Formation of Rogue Waves and Modulational Instability with Zero-Wavenumber Gain in Multicomponent Systems with Coherent Coupling

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It is known that rogue waves (RWs) are generated by the modulational instability (MI) of the baseband type. Starting with the Bers-Kaup-Reiman system for three-wave resonant interactions, we identify a specific RW-building mechanism based on MI which includes zero wavenumber in the gain band. An essential finding is that this mechanism works solely under a linear relation between the MI gain and a vanishingly small wavenumber of the modulational perturbation. The same mechanism leads to the creation of RWs by MI in other multicomponent systems—in particular, in the massive Thirring model.

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Introduction.—The modulational instability (MI) of constant-amplitude continuous waves (cw) against long-wavelength perturbations plays a fundamental role in understanding nonlinear-wave dynamics [1–4]. MI has been predicted and observed in deep water [1,2,5], plasmas [6–8], electric transmission lines [9,10], optics [11–22], matter waves [23–36], and other physical media [37–45].

Theoretical and experimental studies of MI started in the 1960s [1–3,37,46–48]. Nowadays, MI is a subject of great interest as a mechanism underlying the formation of rogue waves (RWs) [49–54], which produce a dramatic impact on the surrounding environment [41,55–57]; see also comprehensive reviews [53,54]. Studies of RWs have been performed in hydrodynamics, optics, plasmas, Bose-Einstein condensates [41,55–72], and other fields [73]. Nevertheless, a complete understanding of RW formation is still an open question. While the generation of RWs is driven by MI, not every kind of MI leads to this outcome [51,74]. Two generic types of MI are baseband and passband ones [51,74]. In the former case, the cw background is unstable against perturbations with infinitesimal wavenumbers Q and vanishingly small gain $|Q|$, while in the latter case MI is absent at $|Q| < Q_{\min}$ with finite $Q_{\min}$. It was found that RWs can be generated solely by MI of the baseband type.

The fact that the gain of the baseband MI vanishes at $Q = 0$ suggests a question if a physically meaningful system can give rise to MI with nonzero gain at $Q = 0$, and whether MI of this type leads to RW formation. Here, using a system for the three-wave resonant interaction, we demonstrate that such zero-wavenumber-gain (ZWG) MI exists, and indeed leads to RW formation, under the condition that an asymptotically linear relation between the MI gain and wavenumber Q holds, see Eq. (11) below.

The three-wave resonant-interaction system and RW existence condition.—We consider the Bers-Kaup-Reiman (BKR) system of equations for three waves $E_{1,2,3}(x,t)$ coupled by the saturable quadratic interaction, which models the resonant three-wave coupling in hydrodynamics, optics, microwaves, and plasmas [75–83]:

$$(E_n)_t + V_n \cdot (E_n)_x = \frac{\sigma_n E_k^* E_l^*}{1 + \epsilon \sum_{n=1}^{3} (|E_n|^2 - a_n^2)^2}. \quad (1)$$

Here $\{n,k,l\}$ are sets of $\{1,2,3\}$ and their transpositions, $V_1 > V_2 > V_3 \equiv 0$ are group velocities, $*$ is the complex conjugate, and σ_j are signs of the interactions, which represent the stimulated-backscattering ($\sigma_1 = \sigma_2 = -\sigma_3 = 1$), soliton-exchange ($\sigma_1 = -\sigma_2 = \sigma_3 = 1$), and explosive ($\sigma_1 = \sigma_2 = \sigma_3 = 1$) regimes. In the latter case, a complicating factor is that in the system is vulnerable to the onset of blowup, therefore it includes the saturation represented by the term $\epsilon \geq 0$ in Eq. (1) [84]. The original system ($\epsilon = 0$) gives rise to cw solution (5) written below, with amplitudes a_n, whose MI and the emerging RWs are the same as in the saturable system ($\epsilon > 0$).

Equation (1) is integrable when $\epsilon = 0$ [75,76], making it possible to produce exact RW solutions via the Hirota method [81,85],

$$E_j = a_j \frac{(\xi + \theta_j)(\xi^* - \theta_j^*) + \eta_0}{|\xi|^2 + \eta_0} e^{i\phi_j}, \qquad (j = 1,2),$$
$$E_3 = i a_3 \frac{(\xi - \theta_1 - \theta_2)(\xi^* + \theta_1^* + \theta_2^*) + \eta_0}{|\xi|^2 + \eta_0} e^{-i(\phi_1 + \phi_2)}, \quad (2)$$

where $a_{1,2,3}$ are nonzero real constants, and $\xi = (\alpha - \beta)x - (V_2\alpha - V_1\beta)t$, $\alpha = [(-\gamma_1)/(V_1 - V_2)p_0^2]$,

$\beta=[(-\gamma_2)/(V_1-V_2)(p_0-i)^2]$, $\phi_j=c_jx+d_jt$, $d_1=d_2=(\gamma_3/2)$, $c_1=-[(2\gamma_1+\gamma_3)/2V_1]$, $c_2=-[(2\gamma_2+\gamma_3)/2V_2]$, $\theta_1=[1/(p_0-i)]$, $\theta_2=-(1/p_0)$, $\eta_0=[1/(p_0+p_0^*)^2]$, with $\gamma_1=\sigma_1a_2a_3/a_1$, $\gamma_2=\sigma_2a_1a_3/a_2$, $\gamma_3=\sigma_3a_1a_2/a_3$, and p_0 taken as a nonimaginary root of a quartic equation,

$$\gamma_3(V_1-V_2)p^2(p-i)^2-\gamma_1V_2(p-i)^2+\gamma_2V_1p^2=0. \quad (3)$$

To make parameters a_1, a_2, a_3, V_1, and V_2 satisfying the condition that the root of Eq. (3) must be nonimaginary for the BKR system of the stimulated-backscattering or explosive type, the sign of the discriminant of Eq. (3),

$$\Delta=-\gamma_1\gamma_2\gamma_3\{[\gamma_1V_2-\gamma_2V_1+\gamma_3(V_1-V_2)]^3 + 27V_1V_2(V_1-V_2)\gamma_1\gamma_2\gamma_3\}, \quad (4)$$

must be subject to constraint $\Delta<0$. For the BKR system of the soliton-exchange type, the constraint securing the existence of the RWs is, instead, $\Delta\geq 0$ [85].

ZWG-MI and the general mechanism for the RW formation.—Equation (1) admits cw solutions

$$E_j=a_j\exp[i(c_jx+d_jt)],$$
$$E_3=ia_3\exp[-i((c_1+c_2)x+(d_1+d_2)t)], \quad (5)$$

where $c_1=-[\sigma_1\sigma_3a_2^2+d_1(d_1+d_2)]/[V_1(d_1+d_2)]$, $c_2=-[\sigma_2\sigma_3a_1^2+d_2(d_1+d_2)]/[V_2(d_1+d_2)]$, $a_3=\sigma_3a_1a_2/(d_1+d_2)$, with free real parameters a_j and d_j representing cw amplitudes and frequencies, respectively. Using invariances of Eq. (1), we fix $a_{1,2}$ to be real, and set $d_1=d_2=\sigma_3a_1a_2/(2a_3)$. Thus, $a_{1,2,3}$ control the cw. Actually, cw (5) is the background supporting the RW states (2). For the same set of a_n, the cw solution and the results for its MI, following below, remain fully valid for $\epsilon>0$ in Eq. (1).

To address MI, the perturbed cw is written as $\widetilde{E_n}=E_n(1+p_n(x,t)/a_n)$, where

$$p_n(x,t)\equiv\eta_{n,1}(t)e^{iQx}+\eta_{n,2}(t)e^{-iQx} \quad (6)$$

are small perturbations with wavenumber Q. The linearized equations for the perturbations amount to a 6×6 system, $d\eta/dt=i\mathbf{M}\eta$, with $\eta=(\eta_{1,1},\eta_{1,2}^*,\eta_{2,1},\eta_{2,2}^*,\eta_{3,1},\eta_{3,2}^*)^T$, nonzero matrix elements of $\mathbf{M}$ being $M_{11}=\sigma_1a_2a_3/a_1-V_1Q$, $M_{22}=-\sigma_1a_2a_3/a_1-V_1Q$, $M_{33}=\sigma_2a_1a_3/a_2-V_2Q$, $M_{44}=-\sigma_2a_1a_3/a_2-V_2Q$, $M_{55}=-M_{66}=\sigma_3a_1a_2/a_3$, $M_{41}=-M_{32}=\sigma_2a_3$, $M_{23}=-M_{14}=\sigma_1a_3$, $M_{61}=-M_{52}=\sigma_3a_2$, $M_{25}=-M_{16}=\sigma_1a_2$, $M_{63}=-M_{54}=\sigma_3a_1$, and $M_{45}=-M_{36}=\sigma_2a_1$.

The stability of $\widetilde{E_n}$ is determined by eigenvalues Ω of $\mathbf{M}$, which are roots of the following characteristic polynomial,

$$B(\Omega)=\Omega^6+\lambda_5\Omega^5+\lambda_4\Omega^4+\lambda_3\Omega^3+\lambda_2\Omega^2+\lambda_1\Omega+\lambda_0, \quad (7)$$

where $\lambda_0=-V_1^2V_2^2\gamma_3^2Q^4$, $\lambda_1=2V_1V_2\gamma_3[V_2(\gamma_1-\gamma_3)+V_1(\gamma_2-\gamma_3)]Q^3$, $\lambda_2=\{V_1V_2[V_1V_2Q^2+6\gamma_3(\gamma_1+\gamma_2-\gamma_3)]-[V_2(\gamma_1-\gamma_3)-V_1(\gamma_2-\gamma_3)]^2\}Q^2$, $\lambda_3=2\{(V_1+V_2)[V_1V_2Q^2+\gamma_3(\gamma_1+\gamma_2-\gamma_3)+\gamma_1\gamma_2]-V_2\gamma_1(\gamma_1-\gamma_3)-V_1\gamma_2(\gamma_2-\gamma_3)\}Q$, $\lambda_4=(V_1^2+V_2^2+4V_1V_2)Q^2-(\gamma_1+\gamma_2-\gamma_3)^2+4\gamma_1\gamma_2$, and $\lambda_5=2(V_1+V_2)Q$.

The six roots of (7) are either real ones or complex-conjugate pairs. The MI emerges in the latter case, being accounted for by the roots with $\mathrm{Im}(\Omega)<0$. There are three different types of the MI: (i) Baseband-MI: $\mathrm{Im}(\Omega)<0$ at $|Q|>0$ and $\mathrm{Im}(\Omega)=0$ at $Q=0$, i.e., the MI band includes small wavenumbers Q but *not* $Q=0$. (ii) Passband-MI: $\mathrm{Im}(\Omega)<0$ at $|Q|>Q_{\min}>0$ with a nonzero boundary $Q_{\min}$ of the MI band, which separates it from $Q=0$. (iii) ZWG-MI: $\mathrm{Im}(\Omega)<0$ at $|Q|<Q_{\max}$ with $Q_{\max}>0$, i.e., the MI band *includes* zero wavenumber, $Q=0$. This situation implies that the mechanical system with three degrees of freedom, which corresponds to Eq. (1) with x-independent fields, is itself unstable, as it represents an amplifying setup.

To address the ZWG-MI, we set $Q=0$ in Eq. (7), obtaining possible nonzero roots $\Omega=\pm\sqrt{\Omega_0^2}$, with

$$\Omega_0^2=(\gamma_1+\gamma_2-\gamma_3)^2-4\gamma_1\gamma_2. \quad (8)$$

Thus, the ZWG-MI exists for $\Omega_0^2<0$, as Eq. (7) has two mutually conjugate imaginary roots Ω at $Q=0$. On the other hand, if $\Omega_0^2\geq0$, Eq. (7) has no imaginary roots at $Q=0$, hence only the baseband or passband MI is possible. A conclusion is that the ZWG-MI occurs if all σ_n in Eq. (1) have the same sign, i.e., solely in the case of the explosive three-wave system. Unless mentioned otherwise, we set $\sigma_1=\sigma_2=\sigma_3=1$ below.

Subsequently, we focus on the MI in the crucially important limit of $Q\to0$. Accordingly, if Eq. (8) yields $\Omega_0^2\neq0$, we approximate (7) as $B(Q\Omega)=Q^4b^{(1)}(\Omega)$, hence Eq. (7) amounts to

$$b^{(1)}(\Omega)=-\Omega_0^2\Omega^4+b_3\Omega^3+b_2\Omega^2+b_1\Omega+b_0=0, \quad (9)$$

where $b_0=-V_1^2V_2^2\gamma_3^2$, $b_1=2V_1V_2\gamma_3[V_2(\gamma_1-\gamma_3)+V_1(\gamma_2-\gamma_3)]$, $b_2=6V_1V_2\gamma_3(\gamma_1+\gamma_2-\gamma_3)-[V_2(\gamma_1-\gamma_3)-V_1(\gamma_2-\gamma_3)]^2$ and $b_3=2\{(V_1+V_2)[\gamma_3(\gamma_1+\gamma_2-\gamma_3)+\gamma_1\gamma_2]-V_2\gamma_1(\gamma_1-\gamma_3)-V_1\gamma_2(\gamma_2-\gamma_3)\}$.

Equation (9) with $b_0<0$ yields, at least, two simple real roots when $\Omega_0^2<0$. If $\Omega_0^2>0$, Eq. (9) has two simple real roots at all values of parameters. Because the discriminant of quartic equation (9) coincides with that of Eq. (3), i.e., Δ [see Eq. (4)], the RW existence condition, $\Delta<0$, can be obtained from the discriminant of Eq. (9).

Thus, for $\Omega_0^2\neq0$, two cases are possible. (i) If $\Delta\geq0$, all roots of Eq. (9) are real, and no baseband-MI occurs. Specifically, if $\Delta\geq0$ and $\Omega_0^2<0$, there exists a ZWG-MI region; if $\Delta\geq0$ and $\Omega_0^2>0$, there is passband-MI or no MI

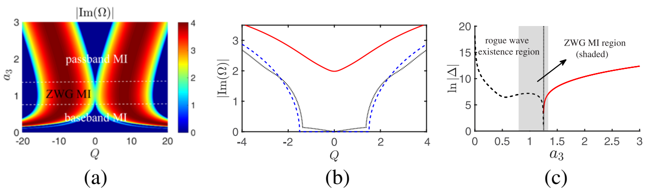

FIG. 1. (a) The map of the MI gain in parameter plane (Q, a_3) of Eq. (1) with fixed parameters $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $V_1 = 2$, $V_2 = 1$, $a_1 = 4$, $a_2 = 1$, and $\epsilon = 0$. (b) The MI gain, $|\mathrm{Im}(\Omega)|$, vs Q, corresponding to panel (a) at $a_3 = 2$, $a_3 = 1$, and $a_3 = 0.5$ (the dashed blue, solid red, and dotted black curves, respectively). Panel (c) shows the RW existence area, $0 < a_3 < 1.247$, no RWs existing at $a_3 > 1.247$. In these areas, separated by the dotted vertical line, the dashed black and solid red curves show dependences of $\ln|\Delta|$ on a_3, with $\Delta < 0$ and $\Delta \geq 0$ in the left and right areas, respectively. ZWG-MI occurs in the shaded region.

takes place. (ii) If $\Delta < 0$, Eq. (9) produces two complex-conjugate roots, and there exists a baseband-MI region at $\Omega_0^2 > 0$, or a ZWG-MI region at $\Omega_0^2 < 0$.

Figures 1 and 2 display the predicted characteristics of the MI and RW existence range. Figures 1(a) and 1(b) show that the MI of the baseband, ZWG, and passband (or maybe no-MI) types exists, respectively, in the regions of $0 < a_3 \leq 4/5$, $4/5 < a_3 \leq 4/3$, and $a_3 > 4/3$. Based on the sign of Δ [see Eq. (4)], as shown in Fig. 1(c), it is seen that the RW existence condition is $a_3 < 1.247$. Namely, when $0 < a_3 < 1.247$, the MI of the baseband or ZWG types occurs and the RWs exist, but when $1.247 < a_3 < 4/3$, the ZWG-MI occurs too, while RWs do not exist. Figures 2(a) and 2(b) show that the passband-MI (or maybe no-MI) is present when $0 < a_2 < 0.5$, while the ZWG-MI occurs at $a_2 > 0.5$. Figure 2(c) demonstrates that the RW existence condition is $a_2 > 0.5$.

Figure 3 shows an example of a fundamental dark-bright-dark RW in the BKR system, as given by solution (2), with the same parameters as in Fig. 2 and $a_2 = 1$. Such

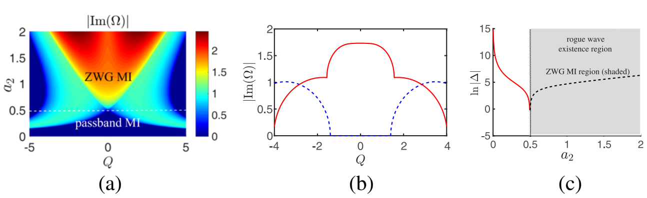

FIG. 2. (a) The map of the MI gain in parameter plane (Q, a_2) of Eq. (1) with fixed parameters $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $V_1 = 2$, $V_2 = 1$, $a_1 = a_3 = 1$, and $\epsilon = 0$. (b) The MI gain vs Q, corresponding to panel (a) at $a_2 = 0.3$ and $a_2 = 1$ (the dashed blue and solid red curves, respectively). Panel (c) shows the RW existence area, $a_2 > 0.5$, no RWs existing at $0 < a_2 < 0.5$. In these areas, separated by the dotted vertical line, the dashed black and solid red curves show dependences of $\ln|\Delta|$ on a_2, with $\Delta \geq 0$ and $\Delta < 0$ in the left and right areas, respectively. ZWG-MI occurs in the shaded region.

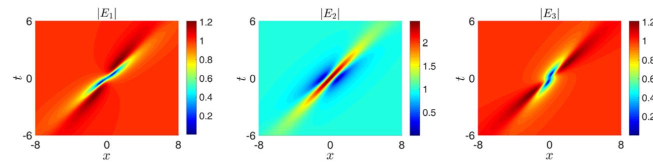

FIG. 3. RWs produced by solution (2) of Eq. (1), with $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $V_1 = 2$, $V_2 = 1$, $a_1 = a_2 = a_3 = 1$, and $\epsilon = 0$. The respective roof of Eq. (3) is $p_0 = 0.930605 - 0.366025i$.

RWs emerge in the ZWG-MI region. Virtually the same RW is produced by direct simulations. In the generic case, multi-RW structures are produced by simulations of Eq. (1) initiated by a chaotically perturbed cw background, as shown in Fig. 4. Following the pattern of Ref. [53], an individual RW selected in the figure is compared to the analytical solution in Fig. 3 of the Supplemental Material [86], which includes Refs. [51,53,84,87].

The above results are established for $\Omega_0^2 \neq 0$. When $\Omega_0^2 = 0$, Eq. (9) is replaced by $B(Q^{1/3}\Omega) = Q^2 b^{(2)}(\Omega)$, and

$$b^{(2)}(\Omega) = \Omega^6 + b_3\Omega^3 = 0. \tag{10}$$

If $b_3 \neq 0$, there are two complex conjugate roots of Eq. (10), and MI is of the baseband type. If $\Omega_0^2 = b_3 = 0$, Eq. (10) is replaced by $B(\sqrt{Q}\Omega) = Q^3 b_3(\Omega)$ and $b^{(3)}(\Omega) = \Omega^6 + b_2\Omega^2 = 0$. We thus infer that, with $b_2 \neq 0$ (b_2, b_3, and Ω_0^2 cannot all be equal to zero), there are at least two complex-conjugate roots, MI being of the baseband type. Therefore, while the baseband-MI occurs at $\Omega_0^2 = 0$, in the case of $\Delta \geq 0$ RWs are absent. Thus, a new feature of the present setting is that RWs may be absent in the baseband-MI region. This situation was not reported before, as it was believed that the presence of baseband-MI always leads to the creation of RWs [51,74].

Thus we arrive at the following conclusions: (i) ZWG-MI generates RWs at $\Delta < 0$, which implies that there exist complex roots Ω of Eq. (7) satisfying

$$\mathrm{Im}(\Omega) = O(Q) \tag{11}$$

(an asymptotically linear dependence) at $Q \to 0$; (ii) the baseband-MI (when $\Omega_0^2 = 0$) cannot generate RWs at

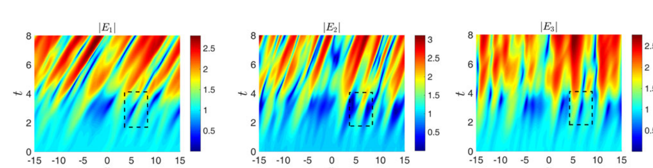

FIG. 4. A multi-RW pattern produced by numerical solution of Eq. (1) with a random perturbation at the 5% level added to the cw background in the ZWG-MI regime with $a_1 = a_1 = a_3 = V_2 = 1$, $V_1 = 2$, and $\epsilon = 0.1$. Dashed-line boxes select an individual RW from the pattern which is compared to the analytical solution in Fig. 3 of the Supplemental Material [86].

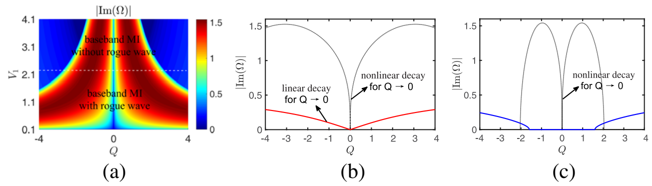

FIG. 5. (a) Maps of the MI gain in parameter plane (Q, V_1) of Eq. (1) with fixed parameters $\epsilon = 0$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $V_2 = 0.1$, $a_1 = \sqrt{3-2\sqrt{2}}$, $a_2 = \sqrt{2(3-2\sqrt{2})}$, and $a_3 = \sqrt{2}$. (b),(c) Dependences of the MI gain, $|\mathrm{Im}(\Omega)|$, as produced by Eq. (7), corresponding to (a), at $V_1 = 1$ and $V_1 = 3$, respectively. Here, the gain branch satisfying Eq. (11) exists in the interval of $0.1 < V_1 < 2.15$.

$\Delta > 0$, which implies that there are no complex roots of Eq. (7) satisfying relation (11); (iii) the baseband-MI (at $\Omega_0^2 \neq 0$) can generate RWs as it satisfies Eq. (11). Therefore, in the regions of MI of the baseband and ZWG types the crucial difference between the presence and absence of RWs is the existence or absence of complex roots of Eq. (9), rather than those of Eq. (7). These facts demonstrate that RWs are generated only when Eq. (11) is valid. Thus, the above analysis implies that solely the MI of the baseband and ZWG types, satisfying condition (11), leads to the formation of RWs. This criterion was not reported previously.

When $Q = 0$, Eq. (7) produces four zero roots and two other ones, $\Omega = \pm\sqrt{\Omega_0^2}$. Condition (11), which produces the asymptotically linear condition of the existence of the rational RW solutions, implies that RWs are related, at $Q = 0$, only to the set of the zero eigenvalues. This fact implies the rational growth of the MI of the respective cw background.

In Fig. 5, we summarize results of the MI analysis produced by varying V_1, while a_n are fixed so as to have $\Omega_0^2 = 0$. As shown in Fig. 5(a), the respective MI is of the baseband type, while RWs exist only in the interval of $0.1 < V_1 < 2.15$. Figures 5(b) and 5(c) show the MI gain, $|\mathrm{Im}(\Omega)|$, as produced by all complex roots of (7) at $V_1 = 1$ and $V_1 = 3$. It is seen, in particular, that Eq. (11) holds for $V_1 = 1$, but not for $V_1 = 3$.

The predicted mechanism of the RW creation can be experimentally realized in amplified three-wave optical, microwave, and hydrodynamic systems. A suitable experimental setup in optics is based on a semiconductor amplifier, providing the generation of light beams with power ~1 W at the standard wavelength, 1.55 μm [88]. For microwave systems, amplifiers using Josephson junctions make it possible to implement the interaction between waves with frequencies ~10 GHz [89,90]. Experiments with water waves can be performed in the frequency range 15–30 Hz, using an apparatus of size ~30 × 30 cm [91]. The boundary conditions which are used to initiate the required wave dynamics are specified in Supplemental Material, Sec. B [86].

Lastly, we present results obtained for the MI and RWs in other integrable systems, that fully agree with the above conclusions. (i) For the BKR system of the soliton-exchange and stimulated-backscattering types, for which condition $\Omega_0^2 > 0$ holds, RWs exist if and only if Eq. (9) has complex roots. Table 1 in Supplemental Material, Sec. A [86] shows the relationship between all possible MI types and RW existence conditions for all types of the BKR system (1). The interpretation of the ZWG-MI in terms of the three-wave mixing, which underlies the BKR system, is additionally considered in Supplemental Material, Sec. C [86]. (ii) In the two-component massive Thirring model, RWs are absent in the case of the ZWG-MI, as Eq. (11) does not hold in that case; RWs do or do not exist in the case of the baseband MI if, respectively, Eq. (11) does or does not hold, as shown in detail analytically and numerically in Supplemental Material, Sec. D [86], which includes Refs. [92–97]. (iii) For other integrable equations which do not give rise to the ZWG-MI, the results concerning the existence of RWs in the case of the baseband-MI amount to a particular case of the above analysis, as Eq. (7) is then the same as Eq. (9), provided that Eq. (11) holds.

Conclusion.—The present work reveals the mechanism for the formation of RWs in multicomponent systems with coherent coupling, i.e., with energy exchange between the components. In the framework of this mechanism, the three-wave BKR system creates RWs in the case of the ZWG-MI, i.e., MI whose gain band includes zero wavenumber. An important finding is that, in both cases of the ZWG and baseband types of MI, the system creates RWs only under the condition of the asymptotically linear relation (11) between the MI gain and small perturbation wavenumber. The same analysis predicts the existence or absence of RWs in other coherently coupled multicomponent systems.

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