Emergence of Time from Quantum Interaction with the Environment

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The nature of time as emergent for a system by separating it from its environment has been put forward by Page and Wootters [Phys. Rev. D **27**, 2885 (1983)] in a quantum mechanical setting neglecting interaction between system and environment. Here, we add strong support to the relational concept of time by deriving the time-dependent Schrödinger equation for a system from an energy eigenstate of the global Hamiltonian consisting of system, environment, *and* their interaction. Our results are consistent with concepts for the emergence of time where interaction has been taken into account at the expense of a semiclassical treatment of the environment. Including the coupling between system and environment without approximation adds a missing link to the relational time approach opening it to dynamical phenomena of interacting systems and entangled quantum states.

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The nature and role of time to decipher the physical world is a basic and persisting research topic, in particular the question of whether time is fundamental or emergent. For the latter, the starting point is a static description of the world. Time emerges from singling out a system from the rest of the world, its environment. As such, time is a meaningful tool to describe the relation of system and environment, both governed by Hamiltonians distinguished in physical or abstract (Hilbert) space. This has lead to two strands of research for the relational approach to time. One strand, initiated by Page and Wootters [1–5] deals with abstract state vectors in Hilbert space and is analytically exact, but remains to date unable to deal with general couplings of system and environment. The second strand uses a semiclassical approach typically in position space, arguing that the environment is "large enough" to allow for semiclassical approximations [6–14]. By these means, time also emerges as the relation between system and environment which may be arbitrarily coupled.

Here, we will show how time emerges quantum mechanically in the relation between system and environment without approximations, more specifically, by retaining arbitrary couplings between them and without the need to resort to semiclassical approximations. That is, starting from a static global state encompassing system and environment we derive the time-dependent Schrödinger equation including an arbitrary, time-dependent potential for the system in a few transparent steps. To this end, we will reformulate the stationary (timeless) Schrödinger equation for the global state as an *invariance principle* and single out a pure state of the system from its inevitable embedding in the environment by projecting a specific state of the environment onto the global state. As a byproduct our approach constitutes a concept for analytical solutions of complicated time-dependent interaction potentials [15].

The invariance principle for the global state $|\Psi\rangle\rangle$ as an eigenstate of the Hamiltonian $\hat{H}$ with global eigenenergy E reads

$$\exp\left[i\lambda(\hat{H}-E)\right]|\Psi\rangle\rangle = |\Psi\rangle\rangle \tag{1}$$

for all complex λ with dimension of inverse energy, where $\langle\langle .|.\rangle\rangle$ stands for the scalar product in the global Hilbert space. Differentiating (1) with respect to λ gives the (timeless) Schrödinger equation $(\hat{H}-E)|\Psi\rangle\rangle = 0$, often referred to as TISE. In the following, we will only consider real-valued λ in (1) which is sufficient to demonstrate the emergence of time. Purely imaginary λ finds its natural application in the emergence of temperature [16]. In order to single out a system state from the global state, we first partition the global Hamiltonian $\hat{H}$ into that of the system $\hat{H}_\mathrm{S}$, its environment $\hat{H}_\mathrm{C}$, and their possible interaction $\hat{V}$,

$$\hat{H} = \hat{H}_\mathrm{S}\otimes\hat{\mathbb{1}}_\mathrm{C} + \hat{\mathbb{1}}_\mathrm{S}\otimes\hat{H}_\mathrm{C} + \hat{V}. \tag{2}$$

We will use environment and clock as synonyms to relate to the aforementioned two strands of research on the emergence of time. While the partition (2) of the global Hamiltonian is natural to define a system in the first place, it is not obvious how to single out a system state from the global, *entangled* state $|\Psi\rangle\rangle$. From a quantum mechanical

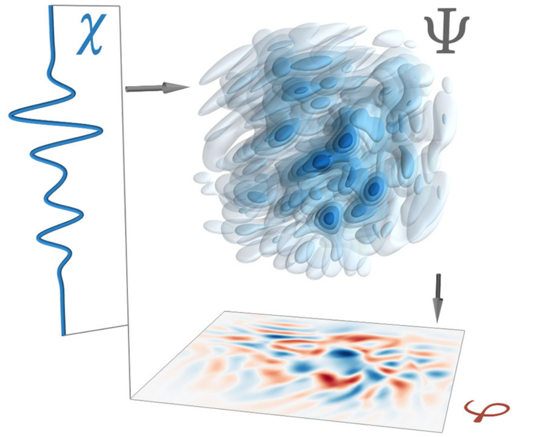

FIG. 1. Sketch of the relational state formalism. A one-dimensional environment state $\chi(x)$ projects out a two-dimensional system state $\varphi(y,z) \propto \int dx \chi^*(x)\Psi(x,y,z)$ from the three-dimensional global state $\Psi(x,y,z)$. Schematically, the clock wave function is multiplied to each vertical column of Ψ and subsequently integrated along this direction to yield each value of φ. With such an inherent clock dependence, the system state generally differs for different clock states.

point of view, the system is inevitably embedded in its environment on which it is therefore conditioned. Hence, a system state $|\varphi\rangle_{\rm S}$ is created by projecting the global state onto a state of the environment, $|\varphi\rangle_{\rm S} = \langle\chi|\Psi\rangle\!\rangle$ [17]. Here and in the following we use the convention that $\langle .|.\rangle$ and $\langle\!\langle .|.\rangle\!\rangle$ denote scalar products in the environment and full Hilbert space, respectively, while $|\varphi\rangle_{\rm S}$ and $|\chi\rangle$ stand for states of system and environment, respectively and $|\Psi\rangle\!\rangle$ is reserved for the global state. A sketch of this relational approach is shown in Fig. 1.

Singling out the system by projection reduces the correlations and in particular breaks the global symmetry such that the system state does not obey the global invariance principle. Rather, the state becomes dependent on the symmetry parameter λ. This can be seen by projecting the invariance equation (1) onto $\langle\chi_0|$, which gives for the interaction free case, $V=0$,

$$\langle\chi_0|e^{i\lambda(\hat{H}_{\rm C}-E)}|\Psi\rangle\!\rangle = e^{-i\lambda\hat{H}_{\rm S}}\langle\chi_0|\Psi\rangle\!\rangle, \tag{3}$$

where we may write

$$|\chi_\lambda\rangle = e^{-i\lambda(\hat{H}_{\rm C}-E)}|\chi_0\rangle \equiv \hat{U}_{\rm C}(\lambda)|\chi_0\rangle. \tag{4}$$

The states $|\chi_\lambda\rangle$ from the environment serve as markers to tag the system state with λ,

$$|\varphi(\lambda)\rangle_{\rm S} \equiv \langle\chi_\lambda|\Psi\rangle\!\rangle. \tag{5}$$

Consistent with $|\varphi(0)\rangle_{\rm S} = \chi_0|\Psi\rangle$, we arrive at

$$|\varphi(\lambda)\rangle_{\rm S} = e^{-i\lambda\hat{H}_{\rm S}}|\varphi(0)\rangle_{\rm S} \equiv \hat{U}_{\rm S}(\lambda)|\varphi(0)\rangle_{\rm S} \tag{6}$$

for all symmetry parameters λ. Hence, we have *derived* from the global invariance (1) without reference to any differential equations how states of the system (6) and the environment (4) evolve. This implies a peculiar consequence on the fundamental level: states with different λ do not have to be related, admitting also discrete symmetries with λ replaced by a set of parameters $\{\lambda_n\}$.

Using the property $\hat{U}^\dagger(\lambda) = \hat{U}(-\lambda)$ of the unitary transformations in (4) and (6) we can rewrite the projected invariance equation (3) as

$$\begin{aligned}\langle\chi_0|\Psi\rangle\!\rangle &= \hat{U}_{\rm S}(-\lambda)\langle\chi_0|\hat{U}_{\rm C}(-\lambda)|\Psi\rangle\!\rangle \\ &= \hat{U}_{\rm S}(-\lambda)\langle\hat{U}_{\rm C}(\lambda)\chi_0|\Psi\rangle\!\rangle,\end{aligned} \tag{7}$$

which has the same form as the invariance for more familiar symmetry transformations, e.g., the invariance of a state $|\psi\rangle$ in coordinate space $\langle\mathbf{r}|\psi\rangle$ if it is rotated by an angle θ about a vector $\mathbf{u}$ with the unitary operator $\hat{D}(\theta) = e^{-i\theta\mathbf{u}\cdot\hat{\mathbf{J}}/\hbar}$ while the coordinate system is rotated backward with the rotation matrix $R(\theta)$: $\hat{D}(\theta)\langle R(-\theta)\mathbf{r}|\psi\rangle = \langle\mathbf{r}|\psi\rangle$. This opposite behavior of states of the system and environment as a consequence of the global invariance was dubbed by Zurek "envariance" and used to motivate, why probabilities correspond to measurements, colloquially known as the Born rule [18].

In our context of letting time emerge by projection of a globally static state, we may conclude that for the projected global invariance (3) the state $|\chi\rangle$ from the environment plays the role of a coordinate which is transformed with $\hat{U}_{\rm C}(\lambda)$ to compensate the transformation of the system state $|\varphi\rangle_{\rm S}$ with $\hat{U}_{\rm S}(-\lambda)$.

Since λ in (1) is a continuous symmetry, (6) can be interpreted as the solution of the differential equation

$$i\frac{d}{d\lambda}|\varphi(\lambda)\rangle_{\rm S} = \hat{H}_{\rm S}|\varphi(\lambda)\rangle_{\rm S} \tag{8}$$

with initial condition $|\varphi(0)\rangle_{\rm S} = \langle\chi_0|\Psi\rangle\!\rangle$. Obviously, (8) is equivalent to the TDSE if time t is introduced through $\lambda = t/\hbar$. What we have described so far is a shortcut derivation of the Page-Wootters relational time approach [1] made possible by recognizing the crucial role of the invariance principle (1).

Strictly speaking, λ is only a label without physical meaning: any reparametrization $\lambda = f(\tilde{\lambda})$ leaves the relations between environment and system invariant. However, one can tag the system's evolution with a reparametrization invariant observable of the environment, $\mathsf{A}_{\rm C}(\lambda) \equiv \langle\chi_\lambda|\hat{A}_{\rm C}|\chi_\lambda\rangle$: $\mathcal{H}_{\rm C} \mapsto \mathbb{R}$. Although $\hat{A}_{\rm C}$ operating on the environment is arbitrary apart from being Hermitian, a good choice is one for which the relation between λ and $\mathsf{A}_{\rm C}$ is simple, for example linear, if the environment is used as a clock. This idea goes back to Poincaré [19]. For instance, the mean position $\mathsf{R}(\lambda) = \lambda\mathsf{P}(0)/M + \mathsf{R}(0)$ of a free particle of mass M with $\hat{H}_C = \hat{P}^2/2M$ can reliably track dynamics for nonvanishing mean momentum $\mathsf{P}(0) \neq 0$ since we can replace $\lambda = M[\mathsf{R}(\lambda) - \mathsf{R}(0)]/\mathsf{P}(0)$ which represents a physical property of the environment, respectively clock. For a state $|\chi_\lambda\rangle$ to clock the system, it must first of all have overlap with the global state (see Fig. 1). To provide a high resolution in λ, the clock state $|\chi_\lambda\rangle \propto \sum_k a_k e^{-i\lambda E_{C,k}}|E_{C,k}\rangle$ must be distributed over many eigenstates $|E_{C,k}\rangle$ of $\hat{H}_C$, with ideally $|a_k| \approx \text{const}$ [3,4,20]. This is easy to realize, if the (physical) dimension of the clock is much larger than that of the system, which also has the effect that the global state can accommodate more complex system dynamics.

We also reemphasize that the entanglement in $|\Psi\rangle\rangle$ with respect to the states of system and environment is crucial for nontrivial system dynamics and requires without interaction $\hat{V}$ the existence of degenerate eigenspaces of the global Hamiltonian. Otherwise, system and environment fulfill separately a "global" invariance principle with λ_S and λ_C, respectively, which leaves the relation $\lambda_S(\lambda_C)$ undetermined.

Finally, it is remarkable that despite the global invariance having been broken by an arbitrary but specific choice of $|\chi_0\rangle$, the properties of the latter do not influence the evolution of the system state other than specifying its initial condition. Hence, the standard procedure of getting rid of properties of the environment to achieve a universal system evolution, namely tracing over the environment, is not necessary. While it is contained in the present description (we could use any kind of mixed state for $|\chi_0\rangle$), choosing a rather structureless $|\chi_0\rangle$ is not suitable for serving the purpose of a clock as just discussed.

So far we have provided a clarification and shortcut to the TDSE for a system not interacting with its environment, enabled by recognizing the power of the invariance principle (1) which was not invoked in [1]. We have detailed our approach since we need it in the following to derive the TDSE for a system interacting with the environment.

In reality, the environment will inevitably interact with the system. This automatically ensures that the global state $|\Psi\rangle\rangle$ is generically entangled. Hence, we should derive the TDSE for the system with interaction $\hat{V} \neq 0$. To this end, we use $|\chi(\lambda)\rangle = e^{-iS(\lambda)}|\chi_\lambda\rangle$ with $|\chi_\lambda\rangle$ from (4) and the complex scalar $S(\lambda) = \int^\lambda d\lambda' \mathcal{E}(\lambda')$, which can be viewed as a λ-dependent phase and normalization. Projected onto this state, the global TISE can be written as

$$\left(-\hat{H}_S + \mathcal{E}(\lambda) + i\frac{d}{d\lambda}\right)\langle\chi(\lambda)|\Psi\rangle\rangle = \langle\chi(\lambda)|\hat{V}|\Psi\rangle\rangle. \quad (9)$$

As a next step we decompose $\langle\chi(\lambda)|\hat{V}|\Psi\rangle\rangle$ into a Hermitian potential $\hat{V}_S(\lambda)$ for the system and a c number which is an expectation value over the global state. The decomposition is facilitated with the operators $\hat{P}_\Psi \equiv |\Psi\rangle\rangle\langle\langle\Psi|$, $\hat{P}_\chi \equiv \hat{1}_S \otimes |\chi(\lambda)\rangle\langle\chi(\lambda)|$ and $\hat{P}_{\Psi_\chi} = \hat{P}_\Psi \hat{P}_\chi / N_\lambda$, where $\hat{P}_{\Psi_\chi}|\Psi\rangle\rangle = |\Psi\rangle\rangle$ since $N_\lambda = \langle\langle\Psi|\hat{P}_\chi|\Psi\rangle\rangle$. We obtain

$$\begin{aligned}\langle\chi|\hat{V}|\Psi\rangle\rangle &= \langle\chi|\hat{V}\hat{P}_{\Psi_\chi}|\Psi\rangle\rangle \\ &= \left[\hat{V}_S(\lambda) - \langle\langle\Psi|\hat{V}\hat{P}_\chi|\Psi\rangle\rangle/N_\lambda\right]\langle\chi(\lambda)|\Psi\rangle\rangle, \quad (10a)\end{aligned}$$

where

$$\hat{V}_S(\lambda) = \frac{\langle\chi|\left(\hat{V}\hat{P}_\Psi + \hat{P}_\Psi\hat{V}\right)|\chi\rangle}{\langle\langle\Psi|\hat{P}_\chi|\Psi\rangle\rangle}. \quad (10b)$$

Inserting (10) into (9), setting $\mathcal{E}(\lambda) \equiv \langle\langle\Psi|\hat{V}\hat{P}_\chi|\Psi\rangle\rangle/N_\lambda$, and rearranging terms gives the TDSE for the system with interaction,

$$\left[\hat{H}_S + \hat{V}_S(\lambda)\right]|\varphi(\lambda)\rangle_S = i\frac{d}{d\lambda}|\varphi(\lambda)\rangle_S. \quad (11)$$

The effective system potential $\hat{V}_S$ from (10b) depends explicitly on λ and implicitly on the state of the environment, $|\chi(\lambda)\rangle = e^{-i\lambda(\hat{H}_C - E) - iS(\lambda)}|\chi_0\rangle$. One can easily retrieve the original TISE $(\hat{H} - E)|\Psi\rangle\rangle = 0$ by inserting the explicit expression for $|\varphi(\lambda)\rangle_S = \langle\chi(\lambda)|\Psi\rangle\rangle$ into (11), performing the differentiation with respect to λ followed by a functional derivative $\delta/(\delta\langle\chi|)$ with respect to the state of the environment.

Equation (11) is the main result of this work and represents, to the best of our knowledge, the first derivation of the time-dependent Schrödinger equation with a *fully general*, Hermitian time-dependent potential $\hat{V}_S$ from a static global state. A pictorial representation of our formalism is shown in Fig. 2.

To stay as general as possible, we have made no further assumptions regarding the interaction potential $\hat{V}$. Of course, it is reasonable (although we have seen not necessary) to assume that the interaction potential has negligible influence on the state $|\chi\rangle$ of the environment. Formally, this can be expressed by $[\hat{V}, \hat{P}_\chi] \approx 0$. Thereby, $|\chi\rangle$ becomes approximately an eigenstate of the interaction $\hat{V}$, turning $|\chi\rangle$ essentially into what has been described as a "pointer state" by Zurek [21]. Then we can write

$$\begin{aligned}\langle\chi|\hat{V}|\Psi\rangle\rangle &= \langle\chi|\hat{P}_\chi\hat{V}|\Psi\rangle\rangle/\langle\chi|\chi\rangle = \langle\chi|\hat{V}\hat{P}_\chi|\Psi\rangle\rangle/\langle\chi|\chi\rangle \\ &= \frac{\langle\chi|\hat{V}|\chi\rangle}{\langle\chi|\chi\rangle}\langle\chi|\Psi\rangle\rangle = \frac{\langle\chi|\hat{V}|\chi\rangle}{\langle\chi|\chi\rangle}|\varphi\rangle_S. \quad (12)\end{aligned}$$

The global state $|\Psi\rangle\rangle$ no longer appears and renders the calculation of $\hat{V}_S$ less involved. Moreover, $\text{Im}[\mathcal{E}(\lambda)] = \langle\langle\Psi|[\hat{V}, \hat{P}_\chi]|\Psi\rangle\rangle/(2iN_\lambda) = 0$, which reflects the negligible influence of the interaction on the environment state.

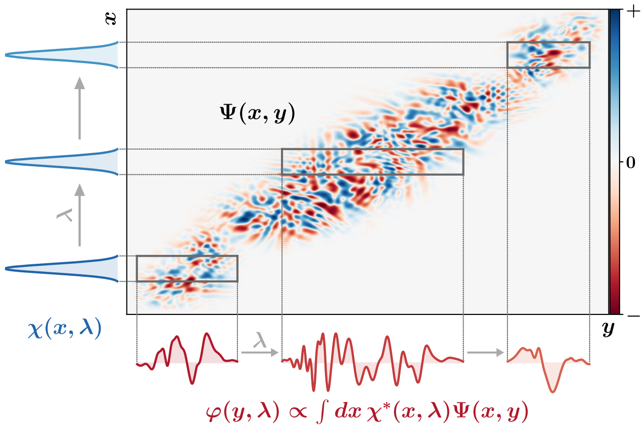

FIG. 2. Emergence of system dynamics by means of the relational formalism. Unitary changes in the clock state induce the system evolution through the correlations contained in the global state. The invariance (1) of Ψ ensures the concurrent system motion, which is governed by an effective clock-dependent system Hamiltonian. Moreover, the entanglement in the global state admits intricate system evolutions even for relatively simple wave functions of the environment.

We close with the promised concept for analytical solutions of TDSEs involving complicated, time-dependent potentials. The following, very simple example of coupled two-level systems gives a flavor for the general strategy. We consider a global Hamiltonian (2) with $\hat{H}_\mathrm{S} = 0$, $\hat{H}_\mathrm{C} = E_\mathrm{C}\hat{\sigma}_{\mathrm{C},z}$, and the interaction $\hat{V} = V_0(\hat{\sigma}_{\mathrm{S},x} + \hat{\sigma}_{\mathrm{S},z}) \otimes \hat{\sigma}_{\mathrm{C},x}$, where $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the three Pauli matrices, with the additional label for system or environment. Setting for simplicity $E_\mathrm{C} = V_0 \equiv 1$, we explicitly get

$$\hat{H} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}, \tag{13}$$

with eigenvalues $E_\pm = \pm\sqrt{3}$, where both of them are doubly degenerate. One eigenvector of E_- in the basis $\{|{\uparrow_\mathrm{S}}{\uparrow_\mathrm{C}}\rangle\!\rangle, |{\uparrow_\mathrm{S}}{\downarrow_\mathrm{C}}\rangle\!\rangle, |{\downarrow_\mathrm{S}}{\uparrow_\mathrm{C}}\rangle\!\rangle, |{\downarrow_\mathrm{S}}{\downarrow_\mathrm{C}}\rangle\!\rangle\}$, we take for the global state, $\Psi = (1, 0, -1, -a)^\mathrm{T}$, where $a = 1 + \sqrt{3}$. Here, we use $S(\lambda) = \int^\lambda d\lambda' \mathrm{Im}\mathcal{E}(\lambda')$ without loss of generality to simplify expressions. With

$$|\chi(\lambda)\rangle = \frac{e^{iE_-\lambda}}{2\sqrt{1 + a\cos^2(\lambda)}}[e^{-i\lambda}|{\uparrow_\mathrm{C}}\rangle + e^{i\lambda}|{\downarrow_\mathrm{C}}\rangle] \tag{14}$$

we obtain from (10b) the effective potential

$$\hat{V}_\mathrm{S} = \mathbf{V}_\mathrm{S}(\lambda)\cdot\hat{\boldsymbol{\sigma}}_\mathrm{S}, \tag{15a}$$

which enters the Schrödinger equation (11), where

$$V_{\mathrm{S},x} = V_{\mathrm{S},z} \equiv \frac{\cos(2\lambda) + a\cos^2(\lambda)}{1 + a\cos^2(\lambda)}, \tag{15b}$$

$$V_{\mathrm{S},y} \equiv -\frac{(a/2)\sin(2\lambda)}{1 + a\cos^2(\lambda)}, \tag{15c}$$

and $\hat{\boldsymbol{\sigma}}_\mathrm{S} \equiv (\hat{\sigma}_{\mathrm{S},x}, \hat{\sigma}_{\mathrm{S},y}, \hat{\sigma}_{\mathrm{S},z})^\mathrm{T}$. A physical realization would be the interaction of an electronic spin-system and a magnetic field, $\hat{V}_\mathrm{S} = -\mathbf{B}(\lambda)\cdot\hat{\boldsymbol{\mu}}$, with magnetic moment $\hat{\boldsymbol{\mu}} = (-e\hbar/2m_e)\hat{\boldsymbol{\sigma}}_\mathrm{S}$ or simply $\hat{\boldsymbol{\mu}} = -\hat{\boldsymbol{\sigma}}_\mathrm{S}/2$ in atomic units. The magnetic field has different time-dependent behavior along different directions, $\mathbf{B}_0 = 2[\cos(2\lambda) + a\cos^2(\lambda)](\mathbf{e}_x + \mathbf{e}_z)/[1 + a\cos^2(\lambda)]$ and $\mathbf{B}_1 = -a\sin(2\lambda)\mathbf{e}_y/[1 + a\cos^2(\lambda)]$.

By construction, we know that the solution of the TDSE with the potential $\hat{V}_\mathrm{S}(\lambda)$ is

$$|\varphi(\lambda)\rangle_\mathrm{S} \equiv \langle\chi(\lambda)|\Psi\rangle\!\rangle = \frac{e^{ia\lambda}}{2\sqrt{1 + a\cos^2(\lambda)}}\left[|{\uparrow_\mathrm{S}}\rangle_\mathrm{S} - \left(ae^{-2i\lambda} + 1\right)|{\downarrow_\mathrm{S}}\rangle_\mathrm{S}\right]. \tag{16}$$

Although the system for which we have constructed the time-dependent potential and the analytical solution of the ensuing TDSE is very simple, it admits, nevertheless, an entire class of time-dependent potentials and corresponding solutions by changing the state $|\chi(\lambda)\rangle$ of the environment.

Replacing the environment with a multilevel system is a straightforward extension with a semiclassical limit if the density of states of the environment in the energy interval defined by the two levels of the system becomes large. This renders the environment "large" as compared to the system and provides a direct link between the two research strands for the emergence of time as discussed in the introduction. One can also construct a more general semiclassical limit without reference to a specific (multilevel) system with a semiclassical state $|\chi(\lambda)\rangle$ from the environment and subsequent application of the stationary phase approximation, breaking implicitly the symmetry of environment and system [22].

While these semiclassical limits are consistent with the corresponding strand for the emergence of time, the semiclassical approach cannot uncover quantum roots of time, as we have worked them out here in the form of two conditions: (i) a global state exists which respects the invariance principle (1) with the global Hamiltonian and (ii) the global Hamiltonian can be decomposed into a Hamiltonian $\hat{H}_\mathrm{S}$ for the system, its environment $\hat{H}_\mathrm{C}$, and their interaction $\hat{V}$. With projecting the invariance principle onto an arbitrary state of the environment and all its λ-dependent variants generated by "rotating" the state with $\hat{H}_\mathrm{C}$, these two conditions suffice to formulate a time-dependent Schrödinger equation for the system with a time-dependent potential. Thereby, we advance the relational approach to time by the crucial inclusion of interaction of system and environment, which so far has been possible only under very special circumstances [20].

Since projection and separation of system and environment as well as entanglement and interaction are also major elements of decoherence, it is not surprising that our theory has points of contact with Zurek's decoherence theory [18] as we have mentioned before. However, decoherence requires time as a prerequisite: the literal meaning of decoherence reveals it as a process *in* time. The successful inclusion of interaction into the emergence of time as outlined here renders our framework suitable to ask if decoherence can be established along with emergent time in the interaction of system and environment, a question we will pursue in future work.

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[1] D. N. Page and W. K. Wootters, Evolution without evolution: Dynamics described by stationary observables, Phys. Rev. D **27**, 2885 (1983).

[2] W. K. Wootters, "Time" replaced by quantum correlations, Int. J. Theor. Phys. **23**, 701 (1984).

[3] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum time, Phys. Rev. D **92**, 045033 (2015).

[4] C. Marletto and V. Vedral, Evolution without evolution and without ambiguities, Phys. Rev. D **95**, 043510 (2017).

[5] P. A. Höhn, A. R. Smith, and M. P. Lock, Trinity of relational quantum dynamics, Phys. Rev. D **104**, 066001 (2021).

[6] J. J. Halliwell and S. W. Hawking, Origin of structure in the Universe, Phys. Rev. D **31**, 1777 (1985).

[7] K. V. Kuchař, Time and interpretations of quantum gravity, *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics* (World Scientific, Singapore, 1992).

[8] C. Kiefer, The semiclassical approximation to quantum gravity, in *Canonical Gravity: From Classical to Quantum*, edited by J. Ehlers and H. Friedrich (Springer, Berlin, Heidelberg, 1994), pp. 170–212.

[9] E. Anderson, Problem of time in quantum gravity, Ann. Phys. (Berlin) **524**, 757 (2012).

[10] C. Kiefer, *Quantum Gravity*, 3rd ed. (Oxford University Press, New York, 2012).

[11] J. S. Briggs and J. M. Rost, Time dependence in quantum mechanics, Eur. Phys. J. D **10**, 311 (2000).

[12] J. S. Briggs and J. M. Rost, On the derivation of the time-dependent equation of Schrödinger, Found. Phys. **31**, 693 (2001).

[13] A. Schild, Time in quantum mechanics: A fresh look at the continuity equation, Phys. Rev. A **98**, 052113 (2018).

[14] L. Chataignier, Beyond semiclassical time, Z. Naturforsch. A **77**, 805 (2022).

[15] S. Gemsheim, The emergence of time with interactions in quantum and classical mechanics, Ph.D. thesis, Technische Universität Dresden, 2023.

[16] S. Gemsheim and J. M. Rost (to be published).

[17] Since the clock projection includes an identity operation on the system part, it is fully described by $(\hat{\mathbb{1}}_S \otimes \langle\chi|)|\Psi\rangle\rangle = \sum_{kl} c_{kl}(\hat{\mathbb{1}}_S|\varphi_k\rangle_S) \otimes \langle\chi|\chi_l\rangle = \sum_k |\varphi_k\rangle_S(\sum_l c_{kl}\langle\chi|\chi_l\rangle) \in \mathcal{H}_S$ for a representation of $|\Psi\rangle\rangle$ in the arbitrary global basis $\{|\varphi_k \otimes \chi_l\rangle\rangle\}_{kl}$. Where obvious, we will omit identity operators in favor of a better readability.

[18] W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Rev. Mod. Phys. **75**, 715 (2003).

[19] C. Callender, *What Makes Time Special?* (Oxford University Press, Oxford, 2017).

[20] A. R. H. Smith and M. Ahmadi, Quantizing time: Interacting clocks and systems, Quantum **3**, 160 (2019).

[21] W. H. Zurek, Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?, Phys. Rev. D **24**, 1516 (1981).

[22] S. Gemsheim and J. M. Rost, From quantum to semiclassical emergence of time (to be published).