Distributed Quantum Incompatibility

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Incompatible, i.e., nonjointly measurable quantum measurements are a necessary resource for many information processing tasks. It is known that increasing the number of distinct measurements usually enhances the incompatibility of a measurement scheme. However, it is generally unclear how large this enhancement is and on what it depends. Here, we show that the incompatibility which is gained via additional measurements is upper and lower bounded by certain functions of the incompatibility of subsets of the available measurements. We prove the tightness of some of our bounds by providing explicit examples based on mutually unbiased bases. Finally, we discuss the consequences of our results for the nonlocality that can be gained by enlarging the number of measurements in a Bell experiment.

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The incompatibility of quantum measurements, i.e., the impossibility of measuring specific observable quantities simultaneously, is one of quantum physics' most prominent and striking properties. First discussed by Heisenberg [1] and Robertson [2], this counterintuitive feature was initially thought of as a puzzling curiosity that represents a drawback for potential applications. Nowadays, measurement incompatibility [3,4] is understood as a fundamental property of nature that lies at the heart of many quantum information processing tasks, such as quantum state discrimination [5–10], quantum cryptography [11,12], and quantum random access codes [13,14]. Even more importantly, incompatible measurements are a crucial requirement for quantum phenomena such as quantum contextuality [15], Einstein-Podolsky-Rosen (EPR) steering [16,17], and Bell nonlocality [18].

Its fundamental importance necessitates gaining a deep understanding of measurement incompatibility from a qualitative and quantitative perspective. By its very definition, measurement incompatibility arises when at least $m \geq 2$ measurements are considered that cannot be measured jointly by performing a single measurement instead. Generally, adding more measurements to a measurement scheme may allow for more incompatibility, hence increasing advantages in certain applications.

However, it is unclear how much incompatibility can be gained from adding further measurements to an existing measurement scheme and on what this potential gain depends. Similarly, it is unclear how the incompatibility of measurement pairs contributes towards the total incompatibility of the whole set. Answering these questions is crucial to understanding specific protocols' power over others, such as protocols involving different numbers of mutually unbiased bases (MUBs) in quantum key distribution [11,19]. While it is known [20] that the different incompatibility structures (e.g., genuine triplewise and pairwise incompatibility) arising for $m \geq 3$ measurements set different limitations on the violation of Bell inequalities and incompatibility structures beyond two measurements have also been studied in [21–23], so far, no systematical way to quantify the gained advantage is known.

The systematical and quantitative analysis of incompatibility structures in this work is inspired by the analysis of the distribution of multipartite entanglement [24] and coherence [25], leading to the observation that these quantum resources behave monogamously across subsets of systems. Despite the mathematical differences, our work follows physically a similar path by studying the distribution of quantum incompatibility across subsets of measurements. Namely, we show how an assemblage's incompatibility depends quantitatively on its subsets' incompatibilities. More specifically, we show how the potential gain of adding measurements to an existing measurement scheme is bounded by the incompatibility of the parent positive operator valued measures (POVMs) that approximate the respective subsets of measurements by a single measurement.

Our results reveal the polygamous nature of measurement incompatibility in the sense that an assemblage of more than two measurements can only be highly incompatible if all its subsets and the respective parent POVMs of the closest jointly measurable approximation of these subsets are highly nonjointly measurable. Our considerations lead to a new notion of measurement incompatibility that accounts only for a specific measurement's incompatibility contribution. We prove the relevance of our bounds on the incompatibility that can maximally be gained by showing that they are tight for particular measurement assemblages based on MUBs. Finally, we show that our results have direct consequences for steering and Bell nonlocality and discuss future applications of our results and methods.

Preliminaries.—We describe a quantum measurement most generally by a POVM, i.e., a set $\{M_a\}$ of operators

$0 \leq M_a \leq \mathbb{1}$ such that $\sum_a M_a = \mathbb{1}$. Given a state ρ, the probability of obtaining outcome a is given by the Born rule $p(a) = \mathrm{Tr}[M_a \rho]$. A measurement assemblage is a collection of different POVMs with operators $M_{a|x}$, where x denotes the particular measurement. We write an assemblage $\mathcal{M}_{(1,2,\ldots,m)} = (\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m)$ of m measurements as an ordered list of POVMs, where $\mathcal{M}_x$ refers to the xth measurement. For instance, $\mathcal{M}_{(1,2,3)} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ refers to an assemblage with three (different) measurements and $\mathcal{M}_{(1,2,2)} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_2)$ denotes an assemblage where the second and the third POVM are equal.

An assemblage $\mathcal{M}$ is called jointly measurable if it can be simulated by a single *parent POVM* $\{G_\lambda\}$ and conditional probabilities $p(a|x,\lambda)$ such that

$$M_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda \quad \forall \; a, x, \tag{1}$$

and it is called incompatible otherwise. Here, we call $G(\mathcal{M})$ a parent POVM of a jointly measurable assemblage $\mathcal{M}$. Various functions can quantify measurement incompatibility [26–28]. The most suitable incompatibility quantifier for our purposes is the recently introduced diamond distance quantifier [29], given by

$$I_\diamond(\mathcal{M}^{\mathbf{p}}) = \min_{\mathcal{F} \in \mathrm{JM}} \sum_x p(x) D_\diamond(\Lambda_{\mathcal{M}_x}, \Lambda_{\mathcal{F}_x}), \tag{2}$$

where JM denotes the set of jointly measurable assemblages, $\Lambda_{\mathcal{M}_x} = \sum_a \mathrm{Tr}[M_{a|x}\rho] |a\rangle\langle a|$ is the measure-and-prepare channel associated to the measurement $\mathcal{M}_x$, and $D_\diamond(\Lambda_1, \Lambda_2) = \max_{\rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})} \frac{1}{2} \| [(\Lambda_1 - \Lambda_2) \otimes \mathbb{1}_d] \rho \|_1$ is the *diamond distance* [30] between two channels Λ_1 and Λ_2, with the trace norm $\|X\|_1 = \mathrm{Tr}[\sqrt{X^\dagger X}]$. Furthermore, $\mathcal{M}^{\mathbf{p}} = (\mathcal{M}, \mathbf{p})$ denotes a weighted measurement assemblage, where $\mathbf{p}$ contains the probabilities $p(x)$ with which measurement x is performed. Note that $I_\diamond(\mathcal{M}^{\mathbf{p}})$ is induced by the general distance $D_\diamond(\mathcal{M}^{\mathbf{p}}, \mathcal{N}^{\mathbf{p}}) \coloneqq \sum_x p(x) D_\diamond(\Lambda_{\mathcal{M}_x}, \Lambda_{\mathcal{N}_x})$ between two assemblages $\mathcal{M}^{\mathbf{p}}$ and $\mathcal{N}^{\mathbf{p}}$.

We denote by $\mathcal{M}^{\#}_{(1,2,\ldots,m)}$ the closest jointly measurable assemblage with respect to $\mathcal{M}_{(1,2,\ldots,m)}$, i.e., the arg min on the rhs in Eq. (2). While $\mathcal{M}^{\#}_{(1,2,\ldots,m)}$ and its underlying parent POVM are generally not unique [22,31], all the results derived in this work hold for any valid choice, as we do not assume uniqueness. If we only approximate a subset of $n < m$ measurements of $\mathcal{M}_{(1,2,\ldots,m)}$ by jointly measurable ones, for instance the first n settings, while keeping the remaining measurements unchanged, we write $\mathcal{M}^{\#(1,2,\ldots,n)}_{(1,2,\ldots,m)}$.

The diamond distance quantifier $I_\diamond(\mathcal{M}^{\mathbf{p}})$ [29] is particularly well suited for our purposes, as it is not only monotonous under the application of quantum channels and classical simulations but it also inherits all properties of a distance [in particular the triangle inequality of $D_\diamond(\mathcal{M}^{\mathbf{p}}, \mathcal{N}^{\mathbf{p}})$], and it is written in terms of a convex combination of the individual measurement's distances.

Besides these technical requirements, the quantifier $I_\diamond(\mathcal{M}^{\mathbf{p}})$ admits the operational interpretation of average single-shot distinguishability of the assemblage $\mathcal{M}$ from its closest jointly measurable assemblage $\mathcal{M}^{\#}$. Furthermore, it can be used to upper bound the amount of steerability and nonlocality that can be revealed by the measurements $\mathcal{M}$ in Bell-type experiments [29].

For pedagogical reasons, we focus in the main text on the scenario $2 \to 3$, i.e., we consider an assemblage of $m = 2$ measurements that is promoted to one with $m' = 3$ settings. Furthermore, we set $p(x)$ to be uniformly distributed and simply use the symbol $\mathcal{M}$ for the weighted assemblage in this case. We refer to the Supplemental Material (SM) [32] for all proofs, more background information, and generalizations to an arbitrary number of measurements and general probability distributions.

Adding a third measurement $\mathcal{M}_3$ to the assemblage $\mathcal{M}_{(1,2)} = (\mathcal{M}_1, \mathcal{M}_2)$ is mathematically described by the concatenation of ordered lists, using the symbol $+\!\!+$, i.e., we write

$$\mathcal{M}_{(1,2,3)} = \mathcal{M}_{(1,2)} +\!\!+ \mathcal{M}_3 = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3). \tag{3}$$

Using the concatenation of ordered lists, we formally define $\mathcal{M}^{\#(1,2)}_{(1,2,3)}$ such that

$$\mathcal{M}^{\#(1,2)}_{(1,2,3)} \coloneqq \mathcal{M}^{\#}_{(1,2)} +\!\!+ \mathcal{M}_3. \tag{4}$$

Three measurements allow for incompatibility structures [20–23] beyond Eq. (1). We define the sets $\mathrm{JM}^{(s,t)}$ with $s \neq t \in \{1,2,3\}$ as those containing assemblages in which the measurements s and t are jointly measurable. This allows us to define pairwise and genuinely triplewise incompatible assemblages [20] as those that are *not* contained in the intersection and the convex hull of the sets $\mathrm{JM}^{(s,t)}$, respectively. See also Fig. 1 for a graphical representation and more details.

Incompatibility gain.—We investigate the incompatibility gain obtained from adding measurements to an already available assemblage. That is, for an assemblage $\mathcal{M}_{(1,2,3)}$ defined via Eq. (3) we want to quantify the gain

$$\Delta I_{(1,2)\to(1,2,3)} \coloneqq I_\diamond(\mathcal{M}_{(1,2,3)}) - I_\diamond(\mathcal{M}_{(1,2)}). \tag{5}$$

Note that $\Delta I_{(1,2)\to(1,2,3)}$ is the difference of two quantities that can be computed via semidefinite programs (SDPs) [29], however, the purely numerical value of the gained incompatibility does only provide limited physical insights by itself. While it seems generally challenging to find an exact analytical expression for the incompatibility gain, we will derive bounds on it in the following.

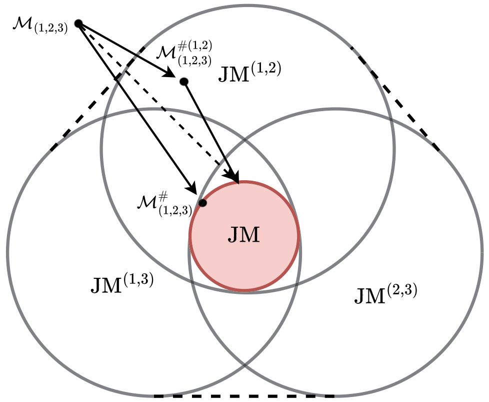

FIG. 1. Different structures of incompatibility for three measurements, see also Ref. [20]. The sets $\mathrm{JM}^{(s,t)}$ contain assemblages of measurements where the pairs (s,t) are compatible. Their intersection $\mathrm{JM}^{\mathrm{pair}} \coloneqq \mathrm{JM}^{(1,2)} \cap \mathrm{JM}^{(1,3)} \cap \mathrm{JM}^{(2,3)}$ contains all pairwise compatible assemblages, with the set JM of all jointly measurable assemblages as a proper subset. Assemblages not contained in the convex hull $\mathrm{JM}^{\mathrm{conv}} \coloneqq \mathrm{Conv}(\mathrm{JM}^{(1,2)}, \mathrm{JM}^{(1,3)}, \mathrm{JM}^{(2,3)})$ of the sets $\mathrm{JM}^{(s,t)}$, i.e., those that cannot be written as a convex combination of assemblages from the sets $\mathrm{JM}^{(1,2)}$, $\mathrm{JM}^{(1,3)}$, and $\mathrm{JM}^{(2,3)}$ are genuinely triplewise incompatible. The incompatibility of $\mathcal{M}_{(1,2,3)}$ is given by the distance to its closest jointly measurable approximation $\mathcal{M}^{\#}_{(1,2,3)}$. This distance can be upper bounded using the triangle inequality via the assemblage $\mathcal{M}^{\#(1,2)}_{(1,2,3)}$.

Our approach relies on a two-step protocol. First, we employ a measurement splitting, i.e., instead of considering the incompatibility of $\mathcal{M}_{(1,2,3)}$, we consider the incompatibility $I_\diamond(\mathcal{M}_{(1,2,1,3,2,3)})$. That is, each measurement of $\mathcal{M}_{(1,2,3)}$ is now split up into two equivalent ones, each occurring with a probability of $\frac{1}{6}$. Furthermore, it holds $I_\diamond(\mathcal{M}_{(1,2,3)}) = I_\diamond(\mathcal{M}_{(1,2,1,3,2,3)})$ since the assemblages can be converted into each other by (reversible) classical postprocessing [32] (Sec. II). The second step involves a particular instance of the triangle inequality and uses specifically that $I_\diamond(\mathcal{M})$ is defined as a convex combination over the individual settings. More precisely, let

$$\mathcal{N} = \mathcal{M}^{\#}_{(1,2)} \text{⧺} \mathcal{M}^{\#}_{(1,3)} \text{⧺} \mathcal{M}^{\#}_{(2,3)}, \tag{6}$$

be an assemblage that contains itself three assemblages (of two measurements each) that are the closest jointly measurable approximations with respect to the individual subsets of $\mathcal{M}_{(1,2,3)}$. We point out that $\mathcal{N}$ itself can be incompatible in general. Using the triangle inequality, it follows that

$$\begin{aligned} I_\diamond(\mathcal{M}_{(1,2,3)}) &= I_\diamond(\mathcal{M}_{(1,2,1,3,2,3)}) \\ &\leq D_\diamond(\mathcal{M}_{(1,2,1,3,2,3)}, \mathcal{N}) + I_\diamond(\mathcal{N}). \end{aligned} \tag{7}$$

Because of our choice of $\mathcal{N}$, the term $D_\diamond(\mathcal{M}_{(1,2,1,3,2,3)}, \mathcal{N})$ evaluates to the average incompatibility of the subsets, as we can split the sum over all six settings into three pairs, i.e., we obtain

$$\begin{aligned} I_\diamond(\mathcal{M}_{(1,2,3)}) \leq \frac{1}{3}[&I_\diamond(\mathcal{M}_{(1,2)}) + I_\diamond(\mathcal{M}_{(1,3)}) \\ &+ I_\diamond(\mathcal{M}_{(2,3)})] + I_\diamond(\mathcal{N}). \end{aligned} \tag{8}$$

That is, the incompatibility of $\mathcal{M}_{(1,2,3)}$ is upper bounded by the average incompatibility of its two-measurement subsets plus the incompatibility $I_\diamond(\mathcal{N})$ that contains the information about how incompatible the respective closest jointly measurable POVMs are with each other. Notice that $I_\diamond(\mathcal{N}) \leq I_\diamond(\mathcal{G})$ holds, where

$$\mathcal{G} = G(\mathcal{M}^{\#}_{(1,2)}) \text{⧺} G(\mathcal{M}^{\#}_{(1,3)}) \text{⧺} G(\mathcal{M}^{\#}_{(2,3)}) \tag{9}$$

is the assemblage that contains the parent POVMs G of the respective subsets, as $\mathcal{N}$ is a classical postprocessing of $\mathcal{G}$ [32] (Sec. II). This shows that the incompatibility of $\mathcal{M}_{(1,2,3)}$ is limited on two different levels through its subsets. Moreover, it reveals a type of polygamous behavior of incompatibility. For high incompatibility of $\mathcal{M}_{(1,2,3)}$ each of the subsets, as well as the underlying parent POVMs of the respective jointly measurable approximations, have to be highly incompatible. Coming back to the incompatibility gain, we are ready to present our first main result.

Result 1.—Let $I_\diamond(\mathcal{M}_{(1,2)}) \geq \max\{I_\diamond(\mathcal{M}_{(1,3)}), I_\diamond(\mathcal{M}_{(2,3)})\}$. It follows that the incompatibility gain as defined in Eq. (5) is bounded such that

$$\Delta I_{(1,2)\to(1,2,3)} \leq I_\diamond(\mathcal{N}) \leq I_\diamond(\mathcal{G}). \tag{10}$$

This means that the potential incompatibility gain is limited by the incompatibility of the assemblage $\mathcal{N}$ in Eq. (6), i.e., the concatenation of the respective closest jointly measurable approximations of the subsets. Physically more intuitive, it is limited by the incompatibility of the assemblage that contains the respective parent POVMs. The assumption $I_\diamond(\mathcal{M}_{(1,2)}) \geq \max\{I_\diamond(\mathcal{M}_{(1,3)}), I_\diamond(\mathcal{M}_{(2,3)})\}$ represents no loss of generality for all practical purposes, as one can simply optimize over all possible two-measurement subsets.

We show in the SM [32] that Result 1 can be generalized to

$$\Delta I_{(1,\ldots,m)\to(1,\ldots,m,m+1)} \leq I_\diamond(\mathcal{N}) \leq I_\diamond(\mathcal{G}), \tag{11}$$

by appropriately redefining $\mathcal{N}$ and $\mathcal{G}$.

We point out that Result 1 allows for the definition of a single maximally incompatible additional measurement, in the sense that it is the measurement $\mathcal{M}_3$ that maximizes the incompatibility gain $\Delta I_{(1,2)\to(1,2,3)}$ for a given assemblage

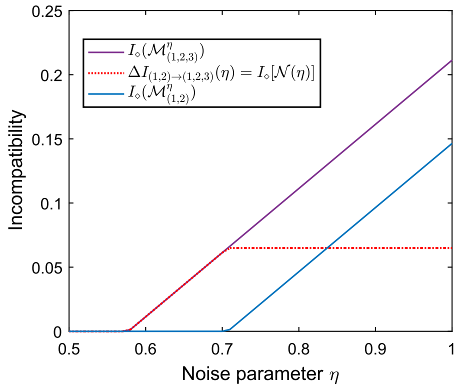

FIG. 2. Incompatibility gain for adding a third Pauli measurement. The gained incompatibility is given by the red (dotted) line. In the regime where $I_\diamond(\mathcal{M}^\eta_{(1,2)}) \neq 0$, the gained incompatibility remains constant. The red (dotted) curve and the blue curve add up to the violet one.

$\mathcal{M}_{(1,2)}$. As an illustrative example, we consider the three projective measurements $\{\Pi_{a|x}\}$ which represent the Pauli X, Y, Z observables subjected to white noise, i.e., we analyze the incompatibility of the assemblage $\mathcal{M}^\eta_{(1,2,3)} = (\mathcal{M}^\eta_1, \mathcal{M}^\eta_2, \mathcal{M}^\eta_3)$ defined via

$$M^\eta_{a|x} = \eta\Pi_{a|x} + (1-\eta)\mathrm{Tr}[\Pi_{a|x}]\frac{\mathbb{1}}{2}, \qquad (12)$$

where $(1-\eta)$ is the noise level. It holds in this particular case that (see Fig. 2)

$$\Delta I_{(1,2)\to(1,2,3)}(\eta) = I_\diamond[\mathcal{N}(\eta)], \qquad (13)$$

which we prove analytically in the SM [32] (Sec. VI). For the regime $(1/\sqrt{2}) \leq \eta \leq 1$ we also show that $I_\diamond[\mathcal{N}(\eta)] = I_\diamond(\mathcal{M}^{1/\sqrt{2}}_{(1,2,3)})$, which means that the gained incompatibility is exactly given by the incompatibility of $\mathcal{M}^\eta_{(1,2,3)}$ at the noise threshold where it becomes pairwise compatible.

Our methods can also be applied to obtain lower bounds. For instance, we show [32] (Sec. III) that $I_\diamond(\mathcal{M}_{(1,2,3)})$ is bounded by the average subset incompatibility:

$$I_\diamond(\mathcal{M}_{(1,2,3)}) \geq \frac{1}{3}[I_\diamond(\mathcal{M}_{(1,2)}) + I_\diamond(\mathcal{M}_{(1,3)}) + I_\diamond(\mathcal{M}_{(2,3)})]. \qquad (14)$$

In general, $I_\diamond(\mathcal{M}_{(1,2,3)}) < I_\diamond(\mathcal{M}_{(1,2)})$ is possible, i.e., adding a measurement to an assemblage can actually decrease the incompatibility, if we do not optimize over the input distribution $\mathbf{p}$. For instance, adding a measurement $\mathcal{M}_3$ that is jointly measurable with $\mathcal{M}^\#_{(1,2)}$, such as an identity measurement, generally decreases the incompatibility.

Another way to see how the incompatibility of an assemblage $\mathcal{M}_{(1,2,3)}$ can be upper bounded in terms of the incompatibility $I_\diamond(\mathcal{M}_{(1,2)})$ plus the gained incompatibility due to measurement $\mathcal{M}_3$ relies on directly applying specific instances of the triangle inequality without splitting the measurements.

A new notion of incompatibility.—Consider the general assemblage $\mathcal{M}_{(1,2,3)}$ as defined in Eq. (3). Because of the triangle inequality, see also Fig. 1, it holds

$$I_\diamond(\mathcal{M}_{(1,2,3)}) \leq D_\diamond(\mathcal{M}_{(1,2,3)}, \mathcal{N}_{(1,2,3)}) + I_\diamond(\mathcal{N}_{(1,2,3)}), \qquad (15)$$

for any assemblage $\mathcal{N}_{(1,2,3)}$. By choosing $\mathcal{N}_{(1,2,3)} = \mathcal{M}^{\#(1,2)}_{(1,2,3)} := \mathcal{M}^\#_{(1,2)} \mathbin{+\!\!\!+} \mathcal{M}_3$, we obtain our second main result.

Result 2.—Let $\mathcal{M}_{(1,2,3)} = \mathcal{M}_{(1,2)} \mathbin{+\!\!\!+} \mathcal{M}_3$ be a concatenated measurement assemblage and $\mathcal{M}^\#_{(1,2)}$ the closest jointly measurable approximation of $\mathcal{M}_{(1,2)}$. It holds

$$I_\diamond(\mathcal{M}_{(1,2,3)}) \leq \frac{2}{3} I_\diamond(\mathcal{M}_{(1,2)}) + I_\diamond\big(\mathcal{M}^{\#(1,2)}_{(1,2,3)}\big). \qquad (16)$$

This means that the incompatibility of $\mathcal{M}_{(1,2,3)}$ is upper bounded by the incompatibility of the subset $\mathcal{M}_{(1,2)}$, weighted with the probability $p = \frac{2}{3}$, plus the incompatibility of the added measurement $\mathcal{M}_3$ with the closest jointly measurable approximation $\mathcal{M}^\#_{(1,2)}$ of $\mathcal{M}_{(1,2)}$. In [32] (Sec. III) we also show that the incompatibility of $\mathcal{M}_{(1,2,3)}$ is lower bounded by

$$I_\diamond(\mathcal{M}_{(1,2,3)}) \geq \frac{2}{3} I_\diamond(\mathcal{M}_{(1,2)}). \qquad (17)$$

The only incompatibility that contributes to $I_\diamond(\mathcal{M}^{\#(1,2)}_{(1,2,3)})$ is the incompatibility of $\mathcal{M}_3$ with the assemblage $\mathcal{M}^\#_{(1,2)}$, which itself is jointly measurable. Therefore, this term in Eq. (16) can be understood as a new notion of incompatibility of the assemblage $\mathcal{M}_{(1,2,3)}$, where all incompatibilities apart of the contribution that comes from the presence of measurement $\mathcal{M}_3$ are omitted.

We show analytically in the SM [32] (Sec. VI) that the bound in Eq. (16) is tight for depolarized Pauli measurements [see Eq. (12)]. Moreover, we show analytically that a similar bound is tight for certain measurements based on d-dimensional MUBs in cases where the number of measurements m is changed such that $m = 2 \to m' = d$, $m = 2 \to m' = d+1$, and $m = d \to m' = d+1$. Namely, we prove and analyze the generalization of Eq. (16):

$$I_\diamond(\mathcal{M}_{(1,2,\ldots,m)}) \leq \frac{|C|}{m} I_\diamond(\mathcal{M}_C) + I_\diamond(\mathcal{M}^{\#C}_{(1,2,\ldots,m)}), \qquad (18)$$

for any assemblage $\mathcal{M}_{(1,2,\ldots,m)}$ and any subset C of measurements with cardinality $|C|$.

Incompatibility decomposition.—Looking at the results in Fig. 2 leads to the question of whether there exists a more general decomposition of $I_\diamond(\mathcal{M}_{(1,2,3)})$ into different incompatibility structures. Indeed, since $I_\diamond(\mathcal{M})$ is a distance-based incompatibility quantifier, our final main result follows.

Result 3.—The incompatibility of any assemblage $\mathcal{M}$ of $m = 3$ measurements is upper bounded such that

$$I_\diamond(\mathcal{M}) \leq I_\diamond^{\mathrm{gen}}(\mathcal{M}) + I_\diamond^{\mathrm{pair}}(\mathcal{M}) + I_\diamond^{\mathrm{hol}}(\mathcal{M}), \quad (19)$$

where $I_\diamond^{\mathrm{gen}}(\mathcal{M})$ is the genuine triplewise incompatibility of $\mathcal{M}$, i.e., its minimal distance to an assemblage $\mathcal{M}^{\mathrm{conv}} \in \mathrm{JM}^{\mathrm{conv}}$. Furthermore, we define $I_\diamond^{\mathrm{pair}}(\mathcal{M}) \coloneqq D_\diamond(\mathcal{M}^{\mathrm{conv}}, \mathcal{M}^{\mathrm{pair}})$ to be the pairwise incompatibility, where $\mathcal{M}^{\mathrm{pair}} \in \mathrm{JM}^{\mathrm{pair}}$ is the closest pairwise compatible assemblage with respect to $\mathcal{M}^{\mathrm{conv}}$. We call the term $I_\diamond^{\mathrm{hol}}(\mathcal{M}) \coloneqq I_\diamond(\mathcal{M}^{\mathrm{pair}})$ the hollow incompatibility, which implicitly depends on $\mathcal{M}$, see also Fig. 1 and Ref. [20].

We emphasize that the bound in Eq. (19) relies crucially on the distance properties of the quantifier $I_\diamond(\mathcal{M})$ and cannot be adapted directly to robustness or weight quantifiers [26,27]. In the SM [32] (Sec. VII) we show that the decomposition in Eq. (19) is tight for the three Pauli measurements, and give numerical indication that this is generally the case for measurements based on MUBs.

Implications for steering and Bell nonlocality.—Because of the mathematical structure of our methods, they can directly be applied to quantum steering and Bell nonlocality. Note that both of these phenomena occur in a scenario that is similar to the one for measurement incompatibility. Namely, they depend on the properties of a set of at least two measurements, while a single measurement by itself does not contain any resource. This distinguishes the above concepts from resource theories of single POVMs (see, e.g., [8,46,47]) where the resource gain can trivially be determined by considering averages of single POVM resources [29]. We describe our results regarding steering and nonlocality in more detail in the SM [32] (Sec. IV). The analysis of the gain in nonlocal correlations in Bell experiments is particularly interesting as it seems fundamentally different from incompatibility and steering. Consider a Bell experiment where Alice performs $m_A = 3$ and Bob $m_B = 2$ measurements. Focusing on dichotomic measurements, we observe the following intriguing effect: Alice cannot find three measurements, such that the three Clauser-Horne-Shimony-Holt (CHSH) inequalities [48] $\mathrm{CHSH}_{(i,j)} \coloneqq \langle A_i \otimes B_1\rangle + \langle A_i \otimes B_2\rangle + \langle A_j \otimes B_1\rangle - \langle A_j \otimes B_2\rangle \leq 2$ with $(i,j) \in \{(1,2),(1,3),(2,3)\}$ are simultaneously maximally violated. That means, $\mathrm{CHSH}_{(1,2,3)} \coloneqq \frac{1}{3}(\mathrm{CHSH}_{(1,2)} + \mathrm{CHSH}_{(1,3)} + \mathrm{CHSH}_{(2,3)}) \leq [(4\sqrt{2}+2)/3] < 2\sqrt{2}$ holds in quantum theory. This implies, that the average two-subset nonlocality is lower than the maximal obtainable nonlocality with two measurements on Alice's side.

Conclusion and outlook.—In this work, we analyzed how much incompatibility can maximally be gained by adding measurements to an existing measurement scheme. We showed that this gain is upper bounded by the incompatibility of the underlying parent POVMs that approximate subsets of measurements. We proved the relevance of our bounds analytically by showing that they are tight for specific measurements based on MUBs. Moreover, we showed that our methods are directly applicable to quantum steering and Bell nonlocality. For nonlocality specifically, we discovered a promising path to understand better why using more than two measurements may not provide any advantage for maximal nonlocal correlations [49,50]. Our results reveal the polygamous nature of distributed quantum incompatibility, in stark contrast to the monogamy of entanglement [24] and coherence [25] across subsystems of multipartite quantum states. While we focused in this text on $m = 3$ measurements, all our findings, in particular, Results 1–3 can be generalized to an arbitrary number of measurements m (see [32], Sec. V).

Our work provides a foundation for several new directions of research. While we focused on a particular distance-based quantifier here, the alternative distance-based quantifiers proposed in [29] do also possess the necessary properties to be used in a similar way. It would be interesting to see whether resource quantifiers such as the incompatibility robustness [27] or weight [26] can also be used to analyze how the incompatibility of an assemblage depends on its subsets. Our methods might also prove helpful to find better bounds on the incompatibility of general assemblages and particularly maximally incompatible assemblages. Finally, it would be interesting to analyze the performance gain of specific cryptography [11,19] or estimation protocols [51] with different numbers of measurements.

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