Nonreciprocal Dicke Model

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(Received 20 February 2023; accepted 15 August 2023; published 13 September 2023)

We investigate the physics of an open two-component Dicke model, where the light field mediates nonreciprocal interactions between two spin species. We show that the model, which we dub nonreciprocal Dicke model, exhibits a discrete parity-time ($\mathcal{PT}$) symmetry and we characterize the emergence of a nonstationary phase, so far explained in terms of dissipation-induced instability, as spontaneous breaking of $\mathcal{PT}$ symmetry. We further show that such $\mathcal{PT}$ symmetry breaking embodies an instance of a nonreciprocal phase transition, a concept recently introduced by Fruchart *et al.* [Nature (London) **592**, 363 (2021)]. Remarkably, the phase transition in our model does not necessitate the presence of any underlying broken symmetry or exceptional points in the spectrum, both believed to be essential requirements for nonreciprocal phase transitions. Our results establish driven-dissipative light-matter systems as a new avenue for exploring nonreciprocal phase transitions and contribute to the theory of nonreciprocal collective phenomena.

DOI: 10.1103/PhysRevLett.131.113602

Introduction.—Newton's third law states that to every action there is always an equal and opposed reaction [1]. For nonequilibrium agents, this principle can be broken, giving rise to nonreciprocal interactions, namely, interactions which are not symmetric upon the exchange of agents. This universal phenomenon has been observed in active matter [2–8], interface growth [9,10], neural systems [11], and social dynamics [12]. Recently, Fruchart *et al.* [13] made a seminal contribution to a general theory of nonreciprocal phase transitions (NRPTs), showing that nonreciprocal interactions can lead to nonstationary phases of matter and exceptional points (EPs). Their findings were illustrated in paradigmatic nonequilibrium models, such as flocking, pattern formation, and synchronization. In contrast, for engineered photonic and coupled light-matter systems, nonreciprocity has been mostly investigated in the context of signal transmission [14–21], with the role of many-body interactions left almost entirely unexplored [22].

In this Letter, we provide the characterization of a NRPT in a many-body light-matter system by studying an open Dicke model [23] featuring two different spin species, with photons mediating nonreciprocal interactions between them. We show that nonreciprocal interactions account for the key features of the model, such as its steady-state phase diagram and dynamics, hence we dub it nonreciprocal Dicke model (NRDM).

The most distinctive feature of the NRDM is arguably the emergence of a nonstationary phase, which we associate with the presence of a NRPT. Going beyond the paradigm of Ref. [13], we show that this transition takes place (i) in the absence of any initial spontaneously broken symmetry and (ii) in the absence of EPs, when nonreciprocal interactions are mediated by a *dynamical* degree of freedom. Excitingly, we find that the NRPT is robust against spin frequency imbalance and spin decay. Our study suggests that NRPTs are a more general phenomenon than currently appreciated.

Specifically, we show that the NRDM is characterized by a parity-time ($\mathcal{PT}$) symmetry, in addition to the parity symmetry associated to the normal-to-superradiant phase transition, which is spontaneously broken in the NRPT. Remarkably, this $\mathcal{PT}$ symmetry breaking occurs at the level of the steady state and thus supersedes standard treatments

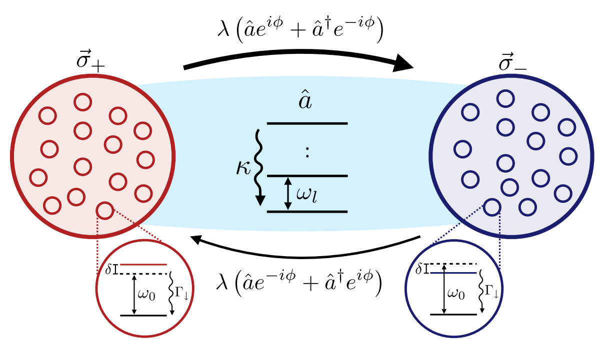

FIG. 1. The nonreciprocal Dicke model. Two spin species, labeled $\pm$, each consisting of N identical spins, interact with a light field $\hat{a}$ with frequency ω_l and decay rate κ, with coupling of modulus λ and phase $\pm\phi$. This results in photon-mediated nonreciprocal interactions between the spins. The two species may have different frequencies $\omega_0 \pm \delta$ and be affected by spin relaxation at a rate $\Gamma_{\downarrow}$.

restricted to transient growth or decay, as obtained from the eigenvalues of associated non-Hermitian Hamiltonians [24].

The nonstationary phase of the NRDM has previously been characterized in terms of a dissipation-induced instability and chiral forces [25–27] and even observed in a spinor Bose-Einstein condensate in an optical cavity [27]. Here, we go beyond these considerations by identifying it as a symmetry broken phase belonging to the novel class of nonequilibrium NRPTs. We further uncover a rich nonstationary behavior, including frequency locking between the light and collective spin oscillations and the coexistence of superradiant and nonstationary behavior.

Our results bridge the fields of nonreciprocal critical phenomena and driven-dissipative light-matter systems and can be tested in state-of-the-art atom-cavity experiments [27–33].

Model and symmetries.—The NRDM is an open Dicke model consisting of two different spin species and a light field featuring complex coupling amplitudes [25–27], see Fig. 1. The coherent dynamics are given by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \hat{H}_0 + \frac{\lambda}{2\sqrt{N}} \sum_{j=1}^{N} \sum_{m=\pm} (e^{-im\phi}\hat{a} + e^{im\phi}\hat{a}^\dagger)\hat{\sigma}^x_{j,m}, \quad (1)$$

with $\hat{H}_0 = \omega_l \hat{a}^\dagger \hat{a} + \frac{1}{2}\sum_{j,m}(\omega_0 + m\delta)\hat{\sigma}^z_{j,m}$, $\hat{a}$ the annihilation operator of the light field, ω_l the photon frequency, $\hat{\sigma}^z_{j,m}$ the z Pauli matrix of the jth spin of species $m = \pm$, ω_0 the mean spin frequency, and 2δ the frequency splitting between species. The interaction is collective, i.e., of Dicke type, with N the number of spins of each species. Crucially, the light-matter coupling amplitudes are complex with modulus λ and a species-dependent phase $\pm\phi$, which cannot be removed from the Hamiltonian by a gauge transformation. Dissipative terms are incorporated in the model via a Lindblad master equation of the form $\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \kappa\mathcal{D}[\hat{a}]\hat{\rho} + \Gamma_\downarrow \sum_{j,m} \mathcal{D}[\hat{\sigma}^-_{j,m}]\hat{\rho}$, where κ is the photon loss rate, $\Gamma_\downarrow$ is the spin decay rate, with $\hat{\sigma}^- = (\hat{\sigma}^x - i\hat{\sigma}^y)/2$ the spin lowering operator, and the dissipator defined as $\mathcal{D}[\hat{R}]\hat{\rho} = \hat{R}\hat{\rho}\hat{R}^\dagger - \frac{1}{2}\{\hat{R}^\dagger\hat{R}, \hat{\rho}\}$.

In the thermodynamic limit, $N \to \infty$, the semiclassical equations of motion become exact

$$\dot{s}_{x,\pm} = -(\omega_0 \pm \delta)s_{y,\pm} - \frac{\Gamma_\downarrow}{2}s_{x,\pm}$$

$$\dot{s}_{y,\pm} = (\omega_0 \pm \delta)s_{x,\pm} - \frac{\Gamma_\downarrow}{2}s_{y,\pm} - \frac{\lambda s_{z,\pm}}{\sqrt{N}}(\alpha e^{\mp i\phi} + \alpha^* e^{\pm i\phi})$$

$$\dot{s}_{z,\pm} = -\Gamma_\downarrow(s_{z,\pm} + 1) + \frac{\lambda s_{y,\pm}}{\sqrt{N}}(\alpha e^{\mp i\phi} + \alpha^* e^{\pm i\phi})$$

$$\dot{\alpha} = -\left(i\omega_l + \frac{\kappa}{2}\right)\alpha - i\frac{\lambda\sqrt{N}}{2}(s_{x,+}e^{i\phi} + s_{x,-}e^{-i\phi}), \quad (2)$$

with $\alpha = \langle\hat{a}\rangle$ and $s_{(x,y,z),\pm} = \langle\hat{\sigma}^{(x,y,z)}_{j,\pm}\rangle$.

As many Dicke models, the NRDM features a $\mathbb{Z}_2$ parity symmetry, $(\alpha, s_{x,\pm}, s_{y,\pm}) \to -(\alpha, s_{x,\pm}, s_{y,\pm})$. Spontaneous breaking of this symmetry leads to a superradiant phase transition [23,34,35]. This corresponds to a transition between a normal phase (NP), where the light field is empty and the spins point down, and a superradiant phase (SP), where the spins acquire a finite x component and the light field is macroscopically populated.

The distinctive trait of the NRDM is that the light field mediates nonreciprocal interactions between the spin species. Their origin can be traced back to the joint presence of the energy nonconserving terms characteristic of the Dicke model, responsible for the gauge-invariant phase 2ϕ in Eq. (1), and photon losses [36]. Nonreciprocal interactions give rise to a region in the phase diagram displaying nonstationary steady states, called the dynamical phase (DP) [25–27]. Nonreciprocal interactions also affect the superradiant region, resulting in the emergence three different phases [25]: one in which the spins are almost aligned ($\text{SP}_{\downarrow\downarrow}$), another in which they are almost anti-aligned ($\text{SP}_{\uparrow\downarrow}$), and a third one corresponding to a coexistence region between the two.

From Eqs. (2) we notice that the NRDM exhibits a second discrete symmetry, associated with the transformation $(s_{j,\pm}, \phi) \to (s_{j,\mp}, -\phi)$ with $j = x, y, z$, which combines a parity transformation that swaps the two species

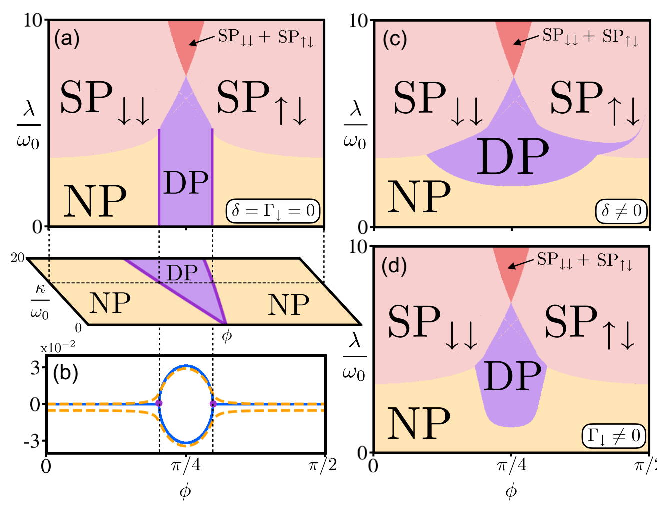

FIG. 2. Effective nonreciprocal interactions. (a) Steady-state phase diagram as a function of coupling λ and phase ϕ, and of photon loss rate κ and ϕ (out-of-plane view), in the limit of adiabatic elimination and for $\delta = \Gamma_\downarrow = 0$. Lines made of EPs are marked in violet. The solid blue line in (b) is the real part of the spectrum around the NP for $\lambda = 2.5\omega_0$; violet dots mark EPs. Photon-mediated nonreciprocal interactions. Phase diagram in the case where nonreciprocal interactions are dynamically mediated by the light field for (c) $\delta = 0.05\omega_0$, $\Gamma_\downarrow = 0$, and (d) $\delta = 0$, $\Gamma_\downarrow = 0.02\omega_0$. The dashed yellow curve (b) is the real part of the spectrum for $\delta = \Gamma_\downarrow = 0$ and $\lambda = 2.5\omega_0$. Parameters: (a)–(d) $\omega_l = 20\omega_0$, $\kappa = 12.5\omega_0$.

with a change of sign of ϕ. Since the phase ϕ acts as a synthetic magnetic flux, a change in sign describes the action of time reversal, so that the combined transformation is equivalent to $\mathcal{PT}$ symmetry. We stress that this is an exact symmetry of the nonlinear set of Eqs. (2), valid for all values of parameters. This has to be contrasted with standard treatments of $\mathcal{PT}$ symmetry in linear models, formulated in terms of non-Hermitian operators. A similar notion of $\mathcal{PT}$ symmetry has been studied in a dimer with saturable loss and gain [37]. $\mathcal{PT}$ symmetry has also been discussed at the level of Lindblad dynamics [38–41]. We will show that, in the absence of explicit symmetry breaking terms, i.e., when $\delta = 0$, the spontaneous breaking of $\mathcal{PT}$ symmetry is the hallmark of the NRPT, as it is spontaneously broken when entering the DP, but unbroken in the NP and SPs.

Nonreciprocal phase transition.—Our analysis begins with the steady-state phase diagram, which we obtain by setting Eqs. (2) to zero, solving the set of algebraic equations, and performing linear stability analysis [36]. We first illustrate the NRPT in the regime $(\omega_l, \kappa) \gg (\omega_0, \delta, \Gamma_\downarrow, \lambda)$, where the photonic degree of freedom can be adiabatically eliminated. We refer to this regime as effective nonreciprocal interactions, because nonreciprocity is encoded as effective asymmetrical coupling constants in the equations of motion for the two spin species [36]. In this scenario we recover the treatment of Ref. [13]. The steady-state phase diagram is shown in Fig. 2(a) for $\delta = \Gamma_\downarrow = 0$. We observe the emergence of the DP in the central region. We characterize this phase exploiting the fact that the spectrum of the fluctuations around the NP can be computed analytically [25]. In Fig. 2(b), we show in blue the real part of the spectrum, associated with the growth rates of fluctuations, as a function of ϕ. As the system crosses the NP-DP boundary, we observe the instability of the NP, heralded by the presence of EPs, resulting from nonreciprocity. While not shown, we also observe eigenvector coalescing, a characteristic feature accompanying EPs. The occurrence of EP and a finite real part of the spectrum is a manifestation of $\mathcal{PT}$ symmetry breaking at the level of fluctuations [13]. Note that for $\lambda = 0$, the system is trivially noninteracting, and thus remains in the NP.

In Fig. 2(a) we also show an out-of-plane section of the phase diagram as a function of the photon loss rate κ and phase ϕ. Starting at $\kappa = 0$, where interactions are reciprocal, we observe how the DP emerges as soon as nonreciprocity ($\kappa \neq 0$) is turned on, with the phase boundaries made of EPs. Again, this agrees with the analysis in [13]: nonreciprocal interactions can open up a nonstationary region in the phase diagram, with lines of EPs present at the boundary. We thus conclude the NP-DP transition indeed corresponds to a NRPT. Nevertheless, there is a fundamental difference between the NRDM we consider and the models in [13]: the NRPT does not necessitate an underlying broken continuous symmetry. In fact, the NRPT here takes place in the absence of *any* initially broken symmetry.

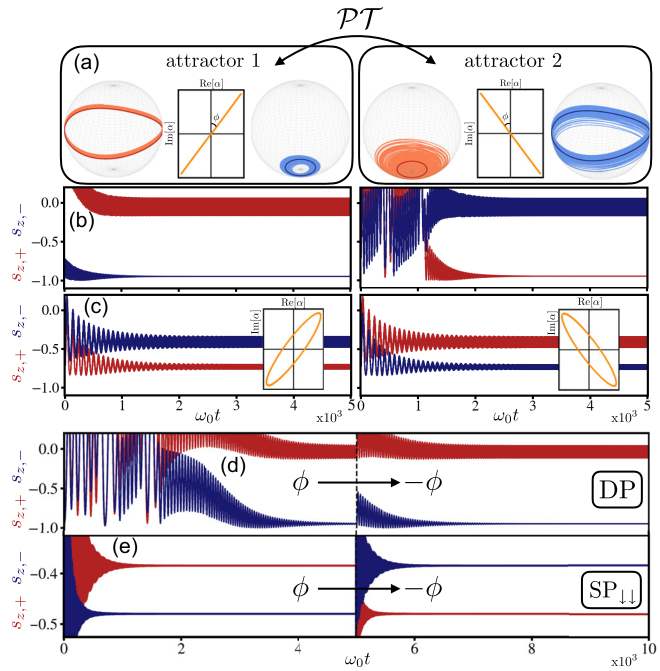

FIG. 3. Dynamics and $\mathcal{PT}$ symmetry breaking. (a) The two steady-state attractors of Eqs. (2) are related by a $\mathcal{PT}$ transformation, for $\phi = \pi/4$, $\lambda = 3\omega_0$, and $\delta = \Gamma_\downarrow = 0$; the light field phase locks at the angles $(\pi/2) \pm \phi$ and the spin trajectories of the two species are depicted on the Bloch sphere. The components $s_{z,\pm}$ are shown in (b) and in (c) for $\Gamma_\downarrow \neq 0$. (d),(e) After the transient has elapsed, quenching the phase $\phi \rightarrow -\phi$ reveals that the DP is a $\mathcal{PT}$ broken phase (d) and $\text{SP}_{\downarrow\downarrow}$ is $\mathcal{PT}$ unbroken (e). Parameters: (a)–(e) $\omega_l = 20\omega_0$, $\kappa = 12.5\omega_0$; (d) $\lambda = 2.5\omega_0$, $\phi = (\pi/4)$; (e) $\lambda = 5.5\omega_0$, $\phi = (\pi/8)$.

We now take our investigation beyond adiabatic elimination and show that NRPTs can take place in the absence of EPs in the spectrum, if the nonreciprocal interactions are mediated by a dynamical degree of freedom; we refer to this scenario as photon-mediated nonreciprocal interactions. We highlight two major findings. First, the DP remains present in the phase diagram, displaying robustness against finite frequency imbalance and spin decay, see Figs. 2(c) and 2(d). In fact, for both $\delta \neq 0$ (c) and $\Gamma_\downarrow \neq 0$ (d), we find the NP to remain stable for a finite region, see also [36] for the corresponding dynamical spectra. We also note that for $\delta = \Gamma_\downarrow = 0$ (not shown), photon fluctuations erase the entire NP (except when interactions are reciprocal, i.e., $\phi = 0, (\pi/2)$) [25] so, strictly speaking, this case does not correspond to a NRPT; we, nevertheless, recover a NRPT for any small perturbation $\delta, \Gamma_\downarrow \neq 0$. Second, and most strikingly, the NP and DP are no longer separated by a boundary of EPs. Instead, the fluctuations of the light field soften this feature, resulting in a smooth spectrum as a function of ϕ [36], see Fig. 2(b). We insist that this still corresponds to a NRPT, as it is nonreciprocal interactions that give rise to the dynamical phase. We expect this behavior to emerge whenever nonreciprocity is mediated by dynamical degrees of freedom.

Steady-state dynamics and $\mathcal{PT}$ symmetry breaking.— We now connect the occurrence of a NRPT to the spontaneous breaking of $\mathcal{PT}$ symmetry. Inside the DP, spins and light field undergo persistent oscillations in the form of limit cycles [25].

For $\delta = \Gamma_\downarrow = 0$, the long-time dynamics is in fact determined by *two* limit-cycle attractors, shown in Fig. 3(a); the z component of the Bloch sphere trajectories is further highlighted in panel (b). We notice that light field oscillates with the phase locked at the angles $(\pi/2) \pm \phi$. Comparing the two attractor solutions we see that they are related by an exchange of the two species and a change of sign in the phase, i.e., via $\mathcal{PT}$. Depending on the initial conditions, the system settles into one of the two available steady states: the $\mathcal{PT}$ symmetry of Eqs. (2) is therefore broken in the steady state, namely the onset of the DP is accompanied by the spontaneous breaking of $\mathcal{PT}$ symmetry. We stress that, due to the nonlinear character of Eqs. (2), this is truly a *spontaneous* breaking of $\mathcal{PT}$ symmetry, unlike for $\mathcal{PT}$ symmetric linear systems.

Noticeably, for $\Gamma_\downarrow \neq 0$, $\delta = 0$, Eqs. (2) are still $\mathcal{PT}$ symmetric, as spin decay acts homogeneously on the spins, yielding again two different steady-state attractors and a $\mathcal{PT}$ breaking phase transition. As shown in Fig. 3(c), both projections $s_{z,\pm}$ now display oscillations, while the light displays imperfect phase locking, however, still about well defined angles $(\pi/2) \pm \phi$. We conclude that the NRPT present in the phase diagram Fig. 2(c) is accompanied by the spontaneous breaking of $\mathcal{PT}$ symmetry. In contrast, a finite frequency imbalance $\delta \neq 0$ explicitly breaks the $\mathcal{PT}$ symmetry of the NRDM at the level of the equations of motion, resulting in the erasure of one of the attractors, and leaving only a single nonstationary solution in the steady-state dynamics (not shown).

To show that $\mathcal{PT}$ symmetry breaking is a phenomenon uniquely associated to NRPTs, and not to normal-to-superradiant phase transitions, we perform additional simulations, in which we first let the system relax to the steady state then quench the phase $\phi \to -\phi$. When the system is in the DP, following the quench, it relaxes back to the same attractor, see Fig. 3(d). This characterizes the steady state as a $\mathcal{PT}$ broken state, since a further swap of the spin species (exchanging their colors) shows that the action of $\mathcal{PT}$ does not leave the steady state invariant. In panel (e) we perform the same numerical experiment starting from $\mathrm{SP}_{\downarrow\downarrow}$, from which we see that the action parity and time reversal (exchanging the species and reversing the sign of ϕ) undo each other, i.e., the superradiant steady state is $\mathcal{PT}$ invariant; the same conclusion applies when starting from $\mathrm{SP}_{\uparrow\downarrow}$ and NP.

Frequency spectrum and dynamical superradiance.— Another remarkable feature shown in Figs. 3(a) and 3(b) is that, inside each attractor, the trajectories of the two spins

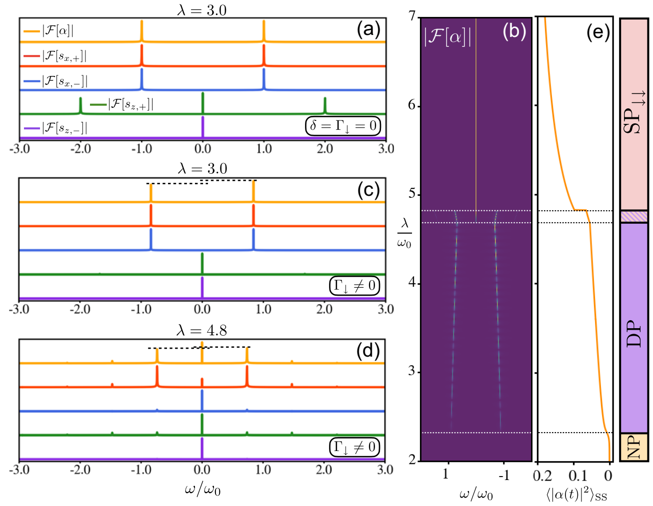

FIG. 4. Frequency spectrum. Absolute value of the Fourier spectrum $\mathcal{F}$ (arbitrary units) of the time evolution inside the DP for (a),(c),(d) the light field and x and z spin components, and (b) for the light field as a function of λ. The color shading in (b) is saturated at a cutoff value for better visualization. Dashed lines in (c) and (d) highlight the height difference between peaks in the spectrum. (e) Time-averaged intensity of the light field in the steady state as a function of λ. Parameters: (a)–(e) $\omega_l = 20\omega_0$, $\kappa = 12.5\omega_0$, and $\phi = \pi/5$; (b)–(e) $\Gamma_\downarrow = 0.02\omega_0$.

are qualitatively different, with one species oscillating in-plane with constant s_z, and the other one featuring oscillations also along s_z. To obtain further insight, we perform a Fourier decomposition of the steady-state dynamics for the case $\delta = \Gamma_\downarrow = 0$, and we find the (x, y) components of both species to oscillate at the natural frequency ω_0, while the nonstationary z component precesses at frequency $2\omega_0$, see Fig. 4(a). This behavior can be understood as follows: the phase of the light field locks at the angles $(\pi/2) \pm \phi$ and decouples from the spin species $\pm$ by spontaneously choosing either $\alpha e^{\mp i\phi} + \alpha^* e^{\pm i\phi} = 0$. This naturally leads to in-plane (constant s_z) coherent dynamics at ω_0 only for the species $\pm$. We also find that the light field oscillation frequency locks to the natural frequency of the spins, as visible from the corresponding spectrum in Fig. 4(a), which peaks at $\pm\omega_0$ [36].

The frequency analysis provides us with additional information and allows us to uncover new features of the NRDM, beyond those captured by the stability analysis. In Fig. 4(b) we show the light frequency spectrum as a function of λ, for $\phi = \pi/5$, $\Gamma_\downarrow \neq 0$, $\delta = 0$; the case corresponds to the phase diagram in Fig. 2(d). Deep into the DP, i.e., for sufficiently small values of the coupling, we observe simple harmonic motion, with positive- and negative-frequency components that move closer to each other for increasing λ. The cut in Fig. 4(c) further shows an asymmetry in the peaks of the two frequency components, in agreement with the ellipses in the inset of Fig. 3(c), and that the spin motion also locks at the same frequency. For large coupling values, on the other hand, we recover the static phase $\mathrm{SP}_{\downarrow\downarrow}$, with its zero-frequency component. However, close to the SP we also find an intermediate range of values where the steady states simultaneously break both parity symmetry and $\mathcal{PT}$ symmetry: although competing, the two symmetry breaking processes are not mutually exclusive. Correspondingly, we observe the two main light frequency components repelling each other, see Fig. 4(b), and the spin motion acquiring additional frequency components, see Fig. 4(d). The coexistence of nonstationary and superradiant behavior, or *dynamical superradiance* for short, is another remarkable finding of our work. The time-averaged intensity of the light field, shown in Fig. 4(e), provides further information on the superradiant behavior of the NRDM: the transition to static superradiance ($\mathrm{SP}_{\downarrow\downarrow}$) is marked by an abrupt jump, while the onset of dynamical superradiance occurs without jumps but is signaled by an increase in the steepness of the intensity as a function of λ, compared to the rest of the DP.

Finally, for values of the phase approaching $\phi = \pi/4$, i.e., maximum nonreciprocity, we also find that the regular attractors are lost and, as λ increases, the system cascades into a chaotic regime with many emerging frequencies, which dominates until the superradiant phase transition occurs (not shown). An in-depth investigation of these regimes is left for future studies.

Conclusion.—We introduced the nonreciprocal Dicke model (NRDM) as a minimal setting to study nonreciprocal interactions in many-body light-matter systems. We identified the presence of a nonreciprocal phase transition (NRPT), showed that NRPTs can occur in a broader class of systems than previously known [13], and linked the NRPT to spontaneous breaking of $\mathcal{PT}$ symmetry at the level of steady states.

Our results can be tested in state-of-the-art atom-cavity experiments [27–32]. These platforms further provide the opportunity to explore the effects of finite-range interactions [31,32], or the possibility to include a lattice and probe the competition between nonreciprocal and Hubbard-type interactions [42]. From a theory point of view, it would also be interesting to contrast nonreciprocal interactions with other ways to stabilize limit-cycle phases in Dicke-type models [43–45].

M. B. acknowledges funding from the Swiss National Science Foundation under Grant No. PCEFP2_194268.

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