Partial Self-Testing and Randomness Certification in the Triangle Network

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Quantum nonlocality can be demonstrated without inputs (i.e., each party using a fixed measurement setting) in a network with independent sources. Here we consider this effect on ring networks, and show that the underlying quantum strategy can be partially characterized, or self-tested, from observed correlations. Applying these results to the triangle network allows us to show that the nonlocal distribution of Renou *et al.* [Phys. Rev. Lett. **123**, 140401 (2019)] requires that (i) all sources produce a minimal amount of entanglement, (ii) all local measurements are entangled, and (iii) each local outcome features a minimal entropy. Hence we show that the triangle network allows for genuine network quantum nonlocality and certifiable randomness.

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Discovered by Bell in the 1960s [1], the phenomenon of quantum nonlocality has been traditionally investigated in a setting where two (or more) separated observers perform local measurements on a shared entangled state [2]. One can then prove, e.g., via Bell inequality violation, that the observed correlations are Bell nonlocal, in the sense that they are incompatible with any physical theory satisfying a natural notion of locality, such as in classical physics. Beyond fundamental aspects, quantum nonlocality is also a strong resource for black-box quantum information processing.

Networks offer an intriguing new platform for exploring quantum nonlocality; see Ref. [3] for a review. The key novelty is that the network structure features several sources, each distributing entanglement to various subsets of the parties. At each party, quantum joint measurements can be performed, which enable the distribution of strong correlations across the entire network. The main idea behind network nonlocality is to investigate the resulting correlations under the assumption that all sources in the network are independent [4,5]. This assumption leads to a formal definition of classical (or network-local) correlations, where each source distributes a classical random variable. This can be viewed as a natural generalization of the notion of Bell locality to networks. Characterizing classical and quantum correlations in such networks is a highly challenging task; see, e.g., Refs. [6–11].

A central question in this research area is to uncover novel forms of quantum nonlocal correlations inherent to the network structure. In turn, one would like to characterize such new forms of nonlocality and explore their potential for applications in quantum information processing. Our Letter brings progress toward these goals.

In 2012, Fritz [12] and Branciard *et al.* [5] discovered that quantum nonlocality can be demonstrated in networks without the need for measurement inputs, i.e., each party performing a single fixed measurement. The example of Fritz considers a simple triangle network, where each pair of parties is connected via a bipartite source; see Fig. 1. While the construction of Fritz can be viewed as a clever embedding of a standard Bell test in the triangle network (see also Ref. [13]), Renou *et al.* [14] presented a strikingly different instance of quantum nonlocality (referred to as RGB4), which they argued is genuine to the triangle network; see also Refs. [15–20]. To formalize this intuition, the concepts of genuine network nonlocality [21] (GNNL) and full network nonlocality [22] (FNNL) were proposed. The first demonstrates the presence of nonclassical joint measurements, while the second ensures that all sources must produce a nonclassical resource. However, the initial question of whether the RGB4 distribution (or any other quantum nonlocal distribution without inputs) has GNNL features remained open so far.

In this Letter, we precisely address these questions. We develop methods for the characterization of quantum distributions in networks without inputs. This allows us

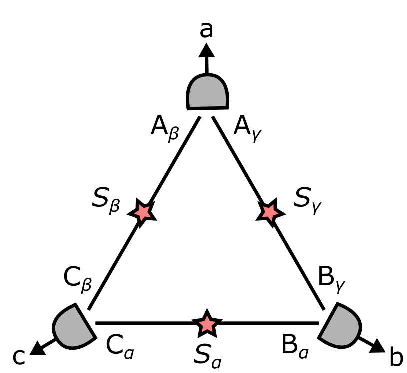

FIG. 1. The triangle network features three parties connected pairwise via three independent sources. The figure illustrates the labels we use for sources and subsystems.

to partially characterize the RGB4 distribution, and prove the following properties: (i) the distribution is GNNL, i.e. all parties must perform a nonclassical measurement, (ii) each source should distribute entanglement, and we obtain a lower bound on the entanglement of formation $\mathcal{E}_F > 2.5\%$, and (iii) certified randomness, via a lower bound on the min-entropy $H_{\min} > 3.8\%$. Our main technical results are self-testing (or quantum rigidity) proofs that apply to quantum (parity) token counting strategies on ring networks. The exposition in the main text will be focused on the triangle; the generalizations are presented at the end.

Triangle network.—The triangle network depicted in Fig. 1, involves three parties A, B, and C. Each pair of parties is connected by a bipartite source, labeled with α, β, and γ. Each party receives two systems (from the neighboring sources) and produces an output a, b, and c. There are in total six involved systems labeled X_ξ, with $X \in \{A, B, C\}$ referring to the party receiving the system and $\xi \in \{\alpha, \beta, \gamma\}$ to the source preparing it.

The set of output probability distributions $P(a, b, c)$ depends on the physical theory. Classically, the output distribution takes the form

$$P(a,b,c) = \sum_{\lambda,\mu,\nu} P_\alpha(\lambda) P_\beta(\mu) P_\gamma(\nu) \times P_A(a|\mu,\nu) P_B(b|\lambda,\nu) P_C(c|\lambda,\mu), \quad (1)$$

where λ, μ, and ν denote the classical random variables distributed by the sources α, β, and γ, respectively. A distribution $P(a, b, c)$ is termed local when such a decomposition can be found, and nonlocal otherwise.

In the quantum case, a Hilbert space is associated to each system; for simplicity we consider here systems of finite (but arbitrary) dimension. Without loss of generality one can assume that the sources distribute pure states [23], and we write the global state as

$$|\Psi\rangle = |\psi_\alpha\rangle_{B_\alpha C_\alpha} |\psi_\beta\rangle_{C_\beta A_\beta} |\psi_\gamma\rangle_{A_\gamma B_\gamma}. \quad (2)$$

The measurements performed by the parties are modeled by positive operator valued measures (POVMs) $\{E_A^a\}_a$, $\{E_B^b\}_b$, $\{E_C^c\}_c$. The output probability distribution is given by

$$P(a,b,c) = \langle\Psi| E_A^a E_B^b E_C^c |\Psi\rangle. \quad (3)$$

Self-testing token counting distributions.—Let us now focus on a particular family of *classical models* on the triangle, called token counting (TC) strategies. In such a strategy a source ξ randomly distributes a fixed number of tokens N_ξ left or right—with probability $p_\xi(i)$ there are i tokens sent to the left and $(N_\xi - i)$ tokens to the right. Each party then outputs the total number of tokens it received. The resulting correlations $P(a, b, c)$ are called TC distributions. By construction, all such distributions fulfill the constraint

$$a + b + c = N, \quad (4)$$

where $N \equiv N_\alpha + N_\beta + N_\gamma$. The TC distributions are known to be *rigid* [17], meaning that among all possible classical strategies on the triangle the TC strategy we just described is essentially the unique model leading to $P(a, b, c)$. We will now show that this result can be generalized to *quantum* strategies.

We first remark that for a quantum model the constraint [Eq. (4)] can be put in the form

$$\langle\Psi| E_A^a E_B^b E_C^c |\Psi\rangle = 0 \quad \text{if } a + b + c \neq N. \quad (5)$$

This guarantees that $\sqrt{E_A^a E_B^b E_C^c}|\Psi\rangle = 0$, and thus also $E_A^a E_B^b E_C^c|\Psi\rangle = 0$, when the sum of the outputs is not equal to N. Next, for each party X let us define an operator

$$U_X \equiv \sum_x e^{\mathrm{i}\varphi_x} E_X^x, \quad \text{with} \quad \varphi_x = \frac{2\pi(x + 1/3)}{N + 1}, \quad (6)$$

where the integer x runs through possible outputs of the party. The action of these operators on the state is

$$U_A U_B U_C |\Psi\rangle = \sum_{a+b+c=N} E_A^a E_B^b E_C^c |\Psi\rangle, \quad (7)$$

since $e^{\mathrm{i}(\varphi_a+\varphi_b+\varphi_c)} = e^{\mathrm{i}(N+1)[2\pi/(N+1)]} = 1$ when $a + b + c = N$ while all the other terms are zero. In addition, by $\sum_{a+b+c=N} E_A^a E_B^b E_C^c \leq \mathbb{1}$ and $\langle\Psi|\sum_{a+b+c=N} E_A^a E_B^b E_C^c|\Psi\rangle = 1$ we obtain

$$U_A U_B U_C |\Psi\rangle = |\Psi\rangle. \quad (8)$$

This implies that the measurements are projector valued, i.e., $E_X^x = \Pi_X^x$ with $(\Pi_X^x)^2 = \Pi_X^x$, and hence operators U_X are unitary. This is shown in the technical Appendix, where we also derive the following result.

Result 1.—Consider a quantum state $|\Psi\rangle = |\psi_\alpha\rangle_{B_\alpha C_\alpha} |\psi_\beta\rangle_{C_\beta A_\beta} |\psi_\gamma\rangle_{A_\gamma B_\gamma}$ on the triangle network, and local unitaries U_A, U_B, and U_C. The condition $U_A U_B U_C |\Psi\rangle = |\Psi\rangle$, implies that all the unitaries are product $U_A = V_{A_\beta} \otimes W_{A_\gamma}$, $U_B = V_{B_\gamma} \otimes W_{B_\alpha}$, $U_C = V_{C_\alpha} \otimes W_{A_\beta}$.

Let us now explore the implications of Result 1 on the measurements, focusing on Alice. The local unitaries can be diagonalized $V_{A_\beta} = \sum_j e^{\mathrm{i}v_j} \Pi_{A_\beta}^j$, $W_{A_\gamma} = \sum_\ell e^{\mathrm{i}w_\ell} \Pi_{A_\gamma}^\ell$. Result 1 guarantees that $\sum_a e^{\mathrm{i}\varphi_a} \Pi_A^a = \sum_{i,\ell} e^{\mathrm{i}(v_j+w_\ell)} \Pi_{A_\beta}^j \otimes \Pi_{A_\gamma}^\ell$. Since the eigenvalues have to match, i.e., $e^{\mathrm{i}(v_j+w_\ell)} = e^{\mathrm{i}\varphi_a}$ for some a, it is easy to see that $e^{\mathrm{i}v_j}$ and $e^{\mathrm{i}w_\ell}$ may take at most $N + 1$ different values each. Hence, the Hilbert space associated to the system A_β (or A_γ) can be split as a direct sum $\mathcal{H}_{A_\beta} = \bigoplus_{j=0}^N \mathcal{H}_{A_\beta}^{(j)}$ of subspaces $\mathcal{H}_{A_\beta}^{(j)}$ on which the different $\Pi_{A_\beta}^j$ project. As it is common in self-testing, one can add enough virtual levels to rewrite the direct sum as a tensor product $\mathcal{H}_{A_\beta} = \mathbb{C}^{N+1}_{\mathbf{A}_\beta} \otimes \mathcal{H}_{J_{A\beta}}$ with $\Pi^j_{A_\beta} = |j\rangle\langle j|_{\mathbf{A}_\beta} \otimes \mathbb{1}_{J_{A\beta}}$. This decomposes the system A_β into a qudit $\mathbf{A}_\beta$ and a "junk system" $J_{A\beta}$ on which the measurements act trivially. The same decomposition can be derived for each system and imposes the following form on any quantum model fulfilling Eq. (5):

$$E^a_A = \left(\sum_{(j,\ell)\in\mathbb{S}(a)} |j\rangle\langle j|_{\mathbf{A}_\beta} \otimes |\ell\rangle\langle \ell|_{\mathbf{A}_\gamma} \right) \otimes \mathbb{1}_{J_{A\beta}J_{A\gamma}}, \quad (9)$$

$$|\psi_\alpha\rangle = \sum_{i,j=0}^{N} \Psi^{(\alpha)}_{ij} |ij\rangle_{\mathbf{B}_\alpha\mathbf{C}_\alpha} |j^{(ij)}_\alpha\rangle_{J_{B\alpha}J_{C\alpha}}, \quad (10)$$

with $\mathbb{S}(a)$ containing all pairs (j,ℓ) such that $e^{i(v_j+w_\ell)} = e^{i\varphi_a}$. Here, the unknown states of the junk may in particular contain a copy of the qudit states $|j^{(ij)}_\alpha\rangle = |ij\rangle$ and remain inside the source. In this case, the quantum model becomes classical, once the junk systems are traced out. Finally, with the help of the rigidity result [17] for classical TC strategies, we arrive at the following result.

Result 2.—Consider a quantum strategy on the triangle with the global state $|\Psi\rangle = |\psi_\alpha\rangle_{B_\alpha C_\alpha}|\psi_\beta\rangle_{C_\beta A_\beta}|\psi_\gamma\rangle_{A_\gamma B_\gamma}$ and the measurements $\{E^a_A\}_a$, $\{E^b_B\}_b$, $\{E^c_C\}_c$ acting on systems $A_\beta A_\gamma$, $B_\gamma B_\alpha$, and $C_\alpha C_\beta$. If the strategy leads to a TC distribution $P(a,b,c)$, arising from a TC strategy with the $N_\alpha, N_\beta, N_\gamma$ tokens distributed by the sources accordingly to probabilities $p_\alpha(i)$, $p_\beta(j)$, $p_\gamma(k)$, then each quantum system $X_\xi = \mathbf{X}_\xi J_{X\xi}$ can be decomposed in subsystems $\mathbf{X}_\xi$ and $J_{X,\xi}$ such that the quantum strategy takes the form

$$E^x_X = \left(\sum_{j+\ell=x} |j\rangle\langle j|_{\mathbf{X}_\xi} \otimes |\ell\rangle\langle \ell|_{\mathbf{X}_{\xi'}} \right) \otimes \mathbb{1}_{J_{X\xi}J_{X\xi'}}$$

$$|\psi_\xi\rangle_{X_\xi Y_\xi} = \sum_{i=0}^{N_\xi} \sqrt{p_\xi(i)}\, |i, N_\xi - i\rangle_{\mathbf{X}_\xi\mathbf{Y}_\xi} |j^{(i)}_\xi\rangle_{J_{X\xi}J_{Y\xi}}, \quad (11)$$

where ξ and ξ' denote the sources connected to party X, and X and Y denote the parties connected to source ξ.

Proof sketch.—The full proof can be found in Sec. C of the Supplemental Material [24] for any ring network. The idea is to observe that any quantum strategy given by Eqs. (9) and (10) defines a unique classical strategy, where each source ξ samples integer local variables (i,j) according to the probability distribution $|\Psi^{(\xi)}_{ij}|^2$, and sends them to the neighboring parties X_ξ and Y_ξ. Upon receiving two such variables from the neighboring sources, each party outputs $x(j,\ell)$ for which $(j,\ell)\in\mathbb{S}(x)$ in Eq. (9). Classical rigidity of TC distributions implies a unique possible $|\Psi^{(\xi)}_{ij}|^2$ and enforces Eq. (11). ∎

RGB4 distribution.—To illustrate the power of these results, we now consider quantum nonlocal distributions $P_Q(a,b,c)$ (with outcomes $a,b,c\in\{0,2,1_0,1_1\}$) on the triangle of Ref. [14]. Here each source distributes a maximally entangled 2-qubit state $|\psi^+\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$, and each party performs the same 2-qubit projective measurement $\{\Pi^0 = |00\rangle\langle 00|, \Pi^2 = |11\rangle\langle 11|, \Pi^{1_0} = |\bar{1}_0\rangle\langle\bar{1}_0|, \Pi^{1_1} = |\bar{1}_1\rangle\langle\bar{1}_1|\}$, with $|\bar{1}_i\rangle = u_i|01\rangle + v_i|10\rangle$, where $u_0 = -v_1 = \cos(\theta)$ and $v_0 = u_1 = \sin(\theta)$ with $\theta\in[0,\pi/4]$. The resulting distributions, which we call RGB4, are given by

$$\begin{aligned} P_Q(1_i,1_j,1_k) &= \frac{1}{8}(u_iu_ju_k + v_iv_jv_k)^2 \\ P_Q(1_i,0,2) &= \frac{1}{8}u_i^2, \qquad P_Q(1_i,2,0) = \frac{1}{8}v_i^2 \qquad ↻ \end{aligned} \quad (12)$$

where ↻ means that the equation is valid up to cyclic permutations of the parties. All the other probabilities $P_Q(a,b,c)$ are zero. From the structural results in the previous section, we can prove our main result, namely a partial self-testing of the RGB4 distribution.

Result 3.—Consider a quantum strategy on the triangle with the global state $|\Psi\rangle = |\psi_\alpha\rangle_{B_\alpha C_\alpha}|\psi_\beta\rangle_{C_\beta A_\beta}|\psi_\gamma\rangle_{A_\gamma B_\gamma}$ and the measurements $\{E^a_A\}_a$, $\{E^b_B\}_b$, $\{E^c_C\}_c$ acting on systems $A_\beta A_\gamma$, $B_\gamma B_\alpha$, and $C_\alpha C_\beta$. If the resulting distribution is of the form of RGB4 (for some value of the parameter θ), then the entanglement of formation of each state $|\psi_\xi\rangle$ is lower bound by $\mathcal{E}_F \geq h_{\mathrm{bin}}(\frac{1}{2}(1-\sqrt{1-16r^2}))$; all measurements are nonseparable (across the natural bipartition); and the output of any party (say A) features randomness, as quantified by the conditional entropy $H_{\min}(A|E) \geq -\log_2(\frac{1}{2}(1+\sqrt{1-4r}))$.

Here the parameter r quantifies the coherence of the observed distribution and can be lower bounded by $r \geq \frac{1}{2}\sin^3(\theta)(3\cos(\theta) + \cos(3\theta) - 6\sin(\theta))$.

To prove Result 3, we first notice that if the outputs of the RGB4 distribution are coarse grained by merging 1_0 and 1_1 into a single outcome 1, the resulting distribution $\bar{P}_Q(a,b,c)$ with $a,b,c=\{0,1,2\}$ becomes TC (with a single token sent left or right at random). Thus by Result 2 we know that the states and the measurement are of the form

$$|\psi_\xi\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{\mathbf{X}_\xi\mathbf{Y}_\xi}|j^c_\xi\rangle_{J_\xi} + |10\rangle_{\mathbf{X}_\xi\mathbf{Y}_\xi}|j^a_\xi\rangle_{J_\xi}) \quad (13)$$

$$\begin{aligned} \Pi^0_X &= |00\rangle\langle 00|_{\mathbf{X}_\xi\mathbf{X}_{\xi'}} \otimes \mathbb{1}_{J_X} \\ \Pi^2_X &= |11\rangle\langle 11|_{\mathbf{X}_\xi\mathbf{X}_{\xi'}} \otimes \mathbb{1}_{J_X} \\ \Pi^1_X &= \Pi^{1_0}_X + \Pi^{1_1}_X = (|01\rangle\langle 01| + |10\rangle\langle 10|)_{\mathbf{X}_\xi\mathbf{X}_{\xi'}} \otimes \mathbb{1}_{J_X}. \end{aligned} \quad (14)$$

Here, a dilation step is in general required to write the projectors $\Pi^{1_0}_X$ and $\Pi^{1_1}_X$ before coarsegraining, and the auxiliary system is absorbed into one of the incoming junk systems; see Sec. D of the Supplemental Material [24] for details. With the help of Eqs. (13) and (14) we express the output probabilities as

$$\begin{aligned} P_Q(1_i,1_j,1_k) &= \frac{1}{8}\|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}(|\Psi^c\rangle + |\Psi^a\rangle)\|^2 \\ P_Q(1_i,0,2) &= \frac{1}{8}\|\Pi_X^{1_i}|\Psi^a\rangle\|^2 \quad \circlearrowright \\ P_Q(1_i,2,0) &= \frac{1}{8}\|\Pi_X^{1_i}|\Psi^c\rangle\|^2 \quad \circlearrowright \end{aligned} \tag{15}$$

where we introduced the global states

$$\begin{aligned} |\Psi^c\rangle &\equiv |01,01,01\rangle_{\mathbf{B}_\alpha\mathbf{C}_\alpha\mathbf{C}_\beta\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{B}_\gamma}|j_\alpha^c,j_\beta^c,j_\gamma^c\rangle_{J_\alpha J_\beta J_\gamma} \\ |\Psi^a\rangle &\equiv |10,10,10\rangle_{\mathbf{B}_\alpha\mathbf{C}_\alpha\mathbf{C}_\beta\mathbf{A}_\beta\mathbf{A}_\gamma\mathbf{B}_\gamma}|j_\alpha^a,j_\beta^a,j_\gamma^a\rangle_{J_\alpha J_\beta J_\gamma} \end{aligned} \tag{16}$$

corresponding to all the tokens sent clockwise (c) or anticlockwise (a). Here, the probabilities

$$\begin{aligned} 8P_Q(1_i,1_j,1_k) = {}& \|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^c\rangle\|^2 + \|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^a\rangle\|^2 \\ & + 2\mathrm{Re}\langle\Psi^c|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^a\rangle \end{aligned} \tag{17}$$

are particularly interesting because they involve a coherence term between the global states $|\Psi^a\rangle$ and $|\Psi^c\rangle$, which only has a quantum interpretation. As $\Pi_X^{1_1}|\Psi^a\rangle = (\mathbb{1} - \Pi_X^{1_0} - \Pi_X^0 - \Pi_X^2)|\Psi^a\rangle = (\mathbb{1} - \Pi_X^{1_0})|\Psi^a\rangle$, and the states $|\Psi^c\rangle$ and $|\Psi^a\rangle$ are locally orthogonal on each party $\langle\Psi^c|\Pi_X^x\Pi_Y^y|\Psi^a\rangle = 0$, the coherence terms in Eq. (17) are equal up to a sign for all possible values i, j, and k. This allows us to quantify the coherence with a single value:

$$r \equiv (-1)^{i+j+k}2\mathrm{Re}\langle\Psi^c|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^a\rangle. \tag{18}$$

Remarkably, by adopting the nonlocality proof of Ref. [14] we derive a lower bound on the coherence

$$r \geq \frac{1}{2}\sin^3(\theta)\Big(3\cos(\theta) + \cos(3\theta) - 6\sin(\theta)\Big), \tag{19}$$

as a function of the parameter θ; see Sec. D of the Supplemental Material [24] for full details. The bound is the most stringent at $\theta_* \approx 0.36$, where $r \geq r_*$ with $r_* \approx 0.025$. The idea behind the derivation is to show that if r is below the bound and Eq. (15) holds, then $q_a(i,j,k) \equiv \langle\Psi^a|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^a\rangle$ and $q_c(i,j,k) \equiv \langle\Psi^c|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^c\rangle$ cannot be valid probability distributions. Since this last step of the argument ignores the network structure, it is not surprising that the bound [Eq. (19)] we obtain is only nontrivial for the subset of distributions with $\theta \in (0, \theta_{\max} \approx 0.48)$—the same subset where the nonlocality of the distribution has been proven in Ref. [14]. The crucial difference is that it now applies to quantum models. Furthermore, by bounding the coherence r we obtain a partial characterization of any quantum model underlying the RGB4 distribution. Quite an insightful one, as we will now see.

Genuine network nonlocality.—Let us first show that the RGB4 distribution is GNNL, i.e., cannot be simulated by wiring of bipartite quantum boxes [21]. In fact, any such wiring results in measurements Π_X^x that are separable for each party, e.g., $\Pi_A^a = \sum_k p_k|\Psi_k^a\rangle\langle\Psi_k^a|_{A_\beta} \otimes |\Phi_k^a\rangle\langle\Phi_k^a|_{A_\gamma}$ for Alice. Since these measurements also satisfy the TC conditions [Eq. (14)], it follows that $\langle 00|\Psi_k^{1_i},\Phi_k^{1_i}\rangle = \langle 11|\Psi_k^{1_i},\Phi_k^{1_i}\rangle = 0$. Hence, these states are either of the form $|\Psi_k^{1_i},\Phi_k^{1_i}\rangle = |01\rangle_{\mathbf{A}_\beta\mathbf{A}_\gamma}|\zeta\rangle_{J_A}$ or $|\Psi_k^{1_i},\Phi_k^{1_i}\rangle = |10\rangle_{\mathbf{A}_\beta\mathbf{A}_\gamma}|\zeta\rangle_{J_A}$ for each k. But such measurements do not erase the information on the direction of each token, and give no coherence $\langle\Psi^c|\Pi_A^{1_i}\Pi_B^{1_j}\Pi_C^{1_k}|\Psi^a\rangle = 0$ (even if only one of the measurements $\Pi_X^{1_x}$ is separable). Hence, the distribution $P_Q(a,b,c)$ is genuinely network nonlocal if $r \neq 0$.

Quantifying source entanglement.—Next we show that all the states distributed by the sources are entangled and quantify the amount of entanglement. To analyze the entanglement distributed by the sources we need a more precise description of the states. Let us decompose the junk system J_ξ into some unknown auxiliary degrees of freedom $X'_\xi Y'_\xi$ that are indeed received and measured by the parties X and Y, and a system E_ξ which can be controlled by an eavesdropper (Eve). Starting with

$$|\psi_\xi\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{\mathbf{X}_\xi\mathbf{Y}_\xi}|j_\gamma^c\rangle_{X'_\xi Y'_\xi E_\xi} + |10\rangle_{\mathbf{X}_\xi\mathbf{Y}_\xi}|j_\gamma^a\rangle_{X'_\xi Y'_\xi E_\xi})$$

and tracing out Eve's systems we define the states

$$\rho^{(\xi)}_{\mathbf{X}_\xi\mathbf{Y}_\xi X'_\xi Y'_\xi} \equiv \mathrm{tr}_{E_\xi}|\psi_\xi\rangle\langle\psi_\xi| \tag{20}$$

received by the parties. Knowing that the measurements act trivially on the system E_ξ kept by the eavesdropper, we want to show that all these states are entangled.

This can be shown by noting that if one state was separable the rigidity constraints would imply $r = 0$. Instead, we will directly proceed to bound the entanglement of formation $\mathcal{E}_F$ [25] of the state $\rho^{(\alpha)}$ (or any of the other two), defined as $\mathcal{E}_F(\rho^{(\alpha)}) \equiv \min\sum_k p_k S(\mathrm{tr}_B|\psi_k\rangle\langle\psi_k|)$ such that $\rho^{(\alpha)} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, where S is the von Neumann entropy. The rigidity constraint [Eq. (13)] guarantees that each state in the partition of $\rho^{(\alpha)}$ is of the form $|\psi_k\rangle = \sqrt{q_k}|01\rangle_{\mathbf{B}_\alpha\mathbf{C}_\alpha}|\phi_k\rangle_{B'_\alpha C'_\alpha} + \sqrt{1-q_k}|10\rangle_{\mathbf{B}_\alpha\mathbf{C}_\alpha}|\zeta_k\rangle_{B'_\alpha C'_\alpha}$ for some unknown states $|\phi_k\rangle$ and $|\zeta_k\rangle$ of the auxiliary systems. Furthermore, the entropy of entanglement of this state is trivially bounded $S(\mathrm{tr}_{\mathbf{B}_\alpha B'_\alpha}|\psi_k\rangle\langle\psi_k|) \geq h_{\mathrm{bin}}(q_k)$ by the entropy $h_{\mathrm{bin}}(q_k)$ of the binary probability distribution $(q_k, 1-q_k)$. Hence we have that $\mathcal{E}_F(\rho^{(\alpha)}) \geq \min\sum_k p_k h_{\mathrm{bin}}(q_k)$. On top of that it is not difficult to see that the inequality [Eq. (18)] implies $\sum_k p_k\sqrt{q_k(1-q_k)} \geq 2r$ for any partition of $\rho^{(\alpha)}$. Minimizing $\sum_k p_k h_{\mathrm{bin}}(q_k)$ under this constraint we get a lower bound on the entanglement of formation

$$\mathcal{E}_F(\rho^{(\alpha)}) \geq h_{\mathrm{bin}}\left(\frac{1}{2}(1-\sqrt{1-16r^2})\right). \quad (21)$$

Hence, all sources must produce entanglement when $r \neq 0$. All details can be found in Sec. E of the Supplemental Material [24]. For the maximal value r_* certified by Eq. (19), we find that $\mathcal{E}_F(\rho^{(\xi)}) > 2.5\%$.

Quantifying output randomness.—Here we lower bound the entropy of an output (say a). To simplify the problem, we use a binary coarse graining: $\bar{a} = 0$ (for $a = 0, 2$) and $\bar{a} = 1$ (for $a = 1_0, 1_1$) encoded in the register $\bar{A}$, since Eq. (14) guarantees the junk degrees of freedom have no influence on $\bar{a}$. When tracing out all the systems but $\bar{A}E$ one finds a simple classical-quantum state

$$\varrho_{\bar{A}E} = \frac{1}{2}|0\rangle\langle 0|_{\bar{A}} \otimes \rho_{E|\bar{a}=0} + \frac{1}{2}|1\rangle\langle 1|_{\bar{A}} \otimes \rho_{E|\bar{a}=1}$$
$$\rho_{E|\bar{a}=0} = \frac{1}{2}(\rho^{ca}_{E_{\beta\gamma}} + \rho^{ac}_{E_{\beta\gamma}}), \quad \rho_{E|\bar{a}=1} = \frac{1}{2}(\rho^{cc}_{E_{\beta\gamma}} + \rho^{aa}_{E_{\beta\gamma}}), \quad (22)$$

where $\rho^{xy}_{E_{\beta\gamma}} = \rho^x_{E_\beta} \otimes \rho^y_{E_\gamma}$ are the conditional states of Eve with $\rho^x_{E_\xi} = \mathrm{tr}_{X'_\xi Y'_\xi}|j^x_\xi\rangle\langle j^x_\xi|_{X_\xi Y'_\xi E_\xi}$. Eve's conditional min-entropy [26] is related by $H_{\min}(\bar{A}|E) = -\log_2(\frac{1}{2}[1+D(\rho_{E|\bar{a}=0}, \rho_{E|\bar{a}=1})])$ to the trace distance D between her marginal states. Clearly, the entropy is not zero, as Eve's perfect knowledge of the direction of tokens ($D = 1$) would imply no coherence ($r = 0$). Nevertheless, we found that the technical challenge of deriving a decent upper bound on D from a lower bound on r is not straightforward. In Sec. E of the Supplemental Material [24] we show that $D(\rho_{E|\bar{a}=0}, \rho_{E|\bar{a}=1}) \geq \sqrt{1-4r}$, leading to

$$H_{\min}(\bar{A}|E) \geq -\log_2\left(\frac{1}{2}(1+\sqrt{1-4r})\right). \quad (23)$$

For the maximal value r_* we find $H_{\min}(\bar{A}|E) \geq 3.8\%$.

Generalizations.—The above partial self-testing (Results 1 and 2) can be generalized to any ring network; see supplementary material [24]. In Sec. F of the Supplemental Material [24] we extend Result 2 for parity token counting (PTC) distributions on the triangle [20]. We expect these results to be helpful to characterize various quantum distributions that become (P)TC upon coarse graining, similarly to our analysis of RGB4.

Conclusion and outlook.—We showed that quantum nonlocal distributions on ring networks without inputs can be partially self-tested, providing a partial characterization of the states and measurements. Applying these methods to the triangle network, we prove that the nonlocal distribution of RGB4 [14] has interesting properties. First, all measurements must be entangled, hence demonstrating GNNL. Also, all states must feature a minimal amount of entanglement. Finally, we obtain a lower bound on the min-entropy for a local outcome, hence quantifying the amount of randomness.

An interesting question is whether the RGB4 can be proven to be FNNL. Here we show a first step in this direction, namely that if the experiment abides by quantum physics then all sources must produce entanglement. But can one prove that all sources must be nonlocal, even if stronger-than-quantum nonsignaling resources are accessible? A related question is to show that the RGB4 distribution is genuine network nonlocal when considering sources that produce nonsignaling correlations and local wirings [21]. Finally, it would be desirable to make our results robust to noise. A first step could be to obtain approximate rigidity results for (P)TC.

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Appendix: Technical.—In this Appendix, we give a formal derivation of Result 1. A first step consists in proving that the operators U_X defined in Eq. (A3) are unitary, which is required to use Result 1.

We start again from the observed probability distribution $P(a,b,c)$. Rather than considering POVM measurements as in Eq. (3), we now present a rigorous derivation using Stinespring's dilation theorem. Specifically, each POVM $\{E^x_X\}_x$ can be dilated to a projective one $\{\bar{\Pi}^x_X\}_x$. This is done by introducing an auxiliary system M_X prepared in the state $|0\rangle_{M_X}$ for each party, so that the projectors $(\bar{\Pi}^x_X)^2 = \bar{\Pi}^x_X$ act on systems $A_\beta A_\gamma M_A$, $B_\gamma B_\alpha M_B$, and $C_\alpha C_\beta M_C$ respectively and satisfy $E^x_X = \mathrm{tr}_{M_X}\bar{\Pi}^x_X|0\rangle\langle 0|_{M_X}$; see Sec. A of the Supplemental Material [24]. The output probability distribution is now given by

$$P(a,b,c) = \langle\Psi,\mathbf{0}|\bar{\Pi}^a_A\bar{\Pi}^b_B\bar{\Pi}^c_C|\Psi,\mathbf{0}\rangle, \quad (A1)$$

with $|\mathbf{0}\rangle = |0,0,0\rangle_{M_A M_B M_C}$. Similarly to the main text derivation, we note that for a quantum model the constraint [Eq. (4)] can be put in the form

$$\bar{\Pi}^a_A\bar{\Pi}^b_B\bar{\Pi}^c_C|\Psi,\mathbf{0}\rangle = 0 \quad \text{if } a+b+c \neq N. \quad (A2)$$

Next, for each party X we define a unitary operator

$$\bar{U}_X \equiv \sum_x e^{i\varphi_x}\bar{\Pi}^x_X, \quad \text{with} \quad \varphi_x = \frac{2\pi(x+1/3)}{N+1}, \quad (A3)$$

where the integer x runs through possible outputs of the party. These definitions allow us to put the constraint [Eq. (A2)] in a particularly simple form,

$$\bar{U}_A\bar{U}_B\bar{U}_C|\Psi,\mathbf{0}\rangle = |\Psi,\mathbf{0}\rangle. \tag{A4}$$

To see this note that $\bar{U}_A\bar{U}_B\bar{U}_C$ is a global unitary with eigenvalues $e^{\mathrm{i}(a+b+c+1)[2\pi/(N+1)]}$, Eq. (A2) guarantees that the state is only supported on the subspace associated to the eigenvalue $e^{\mathrm{i}(a+b+c+1)[2\pi/(N+1)]} = 1$. This condition implies that the dilation of measurements is trivial, as shown in the following result.

Result 0.—For unitaries $\bar{U}_X$ defined in Eq. (A3), the identity [Eq. (A4)] implies that the original measurements are projective:

$$E_X^x = \langle 0|_{M_X}\bar{\Pi}_X^x|0\rangle_{M_X} = \Pi_X^x. \tag{A5}$$

Proof sketch.—The detailed proof is given in Sec. A of the Supplemental Material [24]. The condition [Eq. (A4)] ensures that the unitaries do not change the state of the auxiliary systems and imply that the operators $U_X = \langle 0|_{M_X}\bar{U}_X|0\rangle_{M_X}$ are also unitary. But $U_X = \sum_x e^{\mathrm{i}\varphi_x}E_X^x$ can only be unitary if $\{E_X^x = \Pi_X^x\}_x$ is a projector valued measure. ■

Hence, we can rewrite Eq. (A4) in a simpler form, $U_AU_BU_C|\Psi\rangle = |\Psi\rangle$, where $U_X = \sum_x e^{\mathrm{i}\varphi_x}\Pi_X^x$. This condition implies that the unitaries are product. This is the content of Result 1, which we can now state more rigorously.

Result 1.—Consider a quantum state $|\Psi\rangle = |\psi_\alpha\rangle_{B_\alpha C_\alpha}|\psi_\beta\rangle_{C_\beta A_\beta}|\psi_\gamma\rangle_{A_\gamma B_\gamma}$ on the triangle network, and local unitaries U_A, U_B, U_C acting on the systems $A_\beta A_\gamma$, $B_\gamma B_\alpha$, and $C_\alpha C_\beta$. The condition $U_AU_BU_C|\Psi\rangle = |\Psi\rangle$ implies that all the unitaries are product:

$$\begin{aligned} U_A &= V_{A_\beta}\otimes W_{A_\gamma} \\ U_B &= V_{B_\gamma}\otimes W_{B_\alpha} \\ U_C &= V_{C_\alpha}\otimes W_{A_\beta} \end{aligned}$$

with unitary V_{X_ξ} and W_{X_ξ} acting on the respective systems.

Proof sketch.—The proof is in Sec. B of the Supplemental Material [24] for any ring network. We use the Schmidt decomposition of the states $|\psi_\xi\rangle$ for "moving" an operator to act on the other half of an entangled state (upon transposition and rescaling). Together with the Choi-Jamiołkowski isomorphism, this allows us to express the constraint [Eq. (A4)] as an equality between products of bipartite operators acting on three systems. Finally, we prove a technical lemma showing that these operators are products. ■

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[23] The Hilbert space associated to each system is taken as the support of the state, e.g., $\mathcal{H}_{B_\alpha} = \mathrm{supp}(\mathrm{tr}_{C_\alpha}|\psi_\alpha\rangle\langle\psi_\alpha|)$. Thus, when discussing a system in the following, we refer to the Hilbert space where $|\Psi\rangle\langle\Psi|$ is supported, which is natural in the device-independent framework.

[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.100201. Section A presents the notation for general ring networks and the proof of Result 0. Sections B and C contain the proofs of Results 1 and 2 respectively. Section D presents the detailed derivation of the bound on the coherence parameter r [Eq. (18)]. In Sec. E bounds on the entanglement of the sources (Eq. 20) as well as randomness of the outputs (Eq. 22) are derived. Finally Sec. F presets the extension of Result 2 to parity token counting distributions.

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