Unlimited One-Way Steering

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This work explores the asymmetry of quantum steering in a setup using high-dimensional entanglement. We construct entangled states with the following properties: (i) one party (Bob) can never steer the state of the other party (Alice), considering the most general measurements, and (ii) Alice can strongly steer the state of Bob, in the sense of demonstrating genuine high-dimensional steering. In other words, Alice can convince Bob that they share an entangled state of arbitrarily high Schmidt number, while Bob can never convince Alice that the state is even simply entangled. In this sense, one-way steering can become unlimited. A key result for our construction is a condition for the joint measurability of all high-dimensional measurements subjected to the combined effect of noise and loss, which is of independent interest.

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Introduction.—Nonlocality is among the central features of quantum theory. This effect manifests at different levels: on the one hand, in the mathematical structure of the theory (i.e., the Hilbert space) via the notion of entanglement [1,2] and on the other hand, the measurement statistics of local measurements performed on entangled states can feature strong nonlocal correlations, which are incompatible with any local model [3]. Initially, believed to be two facets of the same phenomenon, it is now clear that entanglement and quantum Bell nonlocality are in fact inherently different; see, e.g., Refs. [4,5].

Another perspective on quantum nonlocality is provided by the notion of quantum steering; see Refs. [6,7] for reviews. This effect, formalized by Wiseman, Jones, and Doherty [8], takes its roots in the early works of Einstein-Podolsky-Rosen [9] and Schrödinger [10,11]. Here, an untrusted party (Bob) wants to convince another party (Alice) that they share an entangled state. By demonstrating his ability to remotely steer Alice's local state in a different measurement basis, Bob can convince Alice. Steering is thus an inherently asymmetrical task, contrary to entanglement and Bell nonlocality.

Interestingly, the asymmetry of steering is also observed at the level of quantum states. Specifically, there exist entangled states that lead to steering from Bob to Alice, but not the other way around [12]; Bob can convince Alice that the shared state is entangled, while Alice can never convince Bob, even if she would use all possible local measurements. This effect, coined "one-way steering" has attracted considerable attention in recent years, with many examples in low-dimensional (mostly 2-qubit) [12–17] and continuous variable Gaussian systems [18,19] as well as experimental demonstrations; see, e.g., Refs. [20–23].

A relevant question is whether there exist different forms of one-way steering, and whether some are stronger than others. So far, this question has not been discussed, due to the lack of an appropriate measure for steering in this context. Here, we tackle this problem, taking advantage of the recently introduced notion of genuine high-dimensional steering [24]; see also Refs. [25,26]. This allows for a dimensional quantification of steering, specifically to lower bound the Schmidt number of an entangled state in a steering scenario. We then ask whether there exist an entangled state with the following properties: (i) Bob cannot steer the state of Alice (even when allowing for all possible local measurements), and (ii) Alice can steer Bob's state strongly, i.e., ensuring the presence of high-dimensional entanglement (Schmidt number at least d). As sketched in Fig. 1, we answer this question in the affirmative by constructing a family of entangled states [of dimension $d \times (d+1)$] with the above two properties for any finite dimension d. To do so, we exploit the connection between steering and measurement incompatibility [27–30], and also its recent generalization [31] to high-dimensional steering and the concept of n simulability

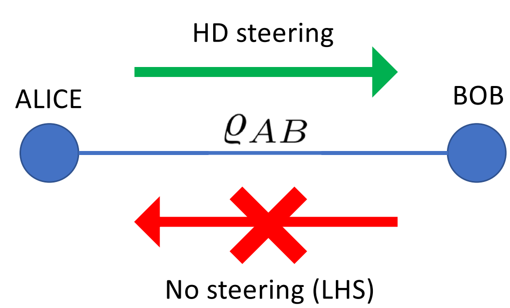

FIG. 1. We present high-dimensional entangled states ϱ_{AB} such that (i) Bob cannot steer Alice and (ii) Alice can strongly steer Bob and demonstrate genuine high-dimensional steering. This shows that the effect of one-way steering can become unlimited.

of measurements [32]. Notably, we give a sufficient condition for the joint measurability of the set of all positive-operator-valued measures (POVMs) subjected to the combined effect of noise and loss.

Question and main result.—We consider a scenario where two distant parties, Alice and Bob, share an entangled state ϱ_{AB}. Each party performs local measurements, represented by sets of POVMs. Specifically, Alice's measurements are described by a set of POVMs $\{A_{a|x}\}$, where x denotes the measurement choice and a the outcome, with the properties that $A_{a|x} \geq 0$ and $\sum_a A_{a|x} = \mathbb{1}$ for all a and x. Similarly, for Bob we define the set of POVMs $\{B_{b|y}\}$.

As the shared state is entangled, the effect of each party's local measurements is to remotely prepare (steer) the other party's state. This is described via two so-called state assemblages: $\{\sigma^A_{b|y}\}$ describe the states of Alice's system conditioned on Bob's measurement while $\{\sigma^B_{a|x}\}$ are Bob's states given Alice's measurement. These are given by

$$\sigma^A_{b|y} = \mathrm{tr}_B[(\mathbb{1} \otimes B_{b|y})\varrho_{AB}], \tag{1}$$

$$\sigma^B_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\varrho_{AB}]. \tag{2}$$

The main question we address here is how different these two state assemblages can become, in other words how asymmetric the effect of steering can be. Loosely speaking we are looking for an entangled state ϱ_{AB} such that one of the state assemblages, say $\{\sigma^A_{b|y}\}$, is classical (in the sense that it can never lead to steering), while the other assemblage $\{\sigma^B_{a|x}\}$ is highly nonclassical (in the sense that it exhibits strong steering, witnessing high entanglement dimensionality).

Before defining the problem more precisely, let us first observe that we are looking for some entangled states that are high dimensional and asymmetrical. Obviously, if the state ϱ_{AB} would be symmetrical under the exchange of Alice and Bob, any assemblage obtainable in one direction can also be obtained the other way around. Moreover, the state ϱ_{AB} should feature a high entanglement dimensionality, as quantified here via the Schmidt number [33,34]: the Schmidt number of ϱ_{AB} is the minimum n such that there exists a decomposition $\varrho_{AB} = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ where all $|\psi_j\rangle$ are pure entangled states of Schmidt rank at most n.

More formally, we are looking for entangled states with the following two properties. (1) The assemblage $\{\sigma^A_{b|y}\}$ admits a local hidden state (LHS) model [8]:

$$\sigma^A_{b|y} = \sum_\lambda p(\lambda)p(b|\lambda, y)\sigma_\lambda \quad \forall \;\; b, y. \tag{3}$$

Importantly, this should hold for any set of measurements for Bob $\{B_{b|y}\}$. Thus, no steering from Bob to Alice is possible, implying that Bob can never convince Alice that the shared state ϱ_{AB} is entangled. (2) The assemblage $\{\sigma^B_{a|x}\}$ is not $(d-1)$ preparable [24], i.e.,

$$\sigma^B_{a|x} \neq \sum_\lambda p(\lambda)\mathrm{tr}_A[(M^\lambda_{a|x} \otimes \mathbb{1})\rho^\lambda_{AB}] \tag{4}$$

with $p(\lambda)$ an arbitrary probability distribution, $M^\lambda_{a|x}$ arbitrary measurements (in $\mathbb{C}^d$), and all ρ^λ_{AB} being arbitrary states of Schmidt number (at most) $d-1$. Thus, genuine d-dimensional steering is demonstrated, so that Alice can convince Bob that the shared state ϱ_{AB} has a large Schmidt number d. For short, we say that ϱ_{AB} is d steerable.

In the following, we construct a family of entangled states in dimension $d \times (d+1)$ with the above two properties, for any d. This shows that one-way steering can become unlimited, in the sense of being maximally asymmetrical. Specifically, we start from a maximally entangled 2-qudit state $|\phi^+_d\rangle = (1/\sqrt{d})\sum_{k=0}^{d-1}|k,k\rangle$ and apply successively a white noise (depolarizing) channel followed by a loss channel, defined by

$$\mathcal{W}_p\colon\; \rho \mapsto p\rho + (1-p)\mathrm{tr}[\rho]\frac{\mathbb{1}_d}{d}, \tag{5}$$

$$\mathcal{L}_\eta\colon\; \rho \mapsto \eta\rho + (1-\eta)\mathrm{tr}[\rho]|\text{ø}\rangle\langle\text{ø}|. \tag{6}$$

The depolarizing (loss) channel replaces the original state with a maximally mixed state (vacuum state $|\text{ø}\rangle\langle\text{ø}|$) with probability $1-p$ $(1-\eta)$. Note that the vacuum state represents an additional "level," orthogonal to the input Hilbert space, so that the output state has dimension $d+1$. After the two channels, we obtain the final state

$$\begin{aligned}\varrho^{(\eta,p)}_{AB} &= \mathrm{id} \otimes (\mathcal{L}_\eta \circ \mathcal{W}_p)[\Phi^+] \\ &= \eta p \Phi^+_d + \eta(1-p)\frac{\mathbb{1}_d \otimes \mathbb{1}_d}{d^2} + (1-\eta)\frac{\mathbb{1}_d}{d} \otimes |\text{ø}\rangle\langle\text{ø}|,\end{aligned} \tag{7}$$

where we use the notation $\Phi^+_d = |\phi^+_d\rangle\langle\phi^+_d|$. Note that $\varrho^{(\eta,p)}_{AB}$ is of dimension $d \times (d+1)$.

Result 1.—The states $\varrho^{(\eta,p)}_{AB}$ defined in Eq. (7) satisfy the following properties: (i) $\varrho^{(\eta,p)}_{AB}$ is unsteerable from Bob to Alice if

$$\eta \leq (1-p)^{d-1}, \tag{8}$$

and (ii) $\varrho^{(\eta,p)}_{AB}$ is d steerable from Alice to Bob if

$$p > \frac{d\sqrt{\frac{d}{d+1}} - 1}{d-1}. \tag{9}$$

Therefore, for any d there are parameter values (η, p) such that conditions (i) and (ii) are both satisfied, thus demonstrating unlimited one-way steering. This regime corresponds to low noise and high loss.

Proof outline.—Statement (i): The first step is to show that the loss channel can be removed, because when acting on the trusted party it does not affect the steering properties of the underlying assemblage. This statement is formalized by the following lemma.

Lemma 1.—Consider an assemblage $\{\sigma_{a|x}\}_{a,x}$ and a loss channel $\mathcal{L}_\eta$ for $\eta > 0$. The assemblage $\{\sigma'_{a|x} = \mathcal{L}_\eta[\sigma_{a|x}]\}_{a,x}$ is n steerable if and only if $\{\sigma_{a|x}\}_{a,x}$ is.

The full proof is in Supplemental Material (SM) [35]. The idea is to show that if the assemblage $\{\sigma'_{a|x}\}_{a,x}$ is n preparable, one can construct an n-preparable model for $\{\sigma_{a|x}\}_{a,x}$. The converse holds trivially (via application of the loss channel).

Hence, to prove that $\rho_{AB}^{(\eta,p)}$ is d steerable from Bob to Alice it is sufficient to show that the isotropic state $\rho_{AB}^{(1,p)} = p\Phi_d^+ + (1-p)[(\mathbb{1}_d \otimes \mathbb{1}_d)/d^2]$ is, which is to be expected for weak enough noise. Indeed, it was shown in Ref. [31] that this is the case for p satisfying Eq. (9), concluding the proof of statement (i). Note that, in the above, d steerability is demonstrated when using the set of all measurements. Instead, one could use a simpler (and practical) witness based on a pair of mutually unbiased bases, certifying d steerability of $\rho_{AB}^{(1,p)}$ [24] for

$$p \geq \frac{(d+\sqrt{d}-1)\sqrt{d-1}-1}{(d-1)(\sqrt{d-1}+1)}. \tag{10}$$

In turn, one shows d steerability of $\rho_{AB}^{(\eta,p)}$ for any $\eta > 0$, via the filtering procedure discussed in the proof of lemma 2, which removes the vacuum component $|\varnothing\rangle$.

Statement (i): Here we exploit the connection [29] between the notions of steering and measurement compatibility, formally introduced in the next section. In particular, for any completely positive trace preserving map $\mathcal{E}$ it is known that a state assemblage given by the operators $\sigma_{b|y} = \mathrm{tr}[(\mathbb{1} \otimes M_{b|y})(\mathrm{id} \otimes \mathcal{E}[\Phi^+])]$ admits a LHS model if and only if the set of measurements $\{\mathcal{E}^*[M_{b|y}]\}$ is jointly measurable. Hence, the state $(\mathrm{id} \otimes \mathcal{E}[\Phi^+])$ is unsteerable if and only if the channel $\mathcal{E}$ is *incompatibility breaking* [30,36,37], that is for any measurement assemblage $\{M_{b|y}\}$ the resulting assemblage $\{\mathcal{E}^*[M_{b|y}]\}$ is jointly measurable. Thus, statement (i) is implied by the following lemma.

Lemma 2.—The channel $\mathcal{E} = \mathcal{L}_\eta \circ \mathcal{W}_p$ is incompatibility breaking if $\eta \leq (1-p)^{d-1}$.

The proof of the lemma is rather straightforward once we have solved the question of joint measurability of all measurements subject to white noise and loss in a given finite dimension d. This is a question of independent interest, and will be exposed in the next section. Then these results are used in SM [35] to give a formal proof of Lemma 2.

Compatibility of all measurements with losses and noise.—Measurement incompatibility is a property of a set of measurements which cannot be performed simultaneously on a single copy of a system. Formally, a set of measurements $\{M_{a|x}\}$ is called incompatible if there exists no "parent" POVM $\{E_\lambda\}$ and classical postprocessings $\{p(a|x,\lambda)\}$ such that

$$M_{a|x} = \sum_\lambda p(a|x,\lambda) E_\lambda. \tag{11}$$

Sets of measurements allowing for such a model are called jointly measurable; see, e.g., Refs. [38,39] for reviews.

Our focus is on the (in)compatibility of measurements which are noisy and lossy. For any POVM $\{M_a\}$ with m outputs $a = 1, \ldots, m$ acting on a system of dimension d, we define its imperfect version $\{\bar{M}_a^{(\eta,p)}\}$ as a POVM with $m+1$ elements given by

$$\bar{M}_a^{(\eta,p)} = \begin{cases} \eta p M_a + \eta(1-p)(\mathrm{tr} M_a)\frac{\mathbb{1}_d}{d} & a = 1, \ldots, m \\ (1-\eta)\mathbb{1}_d & a = \varnothing \end{cases}. \tag{12}$$

The noisy measurement apparatus behaves like the ideal one with probability ηp, produces a random outcome with probability $\eta(1-p)$, and does not click with probability $1-\eta$ (formally this corresponds to the no-click outcome $a = \varnothing$). We are interested in knowing for which values of (η, p) all measurements in a given dimension d become compatible. The following result gives a sufficient condition.

Result 2.—The set of all POVMs $\{\bar{M}_{a|U}^{(\eta,p)}\}_U$ on $\mathbb{C}^d$ is jointly measurable if

$$\eta \leq (1-p)^{d-1}. \tag{13}$$

Proof.—To prove the result we present an explicit construction, inspired by Ref. [40], able to simulate all imperfect POVMs with noise parameters fulfilling Eq. (13). Note that it is sufficient to show that it can simulate all rank-one POVMs $\{M_a = \alpha_a|\varphi_a\rangle\langle\varphi_a|\}$ with $\sum \alpha_a = d$, since they can be postprocessed to simulate all other measurements. The postprocessing is done by mixing POVM elements, i.e., coarse graining the corresponding outputs, by linearity it is consistent with the noisification of the measurements defined in Eq. (12).

We take the parent POVM to be the covariant one—the continuous-valued measurement with the density

$$E_z = d|z\rangle\langle z| \tag{14}$$

where $|z\rangle = \sum_{k=0}^{d-1} z_k|k\rangle$ with $z \in \mathbb{C}^d$ and $|z|^2 = 1$, and $\mathrm{d}z$ is the invariant (under unitary transformations) measure over pure quantum states $|z\rangle$ such that $\int \mathrm{d}z|z\rangle\langle z| = (1/d)\mathbb{1}_d$.

To define the response function $p(a|z)$ simulating a notified rank-one POVM $\{\alpha_a|\varphi_a\rangle\langle\varphi_a|\}_a$ we proceed in two steps. First one samples a possible output a from the probability distribution (α_a/d). Given a and the corresponding state $|\varphi_a\rangle$, one simulates a two outcome POVM $\{N_a^{(a)}, N_\emptyset^{(a)}\}$ by using the deterministic response functions

$$p^{(a)}(a|z) = \begin{cases} 1 & |\langle\varphi_a\rangle z|^2 \geq t \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

and $p^{(a)}(\emptyset|z) = 1 - p^{(a)}(a|z)$, with a parameter $t \in [0,1]$. Let us now compute the resulting operators

$$N_a^{(a)} = \int \mathrm{d}z p^{(a)}(a|z) E_z \tag{16}$$

and $N_\emptyset^{(a)} = \mathbb{1}_d - N_a^{(a)}$. One notes that $N_a^{(a)} = U N_a^{(a)} U^\dagger$ is invariant under all unitary transformations U which leave the state $|\varphi_a\rangle$ unchanged $U|\varphi_a\rangle = |\varphi_a\rangle$, since both the measure $\mathrm{d}z$ and the response function $p^{(a)}(a|z)$ are invariant under such transformations. In other words, any such unitary commutes with our operator $[U, N_a^{(a)}] = 0$. It follows that $N_a^{(a)}$ has $|\varphi_a\rangle$ for an eigenstate, and is furthermore proportional to identity on the orthogonal subspace, i.e., it is of the form

$$N_a^{(a)} = \bar{A}_d(t)|\varphi_a\rangle\langle\varphi_a| + \bar{B}_d(t)\frac{\mathbb{1}_d - |\varphi_a\rangle\langle\varphi_a|}{d-1}, \tag{17}$$

with the scalar functions that can be computed as

$$\bar{A}_d(t) = \mathrm{tr} N_a^{(a)}|\varphi_a\rangle\langle\varphi_a| = d\int \mathrm{d}z p^{(a)}(a|z)|\langle\varphi_a\rangle z|^2$$

$$\bar{A}_d(t) + \bar{B}_d(t) = \mathrm{tr} N_a^{(a)} = d\int \mathrm{d}z p^{(a)}(a|z). \tag{18}$$

These integrals are straightforward to compute. In the SM [35] we show that they give $\bar{A}_d(t) = (1-t)^{d-1}[(d-1)t+1]$ and $\bar{T}_d(t) \equiv \bar{A}_d(t) + \bar{B}_d(t) = d(1-t)^{d-1}$.

Finally, averaging over the sampled value a we see that this strategy simulates $N_a = (\alpha_a/d)N_a^{(a)}$ and $N_\emptyset = \sum_a(\alpha_a/d)N_\emptyset^{(a)}$, leading to

$$N_a = \frac{\alpha_a}{d}\left(\bar{A}_d(t)|\varphi_a\rangle\langle\varphi_a| + \bar{B}_d(t)\frac{\mathbb{1}_d - |\varphi_a\rangle\langle\varphi_a|}{d-1}\right) \tag{19}$$

and $N_\emptyset = \mathbb{1}_d - \sum_a N_a$. On the other hand from Eq. (12) we obtain $\bar{M}_a^{(\eta,p)} = (\alpha_d/d)(d\eta p|\varphi_a\rangle\langle\varphi_a| + \eta(1-p)\mathbb{1}_d)$. Comparing with Eq. (19) we conclude that for

$$\eta = \frac{\bar{T}_d(t)}{d} = (1-t)^{d-1},$$
$$p = \frac{d\bar{A}_d(t) - \bar{T}_d(t)}{(d-1)\bar{T}_d(t)} = t \tag{20}$$

and $t \in [0,1]$ any POVM $\{\bar{M}_a^{(\eta,p)}\}$ can be simulated. Reproducing measurements that are even more noisy is straightforward by adding noise to the above construction. Hence, for any noise level $p = t$ our construction simulates all POVMs $\{\bar{M}_a^{(\eta,p)}\}$ if $\eta \leq (1-t)^{d-1} = (1-p)^{d-1}$. ∎

Finally, we note that Result 2 is only relevant in the presence of losses, i.e., $\eta < 1$. For a channel with white noise but without loss, a condition for being incompatibility breaking is given in Ref. [36].

Discussion and conclusion.—Considering a bipartite steering scenario based on high-dimensional entanglement, we have investigated how asymmetrical the effect of steering can become. We presented entangled states ϱ_{AB} such that Alice can convince Bob that ϱ_{AB} is of high Schmidt number, while Bob can never convince Alice that the state is even entangled. Thus, one-way steering can become unlimited.

Specifically, we constructed families of entangled states that exhibit genuine d-dimensional steering in one direction, while remaining unsteerable (under the most general measurements) in the other direction. We note that this construction can be straightforwardly generalized to states with genuine n-dimensional steering in one direction and unsteerable the other way around, for any $1 < n \leq d$ [for this, one should simply adapt the bounds in Eqs. (9) and (10) to demand only n steerability, following the results of Ref. [31] for Eq. (9) and Ref. [24] for Ref. (10)]. A practical implementation of such states should be feasible, e.g., with the setups of Refs. [41,42] for an experimental demonstration of unlimited one-way steering.

Finally, our work also contributed to the characterization of joint measurability in high-dimensional measurements. In particular, we obtained a criterion for the compatibility of arbitrary measurements subjected to both noise and loss. This result is of independent interest and may find other applications.

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