Horváth and Markoš Reply: 1. In Ref. [1] (Letter) we calculated the infrared (IR) effective counting dimension $d_{\rm IR}$ [2–4] of critical states in 3D Anderson models, inferring two novelties: ($m1$) Space effectively occupied by a critical electron is of dimension $d_{\rm IR} \approx 8/3$. Dimension $d_{\rm IR}$ governs IR scaling of properly defined *effective volume* and is unique in that role [4]. It is the maiden estimate of such a quantity in the Anderson context. ($m2$) Values of $d_{\rm IR}$ in studied classes (O, U, S, AIII) coincide to about two parts per mill with comparable individual errors. We dubbed this superuniversality in $d_{\rm IR}$ since other critical indices differ to a much larger degree.

2. In Ref. [5] (Comment) Burmistrov raises objections to ($m1$) and ($m2$). Burmistrov ($B1$) conjectures that $d_{\rm IR} = f_m(3)$, where $f_m(\alpha)$ is the "singularity spectrum" of critical states in the moment multifractal (mMF) method [6]; ($B2$) combines it with older numerical results on f_m [7] to argue that our $d_{\rm IR}$ analysis is affected by systematics; and ($B3$) proposes that discrepancy arises because our procedure does not assume $\langle \mathcal{N}_\star \rangle_L$ asymptotics predicted by mMF.

3. ($B1$) is conjectural since it lacks the proof that an exact multifractal (MF) representation of $\langle \mathcal{N}_\star \rangle_L$, namely [8]

$$\langle \mathcal{N}_\star \rangle_L = \int_{-\infty}^{\infty} d\alpha\, v(\alpha, L) L^{f(\alpha,L)} \min\{1, L^{D-\alpha}\} \quad (1)$$

implies $\langle \mathcal{N}_\star \rangle_L = h_\star(L) L^{f_m(3)}$, where $h_\star$ is such that $\forall\, \delta > 0$: $\lim_{L\to\infty} L^{-\delta} \max\{h_\star(L), 1/h_\star(L)\} = 0$. The singularity spectrum $f(\alpha) = \lim_{L\to\infty} f(\alpha, L)$ in the MF (not mMF) method is the dimension of the set of points x for which $\psi^+\psi(x) = L^{-\alpha}$. Precise definitions are given in [8]. Without this proof, ($B2$) and ($B3$) are conjectures as well.

4. Moreover, ($B1$) lacks support in raw $d_{\rm IR}$ data. Indeed, the left panel in Fig. 1 shows finite-L dimension $d_{\rm IR}(L) \equiv d_{\rm IR}(L, s = 2)$ ($\lim_{L\to\infty} d_{\rm IR}(L) \equiv d_{\rm IR}$; Eq. (7) in [1]) together with its mMF representations [8,9]. Parabolic mMF severely overestimates $d_{\rm IR}(L)$, while full (quartic) mMF [9] does the opposite. Neither reflects large-L tendencies of $d_{\rm IR}(L)$ at state-of-the-art volumes. The mMF asymptotic term [Eq. (3) in Comment] is almost off scale. Lacking contact (direct or indirect) with data, ($B1$) is in fact a hypothesis rather than conjecture. Note that our data were greatly extended for this purpose.

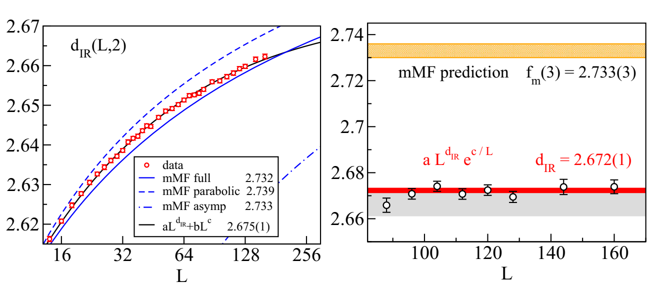

FIG. 1. Left: mMF predictions [8,9] for $d_{\rm IR}(L, 2)$ vs data. The two-power fit describes $\langle \mathcal{N}_\star \rangle_L$ over the entire studied range ($6 \le L \le 160$) and guides the eye. Right: Stability of $d_{\rm IR}$ determination (see text). mMF prediction is based on ($B1$) and Ref. [7].

5. The claim that our analysis assumes "purely power-law-like scaling" of $\langle \mathcal{N}_\star \rangle_L$ is not true. To convey the related basics explicitly, note that $\langle \mathcal{N}_\star \rangle_L = h_\star(L) L^{d_{\rm IR}}$, where $h_\star(L)$ varies slower than any nonzero power for $L \to \infty$. While $h_\star$ is unknown, using *any* such $h(L)$ to model (fit) $\langle \mathcal{N}_\star \rangle_L$ is guaranteed to yield correct $d_{\rm IR}$ if the analysis includes a reliable $L \to \infty$ limit. In the Letter, we used $h(L)$ associated with linear $d_{\rm IR}(1/L)$, namely $h(L) = a \exp(c/L)$. [Both $h(L)$ and $d_{\rm IR}(1/L)$ would be constant in the pure-power case.] To check for sufficiency of available volumes, we fit in a sliding window $[L/2, L]$, with results ($88 \le L \le 160$) in the right panel of Fig. 1. We emphasize that each plotted point is an *extrapolated* $d_{\rm IR}$. Saturation at $L \approx 96$ substantiates the stability of our analysis. The red band arises from an overall fit in the stable range. It lies at 2 parts per mill from 8/3 (gray band) and far away from mMF prediction.

6. Given the reasoning in 3, 4 and 5, we consider Burmistrov arguments based on combination of ($B1$) and ($B2$) a bridge too far, especially when it comes to very fine quantitative resolution needed in the Comment. In their current form, these arguments do not affect the gist of ($m1$) and ($m2$).

7. The present debate opens doors to studies of potential loopholes in saddle point (Gaussian) mMF orthodoxy in the geometric description of Anderson criticality. One possibility is that $f_m \neq f$ [8] which would entail that Anderson multifractals are not exactly self-similar [10]. Recent *multidimensional analysis* [11] favors this scenario.

8. The Comment loses sight of the conceptual novelty in $d_{\rm IR}$ wherein effective numbers lead to well-defined *effective subsets* of a probability sample space and unique effective dimension [4]. This has no analog in MF formalism working with a one-parametric family of *fixed subsets*. In that vein, the (hypothetical) "$d_{\rm IR}$ *is nothing but* $f(d)$" [read $f_m(d)$] reminds us of "π *is nothing but* $22/7 - \int_0^1 dx\, x^4(1-x)^4/(1+x^2)$," with both declarations stripping their subjects of meaning.

Ivan Horváth[1,2,*] and Peter Markoš[3,†]
[1]Nuclear Physics Institute CAS
25068 Řež (Prague), Czech Republic
[2]University of Kentucky
Lexington, Kentucky 40506, USA
[3]Department of Experimental Physics
Faculty of Mathematics, Physics and Informatics Comenius
University in Bratislava
Mlynská Dolina 2, 842 28 Bratislava, Slovakia

Received 5 December 2022; accepted 6 September 2023; published 26 September 2023

DOI: 10.1103/PhysRevLett.131.139702

*ihorv2@g.uky.edu
†peter.markos@fmph.uniba.sk

[1] I. Horváth and P. Markoš, Phys. Rev. Lett. **129**, 106601 (2022).
[2] I. Horváth and R. Mendris, Entropy **22**, 1273 (2020).
[3] A. Alexandru and I. Horváth, Phys. Rev. Lett. **127**, 052303 (2021).
[4] I. Horváth, P. Markoš, and R. Mendris, Entropy **25**, 482 (2023).
[5] I. S. Burmistrov, preceding Comment, Comment on "Super-Universality in Anderson Localization," Phys. Rev. Lett. **131**, 139701 (2023).
[6] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[7] L. Ujfalusi and I. Varga, Phys. Rev. B **91**, 184206 (2015).
[8] I. Horváth and P. Markoš, arXiv:2212.02912.
[9] I. S. Burmistrov, arXiv:2210.10539.
[10] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 3rd ed. (Wiley, New York, 2014).
[11] I. Horváth and P. Markoš, arXiv:2212.09806.