum_{i=1}^3 d_\rho(M_i;N_i)$, where $d_\rho(M_i;N_i)\coloneqq 2\sum_\pm|P(\pm|M_i)-P(\pm|N_i)|$, we arrive at the state-independent measure of incompatibility

$$\Delta_{\mathcal{M}}\coloneqq\min_{\mathcal{N}}\max_{\rho}\Delta_\rho. \tag{2}$$

This worst-case estimate of the inaccuracy characterizes the overall performance of the measurement device.

An elegant lower bound of $\Delta_{\mathcal{M}}$ was proposed in [53] under the strong presumption that the optimal measurements are unbiased. Here considering the most general form of jointly measurable triplet, we strengthen this measurement uncertainty relation by proving that the optimal measurement is actually unbiased [57].

Theorem 1.—(Triplet measurement uncertainty relation) For the most general measurements $\{N_i\}$ that are jointly measurable, their errors of approximation relative to a triplet of ideal qubit observables $\mathcal{M}=\{\vec{m}_i\cdot\vec{\sigma}\}_{i=1}^3$ are tightly lower-bounded as follows:

$$\Delta_{\mathcal{M}}\geq\frac{1}{2}\sum_{k=0}^{3}|\vec{p}_k-\vec{p}_f|-2\coloneqq 2\delta, \tag{3}$$

where $\{\vec{p}_k=\sum_j\gamma_{jk}\vec{m}_j\}$ with $\vec{p}_f$ being its FT point. The lower bound is saturated if and only if $\delta\leq\min_k|\vec{p}_k-\vec{p}_f|$. If the condition is met, the optimal set of jointly measurable triplet reads $\vec{n}_j=\vec{m}_j+(\delta/4)\sum_{k=0}^3\gamma_{jk}[(\vec{p}_f-\vec{p}_k)/|\vec{p}_f-\vec{p}_k|]$ $(k=1,\,2,\,3)$.

As examples of attainability, the triplet with mutually orthogonal Bloch vectors, e.g., $\mathcal{M}_o=\{\vec{\sigma}\cos\gamma\}$, can attain the equality, i.e., the measurement uncertainty relation [Eq. (3)] is optimal for $\gamma\leq\arccos(1/\sqrt{3})\approx 54.74°$, for which $\mathcal{M}_o$ is incompatible. In general, the measurement uncertainty relation [Eq. (3)] cannot be attained, for example, coplanar triplet $\mathcal{M}_p$ with degenerate FT point [51], i.e.,

$\vec{p}_f$ coincides with some $\vec{p}_k$. In these cases, the exact value of incompatibility $\Delta_{\mathcal{M}}$ can be calculated via a convex programming.

Protocol 1.—(Convex programming) The exact value of incompatibility $\Delta_{\mathcal{M}}$ for a triplet $\{\vec{m}_j\}$ of ideal observables is given by the solution to the following convex optimization:

$$\min_{\mathcal{R}_p=\{R_j\}} 2 \max_{k\in\{0,1,2,3\}} \left|\sum_{j=1}^{3} \gamma_{jk}(\vec{m}_j - \vec{n}_j)\right|,$$
$$\text{subj to } R_j \geq 0, \quad (j = 1, 2, \ldots, 8), \quad \sum_j R_j = I, \tag{4}$$

with $\{\vec{n}_j\}$ being the Bloch vectors for three marginal measurements $\{N_i\}_{i=1}^{3}$ of $\mathcal{R}_p$.

Besides the exact value of incompatibility $\Delta_{\mathcal{M}}$, the convex programming critically yields the respective general measurement $\mathcal{R}_p$ and the optimal qubit state. Because the optimal measurement always lies on the boundary, i.e., saturating the joint measurement condition Eq. (1), we can accomplish the optimal joint measurement in a single-qubit experiment [15] as follows:

Theorem 2.—(Implementation) A jointly measurable triplet of unbiased qubit observables $\{N_i\}$ that saturates the joint measurement condition Eq. (1) can be implemented by the following parent measurement $\{R_{\mu_k|k} = P_k O_{\mu_k|k}\}$, where

$$P_k = \frac{|\vec{q}_k - \vec{q}_f|}{4}, \quad O_{\mu_k|k} = \frac{1}{2}\left(1 + \mu_k \frac{\vec{q}_k - \vec{q}_f}{|\vec{q}_k - \vec{q}_f|}\cdot\vec{\sigma}\right), \tag{5}$$

with binary outcomes labeled with $\mu_k = \pm 1$ for each $k = 0$, 1, 2, 3, and $\vec{q}_f$ is the FT point of $\{\vec{q}_k\}$. We obtain the marginal measurements, $N_{\mu|j} = \sum_{k,\mu_k} p_j(\mu|k,\mu_k) R_{\mu_k|k}$ with postmeasurement processing $p_j(\mu|k,\mu_k) = [(1+\mu\gamma_{jk}\mu_k)/2]$, where $j = 1$, 2, 3 and binary outcomes are labeled with $\mu = \pm 1$.

The above discussion essentially lays down an experimental protocol, following which we showcase below several experimental demonstrations of attaining the lower bound of the triplet measurement uncertainty relation.

Experiment.—The experimental schematics is depicted in Fig. 1. In order to prepare the single-photon qubit, we send a laser pulse with $\lambda_p = 779$ nm to a piece of periodically poled MgO doped lithium niobate (PPLN) crystal, where the induced type-0 spontaneous parametric down-conversion (SPDC) process creates a pair of photons at the phase-matched wavelengths of 1560 (signal) and 1556 nm (idler) [58]. We detect the idler photon to herald the presence of a signal photon. Encoding qubit to the polarization state of the signal photon, we use a pair of half- and quarter-wave plates (HWP, QWP) to prepare the optimal qubit state, $|\Phi\rangle_s = \cos\alpha|H\rangle_s + e^{i\phi}\sin\alpha|V\rangle_s$, as prescribed by Protocol 1, where $\alpha/2$ is the angle of the fast axis of a HWP oriented from the vertical, ϕ is the phase,

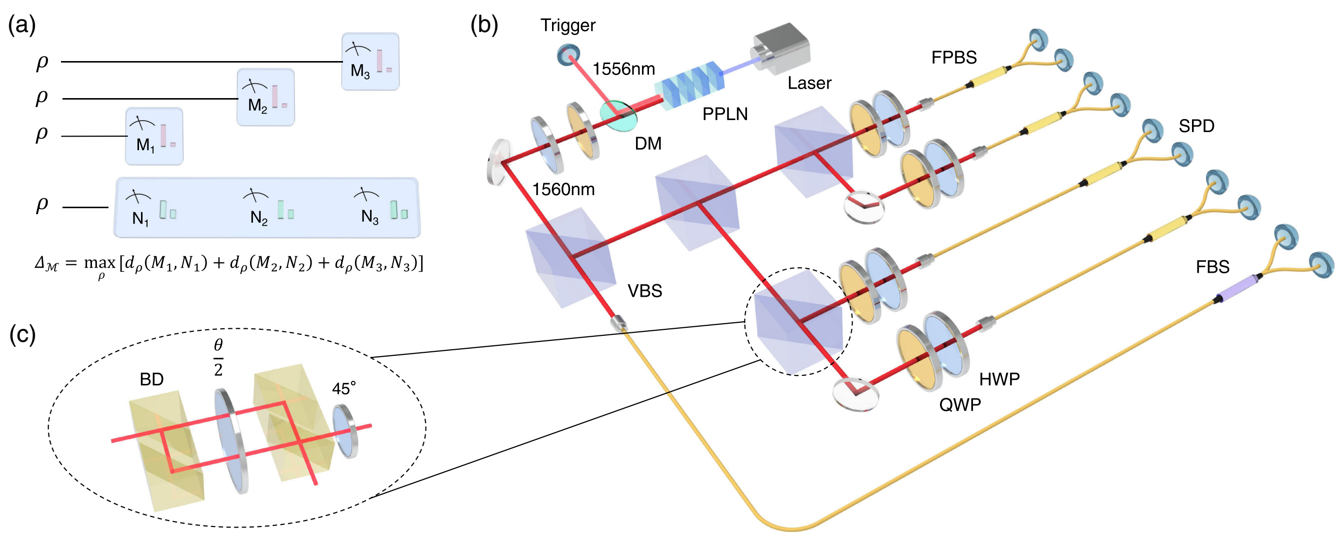

FIG. 1. Experimental optimal joint measurement on triplets of qubit observables. (a) An illustration to approximate measurements $\{M_i\}$ via the implementation of compatible measurement $\{N_i\}$, where the combined approximation errors $\{\Delta_\rho(M_i, N_i)\}$ are maximized over all input quantum states ρ_s and minimized over all triple jointly measurable observables. (b) A schematics of experimental optimal joint measurement on triplets of qubit observables. We generate a pair of correlated photons with $\lambda_{\text{signal}} = 1556$ nm and $\lambda_{\text{idler}} = 1560$ nm via the type-0 spontaneous parametric down-conversion process by pumping a periodically poled MgO doped lithium niobate (PPLN) crystal with a laser light at $\lambda_p = 779$ nm [58]. The detection of a signal photon heralds the presence of an idler photon. With qubit encoded to the polarization state of the idler photon, we install four polarization-projective detection modules to conduct measurements $\{M_i\}$ and $\{O_k\}$ with variable beam splitter (VBS) to adjust the weight, respectively (see Supplemental Material [59] for details). (c) A realization of VBS with adjustable beam-splitting ratio.

and $|H\rangle_s$ and $|V\rangle_s$ stand for horizontal and vertical polarization states, respectively.

Next, we conduct measurements $\{M_i\}$ and $\{R_{\mu_k|k} = P_k O_{\mu_k|k}\}$. We note that both measurements $\{M_i\}$ and $\{O_k\}$ are single-photon polarization-projective measurements in this study. We install four sets of single-photon polarization-projective measurement modules. In each module, we pass the single photon through a QWP, a HWP, a fiber-polarizing beam splitter (FPBS), and feed the outputs of FPBS to single-photon detectors. Employing variable beam splitters (VBS), we, respectively, implement measurements $\{M_i\}$ and measurements $\{O_k\}$ with weight $\{P_k\}$ to obtain measurement statistics, from which we derive the combined approximation errors (see Supplemental Material [59]).

We investigate the optimal joint measurement on triple ideal qubit observables for a few selected scenarios, (i) triplet $\mathcal{M}_o = \{\sigma_Z, \sigma_Y, \sigma_X\}\cos\gamma$ with Bloch vectors pairwise orthogonal; (ii) triplet $\mathcal{M}_\perp = \{\sigma_X\cos\gamma + \sigma_Y\sin\gamma, \sigma_X\cos\gamma + \sigma_Y\sin\gamma, \sigma_Z\}$ with one Bloch vector orthogonal to the plane spanned by the other two; (iii) co-planar triplet $\mathcal{M}_p = \{\sigma_X\cos\gamma + \sigma_Y\sin\gamma, \sigma_X\cos\gamma - \sigma_Y\sin\gamma, \sigma_X\}$; and (iv) triplet $\mathcal{M}_Y = \{[(-\sigma_X + \sqrt{3}\sigma_Y)/2], [(-\sigma_X - \sqrt{3}\sigma_Y)/2], \sigma_X\}\sin\gamma + \sigma_Z\cos\gamma$ with Bloch vectors being neither orthogonal and nor pairwise co-planar. The results are plotted, respectively, in Figs. 2(a)–2(d), with angle parameter $\gamma \in [0°, 90°]$. We draw the lower bounds on the right-hand side (rhs) of Eq. (3) with blue smooth lines, the attainable lower bounds obtained from the convex optimizing program (Protocol 1) with red dashed lines, and experimental results with open dots.

Some remarks are in order. First, comparing the red dashed lines and blue lines, it is evident that the measurement uncertainty relation of Eq. (3) is optimal, i.e., with attainable lower bound, for the entire parameter range of γ in scenario (i) and for parts of the parameter range in (ii) and (iv), and not optimal, i.e., with rhs smaller than that of the attainable lower bound found by Protocol 1, for the other parts of the parameter range in (ii) and (iv) and the entire parameter range in (iii). We note that the region of γ that attains the measurement uncertainty relation can be determined by Theorem 1. Second, experimental results are consistently in good agreement with numerical results obtained via Protocol 1 for all scenarios under study, i.e., we experimentally attain the exact value of incompatibility $\Delta_{\mathcal{M}}$. This justifies the strategy of accomplishing the optimal joint measurement with single qubit given in Theorem 2. We note that one can find analytically the incompatibility measures which coincide with the numerical results for triplets exhibiting certain symmetry [57].

Discussion.—It has been shown that for a pair of compatible observables that are employed to approximate the ideal measurements, joint measurement can be implemented by sequential measurements [55,60,61]. However, this equivalency for three or more observables is not known, i.e., it is not clear at all that whether optimal joint

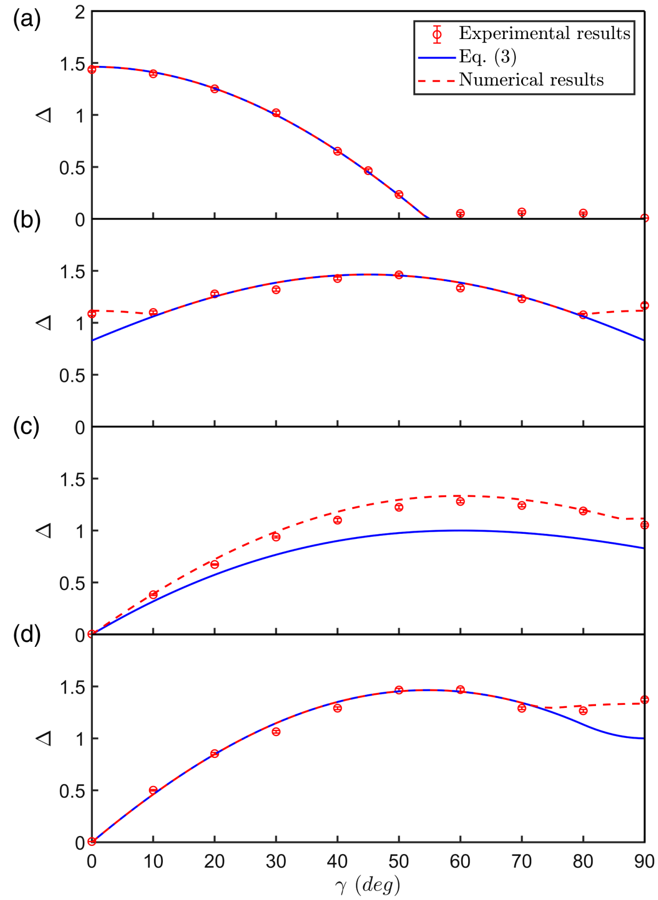

FIG. 2. Exact value of incompatibility of 4 triplets of idea qubit observables: (a) $\mathcal{M}_o$, (b) $\mathcal{M}_\perp$, (c) $\mathcal{M}_p$, (d) $\mathcal{M}_Y$. Red dashed lines are numerical results calculated with protocol 1, blue smooth curves are lower bound of incompatibility in Eq. (3), and red triangles represent experimental results. Error bars stand for 1 standard deviation.

measurable triplet can be measured sequentially. We shall show below that in two types of triplets considered here, namely, the orthogonal triplet $\mathcal{M}_o$ and the triplet $\mathcal{M}_\perp$ with one observable being orthogonal to the other two, the optimal joint measurements can be implemented sequentially.

Consider a sequential measurement $N_1 \to N_2' \to N_3'$ of three compatible observables $\{N_k\}$ as shown in Fig. 3, where N_2' and N_3' are two properly chosen measurements performed on the disturbed states. We perform measurement N_1 first and then measurement N_2' (different from N_2, in general), which is followed by another properly chosen unbiased measurement N_3'. The postmeasurement state at each stage of measurement reads

$$\rho \to \sum_\mu N_{1,\mu}^{\frac{1}{2}} \rho N_{1,\mu}^{\frac{1}{2}} \to \sum_{\mu,\nu} N_{2,\nu}^{\prime\frac{1}{2}} N_{1,\mu}^{\frac{1}{2}} \rho N_{1,\mu}^{\frac{1}{2}} N_{2,\nu}^{\prime\frac{1}{2}}.$$

In this way we actually have performed the following joint measurement:

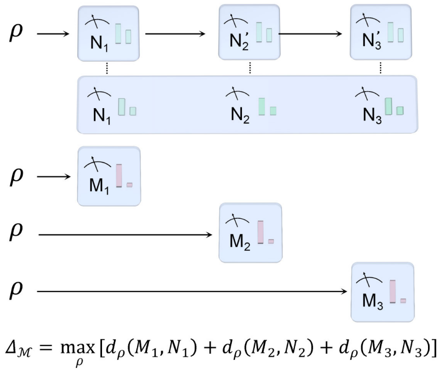

FIG. 3. Realization of compatible measurements $\{N_i\}$ by sequential measurements $N_1 \to N_2' \to N_3'$.

$$M_{\mu\nu\tau} = N_{1,\mu}^{\frac{1}{2}} N_{2,\nu}^{\prime\frac{1}{2}} N'_{3,\tau} N_{2,\nu}^{\prime\frac{1}{2}} N_{1,\mu}^{\frac{1}{2}} \tag{6}$$

as long as the marginal conditions are satisfied. For an example we consider orthogonal triplet $\mathcal{N}_o = \{\vec{n}_j \cdot \vec{\sigma}\}$ with mutually orthogonal measurement directions, i.e., $\vec{n}_j \cdot \vec{n}_k = 0$ for $j \neq k$. To implement a sequential measurement of form Eq. (6) we measure at first observable N_1 with Bloch vector $\vec{n}_1$ and then we choose the two sequential measurements to be unbiased and have the following Bloch vectors:

$$\vec{n}_2' = \frac{\vec{n}_2}{\sqrt{1 - \vec{n}_1^2}}, \qquad \vec{n}_3' = \frac{\vec{n}_3}{\sqrt{1 - \vec{n}_1^2 - \vec{n}_2^2}}.$$

These Bloch vectors correspond to legit qubit measurements because of the joint measurement condition $\sum_k \vec{n}_k^2 \leq 1$. In a more general case of triplet $\mathcal{N}_\perp$ with one observable orthogonal to the other two, the joint measurement can also be implemented sequentially (see Supplemental Material [59]). Thus the triplets $\mathcal{M}_o$ and $\mathcal{M}_\perp$, from the discussions above, can be optimally measured in a sequential manner. Notably, the disturbance induced by the first and second measurement is contained in the errors for the second and third observables from the sequential measurement point of view. The equivalency between the joint measurement and sequential measurement for a general triplet still awaits for a future investigation.

Summary.—Some quantum measurements disturb each other, preventing us from measuring them with a single measurement device without introducing errors. The MUR sets the limit to how well we can perform the joint measurement with the minimal amount of errors according to quantum mechanics. In this Letter, we establish the MUR for triplets of qubit observables. Employing the convex programming, we find the exact value of the incompatibility measure and the optimal measurements to saturate the MUR. This guides us to accomplish the optimal joint measurements on triplets of qubit observables for the first time. The demonstrated strategy is universally applicable to the study of many qubit observables and the case of weighted measurements, which will be considered elsewhere. As a critical step in the study of multiple incompatible quantum measurements, this work may deepen our understanding of Heisenberg's uncertainty principle and lead to innovative applications in quantum metrology and quantum information science.

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*lizd@sustech.edu.cn
†yusixia@ustc.edu.cn
‡fanjy@sustech.edu.cn

[1] W. Heisenberg, Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik, Z. Phys. **43**, 172 (1927).

[2] E. H. Kennard, Zur quantenmechanik einfacher bewegungstypen, Z. Phys. **44**, 326 (1927).

[3] H. P. Robertson, The uncertainty principle, Phys. Rev. **34**, 163 (1929).

[4] M. Ozawa, Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement, Phys. Rev. A **67**, 042105 (2003).

[5] M. Ozawa, Uncertainty relations for noise and disturbance in generalized quantum measurements, Ann. Phys. (Amsterdam) **311**, 350 (2004).

[6] M. Ozawa, Uncertainty relations for joint measurements of noncommuting observables, Phys. Lett. A **320**, 367 (2004).

[7] R. F. Werner, The uncertainty relation for joint measurement of position and momentum, arXiv:quant-ph/0405184.

[8] M. J. W. Hall, Prior information: How to circumvent the standard joint-measurement uncertainty relation, Phys. Rev. A **69**, 052113 (2004).

[9] A. P. Lund and H. M. Wiseman, Measuring measurement–disturbance relationships with weak values, New J. Phys. **12**, 093011 (2010).

[10] K. Fujikawa, Universally valid Heisenberg uncertainty relation, Phys. Rev. A **85**, 062117 (2012).

[11] C. Branciard, Error-tradeoff and error-disturbance relations for incompatible quantum measurements, Proc. Natl. Acad. Sci. U.S.A. **110**, 6742 (2013).

[12] A. Di Lorenzo, Sequential measurement of conjugate variables as an alternative quantum state tomography, Phys. Rev. Lett. **110**, 010404 (2013).

[13] P. Busch, P. Lahti, and R. F. Werner, Proof of Heisenberg's error-disturbance relation, Phys. Rev. Lett. **111**, 160405 (2013).

[14] P. Busch, P. Lahti, and R. F. Werner, Heisenberg uncertainty for qubit measurements, Phys. Rev. A **89**, 012129 (2014).

[15] S. Yu and C. Oh, Optimal joint measurement of two observables of a qubit, arXiv:1402.3785.

[16] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements, Nat. Phys. **8**, 185 (2012).

[17] L. A. Rozema, A. Darabi, D. H. Mahler, A. Hayat, Y. Soudagar, and A. M. Steinberg, Violation of Heisenberg's measurement-disturbance relationship by weak measurements, Phys. Rev. Lett. **109**, 100404 (2012).

[18] G. Sulyok, S. Sponar, J. Erhart, G. Badurek, M. Ozawa, and Y. Hasegawa, Violation of Heisenberg's error-disturbance uncertainty relation in neutron-spin measurements, Phys. Rev. A **88**, 022110 (2013).

[19] S.-Y. Baek, F. Kaneda, M. Ozawa, and K. Edamatsu, Experimental violation and reformulation of the Heisenberg's error-disturbance uncertainty relation, Sci. Rep. **3**, 2221 (2013).

[20] M. M. Weston, M. J. W. Hall, M. S. Palsson, H. M. Wiseman, and G. J. Pryde, Experimental test of universal complementarity relations, Phys. Rev. Lett. **110**, 220402 (2013).

[21] M. Ringbauer, D. N. Biggerstaff, M. A. Broome, A. Fedrizzi, C. Branciard, and A. G. White, Experimental joint quantum measurements with minimum uncertainty, Phys. Rev. Lett. **112**, 020401 (2014).

[22] F. Kaneda, S.-Y. Baek, M. Ozawa, and K. Edamatsu, Experimental test of error-disturbance uncertainty relations by weak measurement, Phys. Rev. Lett. **112**, 020402 (2014).

[23] P. Busch and N. Stevens, Direct tests of measurement uncertainty relations: What it takes, Phys. Rev. Lett. **114**, 070402 (2015).

[24] W. Ma, Z. Ma, H. Wang, Z. Chen, Y. Liu, F. Kong, Z. Li, X. Peng, M. Shi, F. Shi, S.-M. Fei, and J. Du, Experimental test of Heisenberg's measurement uncertainty relation based on statistical distances, Phys. Rev. Lett. **116**, 160405 (2016).

[25] F. Zhou, L. Yan, S. Gong, Z. Ma, J. He, T. Xiong, L. Chen, W. Yang, M. Feng, and V. Vedral, Verifying Heisenberg's error-disturbance relation using a single trapped ion, Sci. Adv. **2**, e1600578 (2016).

[26] T. P. Xiong, L. L. Yan, Z. H. Ma, F. Zhou, L. Chen, W. L. Yang, M. Feng, and P. Busch, Optimal joint measurements of complementary observables by a single trapped ion, New J. Phys. **19**, 063032 (2017).

[27] A. Barchielli, M. Gregoratti, and A. Toigo, Measurement uncertainty relations for discrete observables: Relative entropy formulation, Mathematics **357**, 1253 (2018).

[28] T. Bullock and P. Busch, Measurement uncertainty relations: Characterising optimal error bounds for qubits, J. Phys. A **51**, 283001 (2018).

[29] Y.-L. Mao, Z.-H. Ma, R.-B. Jin, Q.-C. Sun, S.-M. Fei, Q. Zhang, J. Fan, and J.-W. Pan, Error-disturbance trade-off in sequential quantum measurements, Phys. Rev. Lett. **122**, 090404 (2019).

[30] N. Yunger Halpern, A. Bartolotta, and J. Pollack, Entropic uncertainty relations for quantum information scrambling, Commun. Phys. **2**, 92 (2019).

[31] J. T. Monroe, N. Yunger Halpern, T. Lee, and K. W. Murch, Weak Measurement of a Superconducting Qubit Reconciles Incompatible Operators, Phys. Rev. Lett. **126**, 100403 (2021).

[32] T. Heinosaari, T. Miyadera, and M. Ziman, An invitation to quantum incompatibility, J. Phys. A **49**, 123001 (2016).

[33] O. Gühne, E. Haapasalo, T. Kraft, J.-P. Pellonpää, and R. Uola, Colloquium: Incompatible measurements in quantum information science, Rev. Mod. Phys. **95**, 011003 (2023).

[34] P. Busch and C. Shilladay, Complementarity and uncertainty in Mach-Zehnder interferometry and beyond, Phys. Rep. **435**, 1 (2006).

[35] P. Busch, T. Heinonen, and P. Lahti, Heisenberg's uncertainty principle, Phys. Rep. **452**, 155 (2007).

[36] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, Introduction to quantum noise, measurement, and amplification, Rev. Mod. Phys. **82**, 1155 (2010).

[37] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).

[38] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Specker's parable of the overprotective seer: A road to contextuality, nonlocality and complementarity, Phys. Rep. **506**, 1 (2011).

[39] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, The classical-quantum boundary for correlations: Discord and related measures, Rev. Mod. Phys. **84**, 1655 (2012).

[40] P. Busch, P. Lahti, and R. F. Werner, Colloquium: Quantum root-mean-square error and measurement uncertainty relations, Rev. Mod. Phys. **86**, 1261 (2014).

[41] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).

[42] C. Portmann and R. Renner, Security in quantum cryptography, Rev. Mod. Phys. **94**, 025008 (2022).

[43] H. P. Robertson, An indeterminacy relation for several observables and its classical interpretation, Phys. Rev. **46**, 794 (1934).

[44] S. Wehner and A. Winter, Higher entropic uncertainty relations for anti-commuting observables, J. Math. Phys. (N.Y.) **49**, 062105 (2008).

[45] Y. Huang, Variance-based uncertainty relations, Phys. Rev. A **86**, 024101 (2012).

[46] A. A. Abbott, P.-L. Alzieu, M. J. W. Hall, and C. Branciard, Tight state-independent uncertainty relations for qubits, Mathematics **4**, 8 (2016).

[47] L. Maccone and A. K. Pati, Stronger uncertainty relations for all incompatible observables, Phys. Rev. Lett. **113**, 260401 (2014).

[48] S. Kechrimparis and S. Weigert, Heisenberg uncertainty relation for three canonical observables, Phys. Rev. A **90**, 062118 (2014).

[49] W. Ma, B. Chen, Y. Liu, M. Wang, X. Ye, F. Kong, F. Shi, S.-M. Fei, and J. Du, Experimental demonstration of uncertainty Relations for the triple components of angular momentum, Phys. Rev. Lett. **118**, 180402 (2017).

[50] B. Fan, K. Wang, L. Xiao, and P. Xue, Experimental test of a stronger multiobservable uncertainty relation, Phys. Rev. A **98**, 032118 (2018).

[51] S. Yu and C. Oh, Quantum contextuality and joint measurement of three observables of a qubit, arXiv:1312.6470.

[52] R. Schwonnek, D. Reeb, and R. F. Werner, Measurement uncertainty for finite quantum observables, Mathematics **4**, 38 (2016).

[53] H.-H. Qin, T.-G. Zhang, L. Jost, C.-P. Sun, X. Li-Jost, and S.-M. Fei, Uncertainties of genuinely incompatible triple

measurements based on statistical distance, Phys. Rev. A **99**, 032107 (2019).

[54] H. Qin and S.-M. Fei, Optimizing incompatible triple quantum measurements, Eur. Phys. J. Plus **137**, 635 (2022).

[55] T. Heinosaari and T. Miyadera, Universality of sequential quantum measurements, Phys. Rev. A **91**, 022110 (2015).

[56] S. T. Ali, C. Carmeli, T. Heinosaari, and A. Toigo, Commutative POVMs and fuzzy observables, Found. Phys. **39**, 593 (2009).

[57] S. Yu, Y.-L. Mao, C. Niu, H. Chen, Z.-D. Li, and J. Fan, companion article, Measurement uncertainty relation for three observables, Phys. Rev. A **108**, 042208 (2023).

[58] Z.-D. Li, Y.-L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S.-J. Yang, M.-O. Renou, D. Trillo, T. P. Le, N. Gisin, A. Acín, M. Navascués, Z. Wang, and J. Fan, Testing real quantum theory in an optical quantum network, Phys. Rev. Lett. **128**, 040402 (2022).

[59] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.150203 for details about the experiment and theory in the main text.

[60] E. Haapasalo and J.-P. Pellonpä, Optimal quantum observables, J. Math. Phys. (N.Y.) **58**, 122104 (2017).

[61] R. Uola, E. Haapasalo, J.-P. Pellonpää, and T. Kuusela, Retrievability of information in quantum and realistic hidden variable theories, arXiv:2212.02815.