Editors' Suggestion

Gravitational-Wave Phasing of Quasicircular Compact Binary Systems to the Fourth-and-a-Half Post-Newtonian Order

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(Received 16 May 2023; accepted 10 July 2023; published 21 September 2023)

The inspiral phase of gravitational waves emitted by spinless compact binary systems is derived through the fourth-and-a-half post-Newtonian (4.5PN) order beyond quadrupole radiation, and the leading amplitude mode $(\ell, \mathrm{m}) = (2,2)$ is obtained at 4PN order. We also provide the radiated flux, as well as the phase in the stationary phase approximation. Rough numerical estimates for the contribution of each PN order are provided for typical systems observed by current and future gravitational wave detectors.

DOI: 10.1103/PhysRevLett.131.121402

At the time when the LIGO and Virgo gravitational-wave detectors were approved, there was no theoretical prediction available for gravitational waves (GWs) generated by compact binary systems, apart from that of the famous Einstein quadrupole formula [1–3]. However, it was soon realized that, given the frequency band and the expected sensitivity of these ground-based detectors, the waveform modeling was to be drastically improved in order to extract all the potential information from the signal, at least in the case of the inspiral of two neutron stars [4,5]. The breakthrough came with the merging of the post-Newtonian (PN) and the multipolar post-Minkowskian (MPM) expansions into a single formalism [6–10], that was applied with success to derive step by step the waveform of compact binary systems up to 3.5PN order [11–20] (the results are also known at 4PN order for the spin-orbit coupling and the spin-spin coupling; see, e.g., Refs. [21,22]). Since then, many works (outlined below) have aimed at extending the precision of this result to the next level, namely 4PN or even 4.5PN order beyond the Einstein quadrupole formula.

This Letter provides the final results of these efforts, i.e., the GW phasing of nonspinning compact binary systems on quasicircular orbits up to 4.5PN order, as well as the dominant GW mode, given by $(\ell, \mathrm{m}) = (2,2)$, at 4PN order. Ready to be used for building accurate PN template banks for the detection and analysis of the inspiral phase of compact binaries, they should be important for third-generation ground-based detectors (Einstein Telescope and Cosmic Explorer), future space-borne detectors (LISA and TianQin), and of course the current second-generation detectors (LIGO, Virgo, and KAGRA). All results presented in this Letter are to be found in the ancillary file [23] associated with the companion paper [24].

Besides improving the detectors' data analysis, the motivation for computing high PN orders is also to perform high-accuracy tests of general relativity (GR), since the PN coefficients directly probe the nonlinear structure of the theory. By confronting results from the PN expansion against data, one can put constraints on potential deviations from GR [25,26]. This has already allowed for the confirmation of the signature of GW tails [27,28], and is promising for tests with future multiband detections between LISA and ground-based detectors [29].

We denote by $f(t)$ the frequency of the dominant (2,2) mode of the GW as measured by an observer in the asymptotically flat region far from the source (recall that this is twice the orbital frequency), and by $\psi(t) = \pi \int dt\, f(t)$ the corresponding half phase. As usual, we define the directly measurable PN parameter $x = \mathcal{O}(c^{-2})$ by

$$x \equiv \left(\frac{\pi G m f}{c^3}\right)^{2/3}, \tag{1}$$

where $m = m_1 + m_2$ is the binary's total mass, m_1 and m_2 being the constant masses of the progenitors. For circular orbits, x may be defined invariantly from the Killing vector of the helical symmetry in the asymptotically flat spacetime. Since compact binaries tend to have circularized by the time they enter the detector's frequency band [30], we only consider the case of quasicircular orbits, for which the time evolution of the frequency and phase (or "chirp") is entirely driven by the energy flux-balance equation,

$$\frac{dE}{dt} = -\mathcal{F}, \tag{2}$$

where E denotes the invariant energy of the compact binary and $\mathcal{F}$ the total energy flux (or GW luminosity). Both E and $\mathcal{F}$ in the balance equation are unique functions of the PN parameter x and the two masses. They have to be evaluated with the same relative PN precision, in the present case 4.5PN ($\sim x^{9/2}$). From Eq. (2), we derive a simple ordinary differential equation for the frequency as a function of time, and, once it is solved, a further integration yields the phase as a function of frequency.

The invariant energy E follows from the conservative dynamics of the compact binary at 4PN order, which have been obtained by various groups using different methods: (i) the Arnowitt-Deser-Misner (ADM) Hamiltonian formalism [31–34] yielded the first derivation of the 4PN energy, although with an ambiguity parameter obtained by matching the near-zone computation to results imported from gravitational self-force (GSF) [35]; (ii) the Fokker Lagrangian formalism in harmonic coordinates [36–39] derived the complete result, without ambiguity and without resorting to GSF, by using a specific regularization procedure which was proven to be equivalent to dimensional regularization; (iii) the effective field theory approach [40–47] rederived the 4PN energy by using dimensional regularization. From this series of works, the binary's invariant energy was obtained as the Noetherian quantity associated with temporal translation, and reads at 4PN order (see, e.g., Refs. [48,49] for partial results up to 6PN order):

$$\begin{aligned} E = -\frac{m\nu c^2 x}{2}\bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 + \left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right]x^3 \\ & + \left[-\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_\mathrm{E} + \frac{448}{15}\ln(16x)\right)\nu \right. \\ & \left. + \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\right]x^4 + \mathcal{O}(x^5)\bigg\}. \end{aligned} \tag{3}$$

We denote by $\nu \equiv m_1 m_2/m^2$ the symmetric mass ratio (γ_E is the Euler constant). Since there are no terms of half-integer PN order for circular orbits, this expression is actually valid up to 4.5PN order (as indicated by the final error term).

The second input is the energy flux, which we have computed using the PN MPM formalism applied to compact binaries at 4.5PN beyond the leading quadrupole formula. Crucial to this computation was the recently completed source mass quadrupole moment at 4PN order [50–53], the source current quadrupole moment at 3PN order [54], and the nonlinear tail-of-memory effect [55,56]. We provide the technical details of the derivation in the companion paper [24], and report here only the final result:

$$\begin{aligned} \mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5\bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\ & + \left[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \right. \\ & + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\ & \left. + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\right]x^4 \\ & + \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \right. \\ & \left. - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right]\pi x^{9/2} + \mathcal{O}(x^5)\bigg\}. \end{aligned} \tag{4}$$

In the test-mass limit $\nu \to 0$, we exactly retrieve the result of linear black-hole (BH) perturbation theory [57–61]. Since BH perturbations have recently been extended numerically to second order in the mass ratio ν [62–64], it would be interesting to verify the consistency of this numerical result with the PN prediction [Eq. (4)]. Note also that in the case of black holes, the contributions due to the absorption by the BH horizons are not included in the PN calculation, and should be added separately. The BH absorption is a 4PN effect for Schwarzschild black holes [65], and a 2.5PN effect for spinning ones [66–70].

With both Eqs. (3) and (4) in hand, we apply the flux-balance equation (2) and readily obtain the time evolution of the GW frequency. Sophisticated techniques exist to increase the precision of the PN results and the overlap with numerical relativity [71–74], but we do not discuss them here and simply present the results in the form of a fully expanded Taylor PN series. We employ the time variable

$$\tau \equiv \frac{\nu c^3}{5Gm}(t_0 - t), \tag{5}$$

where t is the coordinate time in the asymptotic radiative coordinate system, and t_0 an integration constant. We have the freedom to redefine it as $t_0 \to t_0 + \alpha(Gm/c^3)$ where α is any constant, which amounts to the replacement $\tau \to \tau[1 + \alpha\nu/(5\tau)]$. Although t_0 is not uniquely defined, it might be formally interpreted as the instant of coalescence, when $x \to +\infty$, and then it satisfies $t_0 - t = \mathcal{O}(c^5)$ in the PN regime. At the 4PN order, using the fact that $\tau^{-1} = \mathcal{O}(c^{-8})$ is a small 4PN quantity, we conveniently adjust α so as to simplify as much as possible the result:

$$\begin{aligned}
x = \frac{\tau^{-1/4}}{4}\Bigg\{ & 1 + \left(\frac{743}{4032} + \frac{11}{48}\nu\right)\tau^{-1/4} - \frac{1}{5}\pi\tau^{-3/8} + \left(\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2\right)\tau^{-1/2} + \left(-\frac{11891}{53760} + \frac{109}{1920}\nu\right)\pi\tau^{-5/8} \\
& + \left[-\frac{10052469856691}{6008596070400} + \frac{1}{6}\pi^2 + \frac{107}{420}\gamma_E - \frac{107}{3360}\ln\left(\frac{\tau}{256}\right) + \left(\frac{3147553127}{780337152} - \frac{451}{3072}\pi^2\right)\nu - \frac{15211}{442368}\nu^2 + \frac{25565}{331776}\nu^3\right]\tau^{-3/4} \\
& + \left(-\frac{113868647}{433520640} - \frac{31821}{143360}\nu + \frac{294941}{3870720}\nu^2\right)\pi\tau^{-7/8} \\
& + \Bigg[-\frac{2518977598355703073}{3779358859513036800} + \frac{9203}{215040}\gamma_E + \frac{9049}{258048}\pi^2 + \frac{14873}{1128960}\ln 2 + \frac{47385}{1605632}\ln 3 - \frac{9203}{3440640}\ln\tau \\
& + \left(\frac{718143266031997}{576825222758400} + \frac{244493}{1128960}\gamma_E - \frac{65577}{1835008}\pi^2 + \frac{15761}{47040}\ln 2 - \frac{47385}{401408}\ln 3 - \frac{244493}{18063360}\ln\tau\right)\nu \\
& + \left(-\frac{1502014727}{8323596288} + \frac{2255}{393216}\pi^2\right)\nu^2 - \frac{258479}{33030144}\nu^3 + \frac{1195}{262144}\nu^4\Bigg]\tau^{-1}\ln\tau \\
& + \Bigg[-\frac{9965202491753717}{5768252227584000} + \frac{107}{600}\gamma_E + \frac{23}{600}\pi^2 - \frac{107}{4800}\ln\left(\frac{\tau}{256}\right) \\
& + \left(\frac{8248609881163}{2746786775040} - \frac{3157}{30720}\pi^2\right)\nu - \frac{3590973803}{20808990720}\nu^2 - \frac{520159}{1634992128}\nu^3\Bigg]\pi\tau^{-9/8} + \mathcal{O}(\tau^{-5/4})\Bigg\}.
\end{aligned} \tag{6}$$

Then, the GW half phase ψ of the dominant harmonics is related to the binary's orbital phase ϕ by

$$\psi = \phi - \frac{2\pi G \mathrm{M} f}{c^3}\ln\left(\frac{f}{f_0}\right), \tag{7}$$

where M is the ADM mass of the binary, and f_0 is an arbitrary unphysical scale, reflecting the different origins of time between the local coordinates covering the source and the radiative coordinates. The logarithmic phase modulation was determined in Refs. [75,76] and is physically due to the scattering of GWs on the Schwarzschild background associated with M (i.e., GW tails). While the GW half phase ψ and the corresponding GW frequency $f = \dot{\psi}/\pi$ are directly measurable, the orbital phase ϕ can only be inferred via the theoretical prediction [Eq. (7)]. When expressing the results in terms of the GW observables ψ and f, the arbitrary scale f_0 is canceled out (see Sec. VI B in the detailed paper [24]). The explicit expression of the time-domain GW half phase $\psi(t)$ in terms of $x(t)$ [given by Ref. (6)] reads as

$$\begin{aligned}\psi = \psi_0 - \frac{x^{-5/2}}{32\nu}\bigg\{ & 1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\ln x \\ & + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{856}{21}\ln(16x) + \left(-\frac{15737765635}{12192768}+\frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right]x^3 \\ & + \left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\ & + \left[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_\mathrm{E} - \frac{45245}{756}\pi^2 - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 - \frac{9203}{252}\ln x \right. \\ & + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_\mathrm{E} + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323}\ln 2 + \frac{78975}{392}\ln 3 - \frac{244493}{1323}\ln x\right)\nu \\ & \left. + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\right]x^4 + \left[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_\mathrm{E} + \frac{80}{3}\pi^2 + \frac{856}{21}\ln(16x) \right. \\ & \left. + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\right]\pi x^{9/2} + \mathcal{O}(x^5)\bigg\}, \end{aligned} \tag{8}$$

where the integration constant ψ_0 is determined by initial conditions, e.g., when the wave frequency enters the detector's band. The results above, i.e., Eqs. (6)–(8) give the prediction of Einstein's general relativity for the GW frequency and phase chirp of nonspinning compact binaries up to 4.5PN precision.

Up to now we dealt with the time-domain GW half phase $\psi(t)$. It is useful (especially for data-analysis purposes) to also control the frequency-domain GW half phase, which we denote $\Psi(F)$. Its PN expansion is obtained by using the stationary phase approximation (SPA) [77] and reads as

$$\begin{aligned}\Psi_\mathrm{SPA} = 2\pi F T_0 + \Psi_0 + \frac{3v^{-5}}{128\nu}\bigg\{ & 1+\left(\frac{3715}{756}+\frac{55}{9}\nu\right)v^2 - 16\pi v^3 + \left(\frac{15293365}{508032}+\frac{27145}{504}\nu+\frac{3085}{72}\nu^2\right)v^4 + \left(\frac{38645}{252}-\frac{65}{3}\nu\right)\pi v^5 \ln v \\ & + \left[\frac{11583231236531}{4694215680} - \frac{640}{3}\pi^2 - \frac{6848}{21}\gamma_\mathrm{E} - \frac{6848}{21}\ln(4v) + \left(-\frac{15737765635}{3048192}+\frac{2255}{12}\pi^2\right)\nu + \frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3\right]v^6 \\ & + \left[\frac{77096675}{254016}+\frac{378515}{1512}\nu-\frac{74045}{756}\nu^2\right]\pi v^7 \\ & + \left[-\frac{2550713843998885153}{276808510218240} + \frac{90490}{189}\pi^2 + \frac{36812}{63}\gamma_\mathrm{E} + \frac{1011020}{1323}\ln 2 + \frac{78975}{196}\ln 3 + \frac{18406}{63}\ln v \right. \\ & + \left(\frac{680712846248317}{42247941120} - \frac{109295}{224}\pi^2 + \frac{3911888}{1323}\gamma_\mathrm{E} + \frac{9964112}{1323}\ln 2 - \frac{78975}{49}\ln 3 + \frac{1955944}{1323}\ln v\right)\nu \\ & \left. + \left(-\frac{7510073635}{3048192} + \frac{11275}{144}\pi^2\right)\nu^2 - \frac{1292395}{12096}\nu^3 + \frac{5975}{96}\nu^4\right]v^8 \ln v + \left[\frac{105344279473163}{18776862720} - \frac{640}{3}\pi^2 - \frac{13696}{21}\gamma_\mathrm{E} \right. \\ & \left. - \frac{13696}{21}\ln(4v) + \left(-\frac{1492917260735}{134120448} + \frac{2255}{6}\pi^2\right)\nu + \frac{45293335}{127008}\nu^2 + \frac{10323755}{199584}\nu^3\right]\pi v^9 + \mathcal{O}(v^{10})\bigg\}, \end{aligned} \tag{9}$$

where $v \equiv (\pi G m F/c^3)^{1/3}$ with F being the Fourier frequency, and where T_0 and Ψ_0 are two integration constants. Again we have adjusted T_0 in order to simplify the result (and we have absorbed the usual $-(\pi/4)$ into Ψ_0). The coefficients up to 3.5PN, as well as the 4.5PN piece, are already in use; see, e.g., Appendix A of Ref. [78].

In order to get intuition on the relative contribution of each PN order to the signal, we provide in Table I rough numerical estimates for the number of accumulated GW cycles in the frequency band of current and future detectors. Our naive estimation does not take the various detector noises into account, and a more realistic estimation should be performed [79]. Nevertheless, it can be useful to gain insight on the behavior of the PN expansion, which seems to converge well, as we see from Table I. For all the typical compact binaries in Table I, we find that the 4PN and 4.5PN orders amount to about a tenth of a cycle (less than 1 radian). This suggests that systematic errors due to the PN modeling may be dominated by statistical errors and negligible for LISA. However, this should be confirmed by detailed investigations along the lines of Ref. [80].

TABLE I. Contribution of each PN order to the total number of accumulated cycles inside the detector's frequency band, for typical (but nonspinning) quasicircular compact binaries observed by current and future detectors. We have approximated the frequency bands of LIGO-Virgo, Einstein Telescope (ET), and LISA with step functions, respectively between $[30, 10^3]$ Hz, $[1, 10^4]$ Hz and $[10^{-4}, 10^{-1}]$ Hz. When the merger occurs within the frequency band of the detector, the exit frequency is taken to be the Schwarzschild innermost stable circular orbit (ISCO), $f_{\rm ISCO} = c^3/(6^{3/2}\pi G m)$. The contributions due to the nonlinearities of GR (e.g., tails) increase with the PN order and are detailed in Ref. [24].

Detector	LIGO-Virgo		ET		LISA	
Masses ($M_\odot$)	1.4×1.4	10×10	1.4×1.4	500×500	$10^5\times 10^5$	$10^7\times 10^7$
PN order	Cumulative number of cycles					
Newtonian	2 562.599	95.502	744 401.36	37.90	28 095.39	9.534
1PN	143.453	17.879	4 433.85	9.60	618.31	3.386
1.5PN	−94.817	−20.797	−1 005.78	−12.63	−265.70	−5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	−8.105	−4.604	−17.01	−3.42	−12.47	−1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	−0.627	−0.689	−0.93	−0.59	−0.91	−0.383
4PN	−0.107	−0.064	−0.12	−0.04	−0.12	−0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

Besides the chirp described by the results displayed in Eqs. (6)–(8), it is also important to compute the wave amplitude, in view of the data analysis of LISA [81–83] and high-accuracy comparisons with numerical relativity (see, e.g., Refs. [84–87]). We decompose the waveform, at leading order in the distance R to the source, onto a basis of spin-weighted spherical harmonics (following the conventions of Refs. [88,89]):

$$h_+ - i h_\times = \frac{8 G m \nu x}{R c^2}\sqrt{\frac{\pi}{5}}\sum_{\ell=2}^{+\infty}\sum_{\mathrm{m}=-\ell}^{\ell} H_{\ell\mathrm{m}} e^{-i\mathrm{m}\psi} Y_{-2}^{\ell\mathrm{m}}, \quad (10)$$

where the half-phase variable is given by Eq. (8). All $H_{\ell\mathrm{m}}$ modes are currently known at 3.5PN order for spinning, nonprecessing, quasicircular orbits [88–92]. Although we were able to derive the phase with 4.5PN accuracy, the same precision for the modes is yet out of reach, since, even though the 4.5PN radiation-reaction terms in the equations of motion are known [93,94], neither the source quadrupole moment nor the nonlinear contributions to the GW propagation are fully controlled at 4.5PN order (only the contributions that enter the 4.5PN flux for circular orbits are known). We thus report the extension of the dominant quadrupole mode $(\ell, \mathrm{m}) = (2,2)$ for nonspinning, quasicircular orbits up to 4PN order:

$$\begin{aligned}
H_{22} = 1 &+ \left(-\frac{107}{42}+\frac{55}{42}\nu\right)x + 2\pi x^{3/2} + \left(-\frac{2173}{1512}-\frac{1069}{216}\nu+\frac{2047}{1512}\nu^2\right)x^2 + \left[-\frac{107\pi}{21}+\left(\frac{34\pi}{21}-24i\right)\nu\right]x^{5/2}\\
&+ \left[\frac{27027409}{646800}-\frac{856}{105}\gamma_\mathrm{E}+\frac{428 i\pi}{105}+\frac{2\pi^2}{3}+\left(-\frac{278185}{33264}+\frac{41\pi^2}{96}\right)\nu-\frac{20261}{2772}\nu^2+\frac{114635}{99792}\nu^3-\frac{428}{105}\ln(16x)\right]x^3\\
&+ \left[-\frac{2173\pi}{756}+\left(-\frac{2495\pi}{378}+\frac{14333 i}{162}\right)\nu+\left(\frac{40\pi}{27}-\frac{4066 i}{945}\right)\nu^2\right]x^{7/2}\\
&+ \left[-\frac{846557506853}{12713500800}+\frac{45796}{2205}\gamma_\mathrm{E}-\frac{22898}{2205}i\pi-\frac{107}{63}\pi^2+\frac{22898}{2205}\ln(16x)\right.\\
&+ \left(-\frac{336005827477}{4237833600}+\frac{15284}{441}\gamma_\mathrm{E}-\frac{219314}{2205}i\pi-\frac{9755}{32256}\pi^2+\frac{7642}{441}\ln(16x)\right)\nu\\
&+ \left.\left(\frac{256450291}{7413120}-\frac{1025}{1008}\pi^2\right)\nu^2-\frac{81579187}{15567552}\nu^3+\frac{26251249}{31135104}\nu^4\right]x^4+\mathcal{O}(x^{9/2}). \quad (11)
\end{aligned}$$

Satisfyingly, this result is in perfect agreement with linear black-hole perturbation theory in the limit when $\nu \to 0$; see Appendix B of Ref. [58]. Again, it would be interesting to compare the PN prediction of Eq. (11) with second-order (numerical or analytical) BH perturbation theory.

We thank Adam Pound for discussions on BH perturbation and GSF results, Bala Iyer for his longstanding support and interest in this project, and Alessandra Buonanno for suggestions. F. L. is grateful to the Institut d'Astrophysique de Paris for its hospitality during this project. He received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 817791). G. F. thanks IIT Madras for the support from a Visiting Faculty Fellow position under the Institute of Eminence (IoE) program during the completion of this work.

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