Equivariant Localization in Supergravity

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We show that supersymmetric supergravity solutions with an R-symmetry Killing vector are equipped with a set of equivariantly closed forms. Various physical observables may be expressed as integrals of these forms, and then evaluated using the Berline-Vergne-Atiyah-Bott fixed point theorem. We illustrate with a variety of holographic examples, including on-shell actions, black hole entropies, central charges, and scaling dimensions of operators. The resulting expressions depend only on topological data and the R-symmetry vector, and hence may be evaluated without solving the supergravity equations.

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Introduction.—Starting with the seminal work of [1,2], localization has proved to be an invaluable tool both in mathematics and in supersymmetric quantum field theory. These ideas have notably been applied to the infinite-dimensional supersymmetric path integral (see Ref. [3] for a review, and [4]), but localization in finite dimensions also plays a role. For instance, the localization over instanton moduli spaces in [7], or in the analysis of anomalies and anomaly polynomials. In this Letter we show that localization also plays a role in supergravity. The general structure we uncover explains, unifies, and generalizes many previous results in the literature.

The supergravity theory or relevant part of the geometry will be assumed to have even dimension $d = 2n$. We require that supersymmetric solutions are equipped with an R-symmetry Killing vector ξ, constructed as a bilinear in the Killing spinor ϵ

$$\xi \equiv \bar{\epsilon}\gamma^\mu \gamma_* \epsilon \partial_\mu. \tag{1}$$

Here γ_* is either $\mathbb{1}$ or the chirality operator, depending on the theory considered, with γ_μ generating the Clifford algebra, so $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. With an R symmetry, the space of solutions to the Killing spinor equation generically has one complex dimension, generated by ϵ, and one can then argue (see, e.g., [8]) that $\mathcal{L}_\xi \epsilon = iq\epsilon$, where we refer to the constant q as the R charge.

In such a setup it is natural to introduce the equivariant exterior derivative

$$d_\xi \equiv d - \xi \lrcorner . \tag{2}$$

This acts on differential forms and squares to minus the Lie derivative $d_\xi^2 = -\mathcal{L}_\xi$. Indeed, in this setting differential forms of degree r are constructed naturally as bilinears $\Psi_r \equiv \bar{\epsilon}\gamma_{(r)}\gamma_*\epsilon$, where $\gamma_{(r)} \equiv (1/r!)\gamma_{\mu_1\cdots\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$. From our above assumptions on ξ we immediately have $\mathcal{L}_\xi \Psi_r = 0$. The framework of G structures and intrinsic torsion implies that the Killing spinor equation for ϵ may be recast as a set of differential and algebraic equations on bilinear forms and fields in the theory (see Ref. [9]). In general these equations involve bilinears constructed using the charge conjugate $\bar{\epsilon}^c$ as well as the Dirac conjugate $\bar{\epsilon}$, but only the latter will appear in the present work; the former are charged under ξ when $q \neq 0$, and are thus not invariant under $\mathcal{L}_\xi$.

In the sequel we will be interested in finding polyforms Φ, constructed as polynomials in the bilinears Ψ_r and fields in the supergravity theory, which by virtue of the Killing spinor equation and equations of motion satisfy the equivariantly closed condition

$$d_\xi \Phi = 0. \tag{3}$$

Notice that since ξ is itself a bilinear, the bilinear degree of the $(r-2)$-form component Φ_{r-2} is necessarily one more than Φ_r. Moreover, since only the Dirac bilinears Ψ_r are used to construct Φ, we will only need to impose a subset of the supersymmetry equations and equations of motion to ensure (3) holds. In this sense such structures exist "partially off-shell," and this will also play a role. As we will see, a number of such polyforms can exist in a given theory.

Given Φ satisfying (3) we may integrate it over a ξ-invariant closed submanifold M, and apply the Berline-Vergne-Atiyah-Bott [10,11] fixed point formula. Denoting an embedded connected component of the fixed point set as $f: F \hookrightarrow M$, where $\xi = 0$, we have

$$\int_M \Phi = \sum_{\substack{F \\ \mathrm{codim}\, F = 2k}} \frac{1}{d_F} \frac{(2\pi)^k}{\prod_{i=1}^k \epsilon_i} \int_F \frac{f^* \Phi}{\prod_{i=1}^k \left[1 + \frac{2\pi}{\epsilon_i} c_1(L_i)\right]}. \qquad (4)$$

Here for simplicity [12] we have assumed that the weights ϵ_i of the linear action of ξ on the normal bundle N_F to F in M are generic, so that $N_F = \bigoplus_{i=1}^k L_i$ splits as a sum of complex line bundles, with $c_1(L_i)$ denoting the first Chern classes. The normal space to a generic point on F is taken to be $\mathbb{R}^{2k}/\Gamma$, where Γ is a finite group of order $d_F \in \mathbb{N}$, so that (4) also applies to orbifolds.

What is perhaps surprising is that many BPS physical quantities take the form (4), and furthermore this structure then allows one to evaluate "off-shell," without imposing or solving the supergravity equations. In the remainder of this Letter we give various illustrative examples, both recovering known results and giving some new results, leaving comments on further applications and generalizations for the discussion section.

4d minimal gauged supergravity.—We consider supersymmetric solutions to $4d$, $\mathcal{N} = 2$ minimal gauged supergravity. The bosonic content is Einstein-Maxwell theory with a negative cosmological constant. In Euclidean signature the holographically renormalized action is

$$I = -\frac{1}{16\pi G_4} \left\{ \int_M (R_g + 6 - F^2) \mathrm{vol}_g + \int_{\partial M} 2\mathcal{K} \mathrm{vol}_h - \int_{\partial M} (4 + R_h) \mathrm{vol}_h \right\}. \qquad (5)$$

Here M is a four-manifold, with boundary ∂M with induced metric h, vol denote Riemannian volume forms, R Ricci scalars, and $F = dA$ is the Maxwell field strength. The Gibbons-Hawking-York term involves the trace of the extrinsic curvature $\mathcal{K}$, while the final counterterm renormalizes the on-shell action for asymptotically locally AdS solutions [13]. The Newton constant in dimension d will be denoted G_d.

Supersymmetric solutions to the Euclidean theory were analyzed in [14]. From the Killing spinor ϵ one may construct the following real bilinear forms

$$S \equiv \bar{\epsilon}\epsilon, \quad P \equiv \bar{\epsilon}\gamma_5\epsilon, \quad \xi^\flat \equiv -\mathrm{i}\bar{\epsilon}\gamma_{(1)}\gamma_5\epsilon, \quad U \equiv \mathrm{i}\bar{\epsilon}\gamma_{(2)}\epsilon. \qquad (6)$$

Here $\gamma_5 \equiv \gamma_{1234}$, and the one-form $\xi^\flat$ is dual to ξ, given by (1) with $\gamma_* = -\mathrm{i}\gamma_5$. We then define the polyform

$$\begin{aligned} \Phi &= \Phi_4 + \Phi_2 + \Phi_0 \\ &\equiv (3\mathrm{vol}_g + F \wedge *F) + (U + SF - P * F) - SP, \end{aligned} \qquad (7)$$

where $*$ denotes the Hodge dual. As indicated in the introduction, this is a polynomial in the bilinear forms (6) and supergravity form fields, with Φ_{2j} being degree $2 - j$ in bilinears. The Killing spinor equation implies certain differential conditions on these forms, and using the results in [14] one easily shows that Φ satisfies the equivariantly closed condition (3).

Using the Einstein equation the on-shell action (5) is given by

$$I = \left[\frac{1}{(2\pi)^2} \int_M \Phi\right] \frac{\pi}{2G_4} + \text{boundary terms}. \qquad (8)$$

We cannot directly apply (4) in this case, since M has a boundary. However, following the way in which the fixed point theorem is proven, we notice that the integrand Φ is exact on the complement of the fixed point set and the boundary. Then, assuming [15] ξ has no fixed points on the boundary one can integrate by parts, leading to a boundary term on ∂M together with contributions around the fixed point set in the interior of M. The (divergent) boundary contribution exactly cancels with the boundary integrals in (5). This was shown by explicit computation in [14], but is also expected since the result should be Weyl invariant and for this theory there is no such boundary quantity; equivalently, we cannot construct a finite Weyl invariant counterterm [17]. Thus, from (4) the remaining fixed point contribution in the interior of M gives

$$I = \left\{ \sum_{\substack{\text{fixed} \\ \text{points}}} \frac{\Phi_0}{\epsilon_1 \epsilon_2} + \sum_{\substack{\text{fixed} \\ \Sigma}} \int_\Sigma \frac{\Phi_2}{2\pi\epsilon_1} - \frac{\Phi_0 c_1(L)}{\epsilon_1^2} \right\} \frac{\pi}{2G_4}. \qquad (9)$$

Here L is the normal bundle to the surface Σ in M, and we note that Φ_0 is necessarily constant over Σ.

Following [14] we may write $P = S\cos\theta$, where $\|\xi\| = S|\sin\theta|$. Since the spinor square norm S is necessarily nowhere zero (see Ref. [8] for a general argument), it follows that at a fixed point set $\theta = 0$ or π, and hence correspondingly $P = \pm S$, so that ϵ is necessarily of fixed chirality with $\gamma_5\epsilon = \pm\epsilon$ at such a fixed locus.

Examining first an isolated fixed point, on the tangent space we may write $\xi = \sum_{i=1}^2 \epsilon_i \partial_{\varphi_i}$, where ∂_{φ_i} rotate each copy of $\mathbb{R}_i^2$ in $\mathbb{R}^4 = \mathbb{R}_1^2 \oplus \mathbb{R}_2^2$. Notice here that the overall orientation on $\mathbb{R}^4$ is fixed, but the orientations of each $\mathbb{R}_i^2$ factor are not; this means that the pair (ϵ_1, ϵ_2) is only defined up to overall sign. A local analysis of the bilinears near such a fixed point relates S to the norm of the self-dual and anti-self-dual parts of the two-form $d\xi^\flat$, leading to the general formula $S = |\epsilon_1 \mp \epsilon_2|/2$ [14]. A similar argument leads to $S = -\epsilon_1/2$ for a fixed surface, while formulas in [14] immediately give

$$\int_\Sigma \Phi_2 = 2\int_\Sigma SF = -\epsilon_1 \int_\Sigma F. \qquad (10)$$

A global analysis of spinors near to the fixed surface $\Sigma = \Sigma_\pm$, which have charge $\frac{1}{2}$ under A, then implies [14]

$$\frac{1}{2\pi}\int_{\Sigma_\pm} F = -\frac{1}{2}\int_{\Sigma_\pm} c_1(T\Sigma_\pm) \mp c_1(L), \qquad (11)$$

where $\int_{\Sigma_\pm} c_1(T\Sigma_\pm) = 2(1-g_\pm)$ is the Chern number of the tangent bundle of the Riemann surface $\Sigma_\pm$ of genus $g_\pm$, and substituting into (9) gives

$$I = \left\{ \sum_{\substack{\text{fixed}\\ \text{points}_\pm}} \mp \frac{(\epsilon_1 \mp \epsilon_2)^2}{4\epsilon_1\epsilon_2} + \sum_{\substack{\text{fixed}\\ \Sigma_\pm}} \int_{\Sigma_\pm} \left[\frac{1}{2}c_1(T\Sigma_\pm) \mp \frac{1}{4}c_1(L)\right]\right\} \frac{\pi}{2G_4}. \qquad (12)$$

This is the main result of [14], derived here as a simple application of (4).

6d Romans F(4) gauged supergravity.—A similar structure exists in $6d$ Romans F(4) gauged supergravity, where for the Euclidean theory we follow [18,19]. We work in an Abelian truncation, where in addition to the metric the bosonic content of the theory has a scalar field X, Maxwell field strength $F = dA$, all of which are real, and an imaginary two-form potential B.

From the Killing spinor ϵ one may construct the following bilinear forms

$$S \equiv \bar{\epsilon}\epsilon, \qquad P \equiv \bar{\epsilon}\gamma_7\epsilon, \qquad \xi^\flat \equiv \bar{\epsilon}\gamma_{(1)}\epsilon,$$
$$Y \equiv \mathrm{i}\bar{\epsilon}\gamma_{(2)}\epsilon, \qquad \tilde{Y} \equiv \mathrm{i}\bar{\epsilon}\gamma_{(2)}\gamma_7\epsilon, \qquad (13)$$

where $\gamma_7 \equiv \mathrm{i}\gamma_{123456}$ and the one-form $\xi^\flat$ is dual to ξ, given by (1) with $\gamma_* = \mathbb{1}$. Using these we then define the polyform $\Phi = \Phi_6 + \Phi_4 + \Phi_2 + \Phi_0$, where

$$\Phi_6 \equiv \frac{4}{9}\frac{2+3X^4}{X^2}\mathrm{vol} + \frac{1}{3}X^{-2}F\wedge *F + \frac{\mathrm{i}}{3}B\wedge F\wedge F,$$
$$\Phi_4 \equiv \frac{\sqrt{2}}{3}(XP)X^{-2} * F - \frac{2\sqrt{2}}{3}X * \tilde{Y} - \frac{\sqrt{2}}{3}F\wedge X^{-1}Y + \frac{1}{\sqrt{2}}(XS)F\wedge F + \frac{2\sqrt{2}\mathrm{i}}{3}(XP)B\wedge F,$$
$$\Phi_2 \equiv -\frac{2}{3}PY + \frac{2\mathrm{i}}{3}(XP)^2 B + 2(XS)(XP)F,$$
$$\Phi_0 \equiv \sqrt{2}(XS)(XP)^2. \qquad (14)$$

Notice that Φ_{2j} has bilinear degree $3-j$. The differential equations satisfied by (13) may be found in [19], and using these one can show that Φ satisfies the equivariantly closed condition (3).

The on-shell action may be written as

$$I = \left[\frac{1}{(2\pi)^3}\int_M \Phi\right]\frac{\pi^2}{2G_6} + \text{boundary terms}, \qquad (15)$$

where the full set of boundary counterterms may be found in [18]. Similar to the case of $4d$ minimal gauged supergravity, for this theory there are again no finite Weyl invariant counterterms. Thus, assuming that the fixed point set lies within the interior of M, the boundary terms will cancel leaving only a fixed point contribution in the interior. It is then straightforward to write down an explicit expression for the on-shell action using localization. For brevity, we just give the result for the class of solutions in which there are only isolated fixed points:

$$I = \sum_{\substack{\text{fixed}\\ \text{points}}} \pm \frac{(\epsilon_1+\epsilon_2+\epsilon_3)^3}{\epsilon_1\epsilon_2\epsilon_3}\frac{\pi^2}{4G_6}, \qquad (16)$$

where $\mathbb{R}^6 = \oplus_{i=1}^3 \mathbb{R}_i^2$, and as commented in the previous section only the overall orientation is fixed. Here we have used the fact that at a fixed point set again $|P| = S$, together with the local analysis in [19] which shows that $(XS)|_{\text{fixedpoint}} = (\epsilon_1+\epsilon_2+\epsilon_3)/\sqrt{2}$, in appropriate orientation conventions for each $\mathbb{R}_i^2$. Having chosen such conventions, the orientation on $\mathbb{R}^6$ then either agrees with the fixed orientation or not, which we write as a $\pm$ sign in (16).

Remarkably, we have recovered the conjectured result for the on-shell action given in [18]. This conjecture was known to hold for at least three different families of examples, including the nonrotating black hole solutions in [20] (with hyperbolic space horizons), but here we have obtained a general proof, which goes beyond these examples.

As a further illustration, we show that (16) correctly gives the on-shell action and hence entropy of supersymmetric rotating black hole solutions in this theory. We analyze the complex branch of supersymmetric black hole solutions studied in [21]. The spacetime M has the topology $\mathbb{R}^2\times S^4$, and the R-symmetry Killing vector is

$$\xi = \sum_{i=1}^{2}\mathrm{i}\frac{\omega_i}{\beta}\partial_{\varphi_i} + \frac{2\pi}{\beta}\partial_{\varphi_3}. \qquad (17)$$

Here ω_i, $i = 1, 2$ are (complex) angular velocity chemical potentials, and we have embedded $S^4 \subset \mathbb{R}_1^2 \oplus \mathbb{R}_2^2 \oplus \mathbb{R}$, with ∂_{φ_i} rotating the $\mathbb{R}_i^2$ factors, $i = 1,\ 2$, while $\varphi_3 = 2\pi\tau/\beta$, with τ the Euclidean time circle, of period β. For generic values of parameters the fixed point set consists of the north and south poles of the S^4 horizon. These give an equal contribution, and (16) (with an overall plus sign) gives the on-shell action

$$I = \frac{\pi[2\pi + \mathrm{i}(\omega_1+\omega_2)]^3}{4\omega_1\omega_2 G_6}. \qquad (18)$$

This agrees with the result in [21], after taking into account the fact that the AdS radius in our conventions is $\ell = 3/\sqrt{2}$. The entropy of the black hole may then be computed via a Legendre transform, extremizing

$$S = -I - \sum_{i=1}^{2}\omega_i J_i - \frac{\ell}{3}(-2\pi\mathrm{i} + \omega_1 + \omega_2)Q, \qquad (19)$$

over the chemical potentials ω_i, where J_i, Q are the angular momenta and electric charge, respectively.

$\mathrm{AdS}_5\times M_6$ *solutions.*—In this section we consider supersymmetric AdS_5 solutions to $11d$ supergravity, as analyzed in [22]. The $11d$ metric takes the warped product form

$$ds^2_{11} = e^{2\lambda}(ds^2_{\mathrm{AdS}_5} + ds^2_{M_6}), \tag{20}$$

where we take AdS_5 to have unit radius, and will assume that M_6 is compact without boundary. The four-form flux G and function λ are pullbacks from M_6.

Denoting $\epsilon = \epsilon^+_{\mathrm{there}}$ in [22], we have bilinears on M_6:

$$1 = \bar{\epsilon}\epsilon, \qquad \sin\zeta \equiv -\mathrm{i}\bar{\epsilon}\gamma_7\epsilon, \qquad \xi^\flat \equiv \frac{1}{3}\bar{\epsilon}\gamma_{(1)}\gamma_7\epsilon,$$
$$Y \equiv -\mathrm{i}\bar{\epsilon}\gamma_{(2)}\epsilon, \qquad Y' \equiv \bar{\epsilon}\gamma_{(2)}\gamma_7\epsilon, \tag{21}$$

where $\gamma_7 \equiv \gamma_{123456}$. There are some immediate differences with the applications in the previous sections: the spinor necessarily has constant norm, which we take to be 1, but there is instead a warp factor function $e^{2\lambda}$. We have also normalized the Killing vector ξ so that the R charge is $q=\frac{1}{2}$. In the notation of [22] then $\xi = \partial_\psi$. An important role is played by the function

$$y \equiv \frac{1}{2}e^{3\lambda}\sin\zeta, \tag{22}$$

which was used as a canonical coordinate in [22].

We find the following collection of equivariantly closed forms under d_ξ

$$\Phi \equiv e^{9\lambda}\mathrm{vol} + \frac{1}{12}e^{9\lambda} * Y - \frac{1}{36}ye^{6\lambda}Y - \frac{1}{162}y^3,$$
$$\Phi^G \equiv G - \frac{1}{3}e^{3\lambda}Y' + \frac{1}{9}y,$$
$$\Phi^Y \equiv e^{6\lambda}Y + \frac{1}{3}y^2. \tag{23}$$

We emphasize that closure under d_ξ uses the differential conditions on the Dirac bilinears only, which is a strict subset of the equations in [22]. Also, since ϵ here has unit norm the bilinear degrees in (23) are less transparent, although one could choose to keep this norm arbitrary. The a central charge for such a solution is [23]

$$a = \frac{1}{2(2\pi)^6\ell_p^9}\int_{M_6}\Phi, \tag{24}$$

where ℓ_p is the $11d$ Planck length; this then localizes using (4).

As an example, let us consider the near-horizon limit of N M5-branes wrapped on a spindle. The full supergravity solutions were constructed in [24], with M_6 being the total space of an S^4 bundle fibered over a spindle $\Sigma = \mathbb{WCP}^1_{[n_+,n_-]}$. The latter is topologically a two-sphere, but with conical deficit angles $2\pi(1-1/n_\pm)$ at the poles. We write the R-symmetry vector as

$$\xi = \sum_{i=1}^{2} b_i\partial_{\varphi_i} + \varepsilon\partial_{\varphi_3}, \tag{25}$$

where b_i and ε are constants, arbitrary at this stage, which define the Killing vector. Here ∂_{φ_i} rotate the two copies of $\mathbb{R}^2_i$ in $S^4 \subset \mathbb{R}^2_1 \oplus \mathbb{R}^2_2 \oplus \mathbb{R}$, while ∂_{φ_3} is a lift of the vector field that rotates the spindle, where we use the construction of such a basis in [25].

Consider first fixing one of the poles on Σ, say the plus pole with orbifold group $\mathbb{Z}_{n_+}$, and consider a linearly embedded $S^2_i \subset \mathbb{R}^2_i \oplus \mathbb{R} \subset \mathbb{R}^5$ in the (covering space of the) fiber over it. The homology class of this S^2_i is trivial, so it follows that

$$0 = \int_{S^2_i}\Phi^Y = \frac{2\pi}{b_i^\pm}\frac{1}{3}[(y_N^+)^2 - (y_S^+)^2], \tag{26}$$

where the N and S subscripts refer to the poles in the fibre sphere S^4, and $b_i^\pm$ are the weights of the Killing vector at these poles [i.e., the ϵ_i in (4)], which we shall determine below. This immediately implies that $|y_N^\pm| = |y_S^\pm|$.

We next consider flux quantization through the fibers $S^4/\mathbb{Z}_{n_\pm}$ over the poles of Σ. This reads

$$N_\pm = \frac{1}{(2\pi\ell_p)^3}\int_{S^4/\mathbb{Z}_{n_\pm}}\Phi^G$$
$$= \frac{1}{(2\pi\ell_p)^3}\frac{1}{n_\pm}\frac{(2\pi)^2}{b_1^\pm b_2^\pm}\frac{1}{9}(y_N^\pm - y_S^\pm). \tag{27}$$

With $N_\pm > 0$, this fixes the signs to be $y_N^\pm = -y_S^\pm > 0$. Moreover, from the homology relation between these cycles we deduce

$$N \equiv n_+N_+ = n_-N_-. \tag{28}$$

The central charge (24) may then also be computed by localizing

$$\int_{M_6}\Phi = -(2\pi)^3\left[\frac{1}{n_+}\frac{(y_N^+)^3 - (y_S^+)^3}{162}\frac{1}{(-\varepsilon/n_+)b_1^+b_2^+}\right.$$
$$\left. + \frac{1}{n_-}\frac{(y_N^-)^3 - (y_S^-)^3}{162}\frac{1}{(\varepsilon/n_-)b_1^-b_2^-}\right]. \tag{29}$$

Using (27) this remarkably simplifies to

$$a = \frac{9[(b_1^+b_2^+)^2 - (b_1^-b_2^-)^2]}{16\varepsilon}N^3. \tag{30}$$

This takes a "gravitational block" form (see Ref. [26]), involving a difference of M5-brane anomaly polynomials in the numerator, one associated to each $\pm$ pole of Σ.

In order to evaluate (30) further we need to first describe the fibration structure in more detail. The normal bundle to the M5-brane wrapped on Σ is $N_\Sigma = \mathcal{O}(-q_1) \oplus \mathcal{O}(-q_2)$, where in order for the total space to be Calabi-Yau (giving a topological twist) we have

$$q_1 + q_2 = n_+ + n_-. \tag{31}$$

The weights $b_i^\pm$ may then be computed using the results in [25]. We have

$$b_1^\pm + b_2^\pm = 1 \mp \frac{\varepsilon}{n_\pm}, \qquad b_i^+ - b_i^- = -\frac{q_i}{n_+ n_-}\varepsilon, \tag{32}$$

the first equation coming from the charge of the holomorphic (3, 0)-form on the Calabi-Yau, and the second equation being (3.24) of [25] (with $q_i = -p_i^{\mathrm{there}}$). We may then solve these constraints by introducing new variables ϕ_i via

$$b_i^\pm = \frac{1}{2}\left(\phi_i \mp \frac{q_i}{n_+ n_-}\varepsilon\right), \tag{33}$$

with the constraint

$$\phi_1 + \phi_2 = 2 + \frac{n_+ - n_-}{n_+ n_-}\varepsilon. \tag{34}$$

Our final central charge is then

$$a = -\frac{9[(q_2\phi_1 + q_1\phi_2)(q_1 q_2 \varepsilon^2 + n_+^2 n_-^2 \phi_1\phi_2)]}{64 n_+^3 n_-^3} N^3. \tag{35}$$

This derives the conjectured gravitational block formula in [27], where we have corrected the overall sign. In that reference it was shown that extremizing a over the variables ϕ_i [subject to (34)] gives the central charge as well as determines the R-symmetry Killing vector of the explicit supergravity solutions constructed in [24]. Moreover, (35) agrees *off-shell* with the trial a-function in field theory, obtained by integrating the M5-brane anomaly polynomial over the spindle.

This extremization is also explained by our gravity formalism: we have imposed a subset of the supersymmetry equations to obtain (35). Substituting this back into the action, one then extremizes the resulting expression over any remaining degrees of freedom to obtain the on-shell result. This is the same general idea used in [28,29], and will be discussed further in the present context in [30].

Finally, let us consider the conformal dimensions of chiral primary operators in the dual SCFT that are associated with M2-branes wrapped over the copies of the spindle Σ_N, Σ_S at the poles of the S^4. We have the general result [23]

$$\Delta(\Sigma) = \frac{1}{(2\pi)^2 \ell_p^3}\int_\Sigma e^{3\lambda} Y', \tag{36}$$

where Σ is calibrated by Y'. Since the latter is always closed when restricted to Σ, Φ^G in (23) defines an equivariantly closed form on Σ, and localization gives

$$\begin{aligned}\Delta(\Sigma_N) &= \frac{-1}{(2\pi)^2\ell_p^3}\left[\frac{1}{n_+}\frac{2\pi}{(-\varepsilon/n_+)}\frac{y_N^+}{3} + \frac{1}{n_-}\frac{2\pi}{(\varepsilon/n_-)}\frac{y_N^-}{3}\right],\\ &= \frac{3(b_1^+ b_2^+ - b_1^- b_2^-)}{2\varepsilon}N = -\frac{3(q_2\phi_1 + q_1\phi_2)}{4n_+ n_-}N,\end{aligned} \tag{37}$$

with the same result for Σ_S (up to orientation). Evaluating this on the extremal values ϕ_i^*, ε^*, one can verify the result agrees with that computed using the explicit supergravity solutions in [24].

Similar calculations reproduce central charges and scaling dimensions for many other classes of $\mathrm{AdS}_5 \times M_6$ solutions, including all those in [22], the M5-branes wrapped on general Riemann surfaces in [31], and also new results for which explicit supergravity solutions have not been constructed [30]. One only needs to input topological data for the solutions, as we have done above.

Discussion.—The general structure we have uncovered in supergravity is ripe for many further applications and generalizations. We certainly expect analogous results to hold for more general supergravity theories, including coupling to matter multiplets, and including higher derivative corrections (see, in particular, [32–34]).

We have focused on supergravity geometries in even dimensions, but generalizations to odd dimensions are also possible. This should lead to a derivation of entropy functions for supersymmetric black holes in diverse dimensions, generalizing our derivation of the on-shell action (18) and entropy function (19) (see also [35]), and also gravitational block formulas that have been discovered in GK geometry [25,36], and generalizations thereof. The latter is very much related to the computation of anomaly polynomials in field theory, and it would be interesting to make contact with the gravitational approach in [37]. We will report on many of these topics in the near future [30].

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