Using Adaptiveness and Causal Superpositions Against Noise in Quantum Metrology

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(Received 5 January 2023; accepted 11 July 2023; published 30 August 2023)

We derive new bounds on achievable precision in the most general adaptive quantum metrological scenarios. The bounds are proven to be asymptotically saturable and equivalent to the known parallel scheme bounds in the limit of a large number of channel uses. This completely solves a long-standing conjecture in the field of quantum metrology on the asymptotic equivalence between parallel and adaptive strategies. The new bounds also allow us to easily assess the potential benefits of invoking nonstandard causal superposition strategies, for which we prove, similarly to the adaptive case, the lack of asymptotic advantage over the parallel ones.

DOI: 10.1103/PhysRevLett.131.090801

Introduction.—In the field of quantum information and quantum technologies, one can distinguish three levels of quantumness that are behind the boost in performance of various communication [1,2], computational [3], or metrological tasks [4–6]. The most rudimentary one is quantum coherence (C), which refers to the potential of having a single quantum system in the state of quantum superposition. This is already enough to implement secure quantum key distribution protocols [7] or even reach the Heisenberg limit in noiseless quantum metrology, provided a given quantum probe can pass through a sensing channel multiple times [8,9]. The next level is entanglement (E), where quantum coherence present in multipartite systems manifests itself in the form of nonclassical correlations. This quantumness level is crucial to guarantee quantum speedup in computational tasks [10] as well as to assure the ultimate security in the so-called device-independent quantum key distribution [11]. In quantum metrology, it had long been appreciated as the way to boost the precision in optical and atomic interferometric tasks [12–16], either in the form of N00N states [17,18] or much more practical optical and atomic squeezed states [19,20]. Finally, exploiting the quantum potential to its limits, one can consider adaptive (AD) or active quantum feedback strategies, where the probes are entangled with noiseless ancillary systems, and quantum control operations may actively modify the probe system that will be sent to the subsequent channel based on the information obtained so far [21–25], see Fig. 1. Such protocols represent the most general channel sensing schemes, containing E as a special case and encompassing in particular all quantum error-correcting strategies widely used in the whole field of quantum information processing to counter noise [26–29].

Interestingly, in the absence of noise, AD strategies provide no advantage over optimal E strategies [30]. In the presence of noise, however, some advantages have been observed in the small-number-of-uses regime where a direct search of optimal metrological protocols could be carried out [21,31–35]. In 2014 a conjecture was formulated [21] predicting no asymptotic advantage of AD over E. A notable progress in answering this fundamental question was made in 2021 [36,37], when it was demonstrated that in the models where quantum coherence cannot be protected against noise on an arbitrary scale, and hence the Heisenberg scaling (HS) is not achievable, AD strategies offer no asymptotic advantage over E. Still, the full answer to the question was lacking, mainly due to the fact that the bounds used there were not tight enough.

In this Letter, utilizing our new bounds, we indeed answer the conjecture in an affirmative way, proving in full

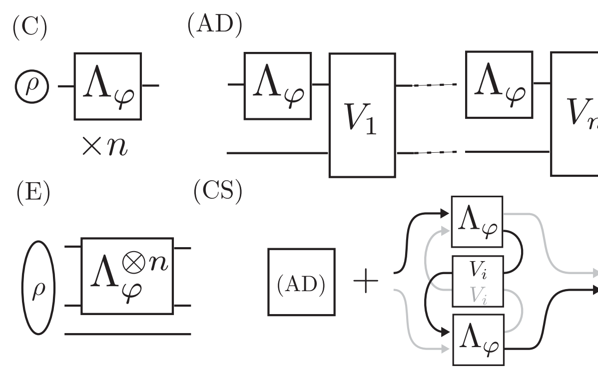

FIG. 1. Metrological schemes utilizing "four levels of quantumness": (C) channels probed independently (basic use of quantum coherence); (E) channels probed in parallel using a general entangled state, with ancillary systems potentially involved; (AD) general adaptive (active quantum feedback) strategies; (CS) causal superposition strategies, where additionally channels may be probed in a superposition of different causal orders.

generality that AD strategies provide no asymptotic advantage over E. As negative as it may sound, the result by no means implies that AD strategies are useless. In fact, our bounds allow us to clearly pinpoint the potential advantage one may expect in the finite number-of-uses regime, and easily observe how the advantage fades away when approaching the asymptotic limit of a large number of channel uses. On a more practical side, adaptive strategies may sometimes be in fact easier to implement than parallel, as they may not necessarily require entangling a large number of particles, while obtaining the same effect via small scale entanglement and active feedback. Even though the "three levels of quantumness" listed above appear to cover all quantum aspects of metrological protocols, an intriguing idea was put forward of considering causal superposition (CS) strategies where different channels are being probed in a superposition of different causal orders [35,38–44]. Advantages of such a strategy over the most general AD strategy have been observed, but no efficiently computable bounds have been proposed. In this Letter, we provide bounds valid also for this more general class of protocols and show their asymptotic equivalence to AD and E, which also means that CS strategies cannot surpass the HS [45].

Introductory example.—Let us start with the most elementary yet very illuminating example of a noisy metrological model, where it is possible to remove noise while assuring the preservation of HS of precision in the asymptotic regime. Consider a single qubit channel $\Lambda_\varphi(\cdot) = \sum_k K_{\varphi,k} \cdot K_{\varphi,k}^\dagger$, where $K_{\varphi,k} = U_\varphi K_k$,

$$U_\varphi = e^{-\frac{i}{2}\sigma_z \varphi}, \qquad K_1 = \sqrt{p}\mathbb{1}, \qquad K_2 = \sqrt{1-p}\sigma_x. \quad (1)$$

The channel represents dephasing of a qubit along the x axis of the Bloch ball (the operator K_2 may be understood as a σ_x error occurring with probability $1-p$) and the subsequent rotation U_φ of the state around the z axis by angle φ, where φ is the parameter to be estimated—a similar model has been used in an experimental demonstration of quantum error-correction enhanced metrology in NV-center sensing setups [46], as well as in [47] where the possibility of beating the standard scaling (SS) in the presence of transversal noise was shown. In the case of a single channel use, $n=1$, the effect of noise may be completely mitigated by choosing the input state as $|\psi^{(1)}\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. This state is not affected by the σ_x error and the output state $|\psi_\varphi\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$ represents a noiseless phase encoding. We will quantify the performance of a given protocol using the quantum Fisher information (QFI) [48,49] of the output state, which in this case is $F^{(1)} = 1$ (we recall the definition of the QFI in [50], Section A).

Assume now that we can use the channel twice, $n=2$. If we send the optimal single qubit probes independently to each of the channels, we get the QFI value $F_C^{(2)} = 2$. We can, however, also consider a parallel strategy involving an entangled input state $|\psi^{(2)}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (the N00N state [52]). In this case if either zero or two σ_x errors occur, the final state will again correspond to the noiseless phase encoding $|\psi_\varphi^{(2)}\rangle = U_\varphi^2|\psi^{(2)}\rangle = (|00\rangle + e^{2i\varphi}|11\rangle)/\sqrt{2}$ for which the QFI equals 4. Whereas, if only a single σ_x occurs the state will contain no information about the phase at all. As a result the final QFI reads $F_E^{(2)} = 4[p^2 + (1-p)^2] \geq F_C^{(2)}$. Interestingly, this result may be further improved via a simple adaptive strategy. The protocol involves entangling the initial single probe qubit with a single ancillary qubit, so that the input state is again $|\psi^{(2)}\rangle$. After a single action of the channel, $\Lambda_\varphi \otimes \mathcal{I}$, an error correction operation is performed, where we check if a σ_x error occurred and correct the error accordingly. Then the channel acts on the probe state again, and with probability p yields the ideal state $|\psi_\varphi^{(2)}\rangle$; while if another σ_x error occurs, the final unitary rotation U_φ removes all the phase information from the state. Consequently, the protocol yields a QFI equal to $4p$. This protocol is actually the optimal one provided $p \geq 0.5$. If $p < 0.5$, then one simply needs to modify the recovery operation in a way that instead of correcting a single σ_x error on the probe system the σ_x operation is applied to the ancillary qubit. In the end the optimal QFI reads $F_{\mathrm{AD}}^{(2)} = 2(1 + |1-2p|) \geq F_E^{(2)}$ (see Ref. [50] Section A for details).

With this example in mind, one may wonder how to prove that the actual protocols are indeed optimal and what is (if any) the potential benefit of using even more general CS strategies ($F_{\mathrm{CS}}^{(2)} \overset{?}{>} F_{\mathrm{AD}}^{(2)}$). For larger n the task becomes even more challenging, and no brute-force optimization approach can tell what happens in the asymptotic limit $n \to \infty$. The methods developed in this Letter allow us to answer all these questions.

State-of-the-art bounds.—The most powerful state-of-the-art bounds for the performance of E as well as AD strategies, are based on the concept of minimization of certain operator norm expressions over different Kraus representations of the channel $\Lambda_\varphi = \sum_k K_{\varphi,k} \cdot K_{\varphi,k}^\dagger$ [21,22,28,31,37,53–56]—in what follows we drop subscript φ in Kraus operators for conciseness. For E strategies, the upper bound on the achievable QFI reads

$$F_E^{(n)} \leq \min_{\{K_k\}} 4[n\|\alpha\| + n(n-1)\|\beta\|^2], \quad (2)$$

where $\|\cdots\|$ denotes the operator norm, $\alpha = \sum_k \dot{K}_k^\dagger \dot{K}_k$, $\beta = \sum_k \dot{K}_k^\dagger K_k$, and $\dot{K}_k = \partial_\varphi K_k$. If a Kraus representation exists for which $\beta = 0$, the QFI scales asymptotically at most linearly with n—SS models—and the optimal quantum enhancement amounts to a constant factor improvement [21,54–56]. If no such representation exists, then the

HS can be preserved asymptotically [28,37]. Interestingly, the above bound has been proven to be asymptotically tight for both SS ($\beta = 0$) and HS ($\beta \neq 0$) models [37].

Moving to AD strategies, the best state-of-the-art universally valid bound reads [21,31,37]

$$F_{\mathrm{AD}}^{(n)} \leq \min_{\{K_i\}} 4\left[n\|\alpha\| + n(n-1)\|\beta\|\left(\|\beta\| + 2\sqrt{\|\alpha\|}\right)\right]. \quad (3)$$

It is asymptotically equivalent to the parallel bound, Eq. (2), in the case of SS models ($\beta = 0$), and, since the parallel bound is asymptotically saturable, this implies no asymptotic advantage of AD strategies over E. Still, the bound leaves space for improvement for finite n and does not exclude an asymptotic advantage for HS models—the term quadratic in n has a larger coefficient than the one in Eq. (2).

Iterative bound.—Below, we derive a tighter adaptive bound than the one given above, and prove it is asymptotically equivalent to the parallel one—consequently, this implies no asymptotic advantage of AD over E for all models (both SS and HS).

Let $\Lambda_\varphi^{(n)}(\cdot) = \sum_{\boldsymbol{k}^{(n)}} K_{\boldsymbol{k}^{(n)}} \cdot K_{\boldsymbol{k}^{(n)}}^\dagger$ represent a combined action of n channels Λ_φ in a general adaptive strategy where they are intertwined with control operations V_i acting on probe and ancillary systems, as in Fig. 1 (AD). $K_{\boldsymbol{k}^{(n)}}$ denote the corresponding Kraus operators, which can be computed via the following iteration relation: $K_{\boldsymbol{k}^{(1)}} = V_1(K_{k_1} \otimes \mathbb{1})$,

$$K_{\boldsymbol{k}^{(i+1)}} = V_{i+1}(K_{k_{i+1}} \otimes \mathbb{1})K_{\boldsymbol{k}^{(i)}}, \quad (4)$$

where $\boldsymbol{k}^{(i)} = (k_i, \ldots, k_1)$, and $\mathbb{1}$ is acting on the ancillary system (we will drop it in what follows for conciseness of notation).

The starting point for the derivation of the state-of-the art bounds as reported in Eqs. (2), (3), is an observation that, given a channel $\Lambda_\varphi^{(n)}$, maximization of the QFI of the output state over all inputs and sets of control operations can be upper bounded by [53]

$$F_{\mathrm{AD}}^{(n)} = \max_{\rho_0, \{V_i\}} F\left[\Lambda_\varphi^{(n)}(\rho_0)\right] \leq \max_{\{V_i\}} \min_{\{K_{\boldsymbol{k}^{(n)}}\}} 4\|\alpha^{(n)}\|, \quad (5)$$

where $\alpha^{(n)} = \sum_{\boldsymbol{k}^{(n)}} \dot{K}_{\boldsymbol{k}^{(n)}}^\dagger \dot{K}_{\boldsymbol{k}^{(n)}}$, the minimization is performed over all equivalent Kraus representations of $\Lambda_\varphi^{(n)}$. Note that for a large enough ancillary system the inequality becomes equality. As such, this inequality is not of much practical use due to the infeasibility of performing the minimization over all Kraus representations for larger values of n, as well as the need to additionally perform the optimization over the control operations V_i. The usefulness of this inequality stems from the fact, that it is possible to further upper bound the rhs of Eq. (5) with norms of operators defined in terms of Kraus operators of the *elementary channel* Λ_φ. This is how bounds (2) and (3) were obtained [21,31,37,54,55].

In what follows we provide a novel step-by-step approach, where at each step we bound the maximal *increase* in the final QFI thanks to the additional usage of a single quantum channel [57]. Using Eq. (4) we have

$$\begin{aligned}\alpha^{(i+1)} &= \sum_{k_{i+1}, \boldsymbol{k}^{(i)}} \left(K_{\boldsymbol{k}^{(i)}}^\dagger \dot{K}_{k_{i+1}}^\dagger + \dot{K}_{\boldsymbol{k}^{(i)}}^\dagger K_{k_{i+1}}^\dagger\right) \times \mathrm{H.c.} \\ &= \sum_{\boldsymbol{k}^{(i)}} K_{\boldsymbol{k}^{(i)}}^\dagger \alpha K_{\boldsymbol{k}^{(i)}} + K_{\boldsymbol{k}^{(i)}}^\dagger \beta \dot{K}_{\boldsymbol{k}^{(i)}} + \dot{K}_{\boldsymbol{k}^{(i)}}^\dagger \beta^\dagger K_{\boldsymbol{k}^{(i)}} + \alpha^{(i)}.\end{aligned} \quad (6)$$

We will now use the following operator norm inequality (see Ref. [50] Section B for the proof):

$$\left\|\sum_k L_k^\dagger A Q_k\right\| \leq \sqrt{\left\|\sum_k L_k^\dagger L_k\right\|}\|A\|\sqrt{\left\|\sum_k Q_k^\dagger Q_k\right\|}, \quad (7)$$

which, together with the triangle inequality and the trace preservation condition, $\sum_{\boldsymbol{k}^{(i)}} K_{\boldsymbol{k}^{(i)}}^\dagger K_{\boldsymbol{k}^{(i)}} = \mathbb{1}$, yields

$$\|\alpha^{(i+1)}\| \leq \|\alpha^{(i)}\| + \|\alpha\| + 2\|\beta\|\sqrt{\|\alpha^{(i)}\|}. \quad (8)$$

Let us define the following iteration:

$$a^{(i+1)} = a^{(i)} + \|\alpha\| + 2\|\beta\|\sqrt{a^{(i)}}, \qquad a^{(0)} = 0, \quad (9)$$

which, in light of Eqs. (5) and (8), yields $F_{\mathrm{AD}}^{(n)} \leq 4a^{(n)}$. The resulting bound $4a^{(n)}$ may be optimized over the choice of Kraus representation of the elementary channel in each iteration *separately* (how to efficiently implement this iteration numerically is described in [50], Section D) or, in a weaker variant, over a *single* Kraus representation identically used in each step (for which the resulting bound will also be valid for CS strategies—see Ref. [50] Section C for the proof). Since $a^{(n)}$ is strategy independent, the maximization over $\{V_i\}$, or, more generally, over all CS strategies, is no longer necessary. This finally yields

$$F_{\mathrm{AD}}^{(n)} \leq \min_{\{K_k\}^{\times n}} 4a^{(n)}, \qquad F_{\mathrm{CS}}^{(n)} \leq \min_{\{K_k\}} 4a^{(n)}. \quad (10)$$

Interestingly, the possibility to use a different Kraus representation for each channel use allows us to tighten the bound also for parallel strategies, see Ref. [50] Section D3.

Closed formula bounds.—In order to appreciate how much tighter the obtained bounds are compared to the state-of-the-art bounds, we will provide some closed formulas for the bounds that result from a relaxed variants of the iteration procedure. First, observe that from Eq. (7) we get

$\|\beta\| \le \sqrt{\|\alpha\|}$. From Eq. (9) it then follows that $a^{(n)} \le n^2\|\alpha\|$ (the bound obtained in [34]), which when put back into the iteration formula results in

$$F^{(n)}_{\mathrm{AD,CS}} \le \min_{\{K_k\}} 4\left(n\|\alpha\| + n(n-1)\|\beta\|\sqrt{\|\alpha\|}\right). \tag{11}$$

Note, that the bound is noticeably tighter than Eq. (3) and is also valid for CS strategies, as the same Kraus representation is used in each step. We also see that the difference between this bound and Eq. (2) amounts to replacing one $\|\beta\|$ with $\sqrt{\|\alpha\|}$. It might be tempting to conjecture that this difference reflects the asymptotic gain of AD over E strategies. This is not the case, however, as we demonstrate below.

For any fixed $\|\alpha\|$, $\|\beta\|$ consider the following function $f(n) = n\|\alpha\| + n(n-1)\|\beta\|^2 + n\log n(\|\alpha\| - \|\beta\|^2)$. For $n \ge 0$ it can be shown (see Ref. [50] Section E) that $f(n+1) \ge f(n) + \|\alpha\| + 2\|\beta\|\sqrt{f(n)}$. Hence, in light of Eq. (9) we get $f(n) \ge a^{(n)}$ and as a result

$$F^{(n)}_{\mathrm{AD,CS}} \le \min_{\{K_k\}} 4\left[n\|\alpha\| + n(n-1)\|\beta\|^2\left(1 + \frac{c\log n}{n-1}\right)\right], \tag{12}$$

where $c = (\|\alpha\| - \|\beta\|^2)/\|\beta\|^2$. Since we know that the parallel bound, Eq. (2), is asymptotically saturable this implies that

$$\lim_{n\to\infty}\left(F^{(n)}_{\mathrm{AD,CS}}/F^{(n)}_{E}\right) = 1 \tag{13}$$

and, hence, there is no asymptotic advantage of AD nor CS over E.

Interestingly, lack of asymptotic advantage thanks to adaptiveness has also been demonstrated for continuous-time models [25], a result which can be regarded as a limiting case of the theory we develop here (see Ref. [50] Section F for details).

Examples.—In order to illustrate the practical applications of the bounds, we compute them for four representative models and compare the results with the actual performance of the optimal protocols that can be determined numerically for a small number of channel uses ($n \le 4$) via semidefinite programming (SDP) as described in [21] (parallel strategies), [33] (adaptive protocols), and [35] (causal superposition protocols). The results are presented in Fig. 2. As a figure of merit we plot the achievable QFI with n uses of a channel normalized by n times $F^{(1)}$ (the maximal QFI for single-channel sensing with a possible use of ancillary systems).

Figure 2(a) presents results corresponding to the introductory example of the perpendicular dephasing model, Eq. (1)—in all the models that follow we also assume the convention that $K_{\varphi,k} = U_\varphi K_k$ (signal comes after noise). Among the four models presented, this is the only one that admits asymptotic HS—hence the linear increase of the figure of merit. Interestingly, the bounds are saturated for $n = 2$ and the optimal QFI values are equal to the ones obtained for the protocols discussed in the introductory example, proving they are indeed optimal. For larger n, the bounds are very tight, and, as expected, the bounds for AD and CS converge asymptotically to the E bound (unlike the state-of-the-art bound).

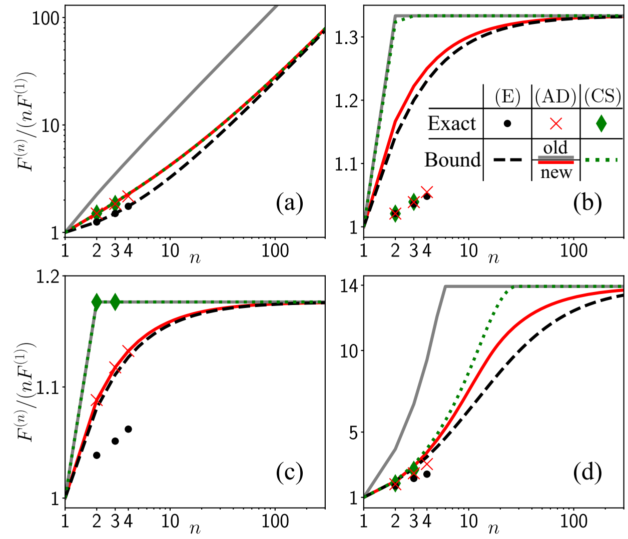

FIG. 2. Achievable QFI as a function of the number of channels probed for parallel (E, black), adaptive (AD, red), and causal superposition strategies (CS, green) normalized by n times the single-channel QFI. Points represent the result of the exact optimization, while curves represent the respective bounds. The best previously known adaptive bound (gray) is depicted for comparison. The four plots correspond to different metrological models with a qualitatively different behavior: (a) dephasing perpendicular to the signal, Eq. (1) ($p = 0.75$); (b) dephasing parallel to the signal, Eq. (14) ($p = 0.75$); (c) damping perpendicular to the signal, Eq. (15) ($p = 0.15$); (d) damping parallel to the signal, Eq. (16) ($p = 0.75$).

Results depicted in Fig. 2(b) refer to the parallel dephasing model (both the unitary encoding and the dephasing are with respect to the z axis), where the Kraus operators read

$$K_1 = \sqrt{p}\mathbb{1}, \qquad K_2 = \sqrt{1-p}\sigma_z. \tag{14}$$

In this case, gains due to adaptiveness or causal superpositions are very modest, and the bounds are not particularly tight for low n—still, thanks to the general theorem, we know they are tight asymptotically.

Figure 2(c) illustrates results for the perpendicular amplitude damping model (unitary encoding with respect to the z axis, amplitude damping with respect to the x axis):

$$K_1 = |-\rangle\langle-| + \sqrt{p}|+\rangle\langle+|, \qquad K_2 = \sqrt{1-p}|-\rangle\langle+|, \tag{15}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ are the eigenvectors of σ_x. This model is of particular interest as the finite-n bounds are saturated here both for AD and CS for all n. This suggests that it is highly unlikely that any tighter metrological bounds can be derived solely from the properties of the single-channel Kraus operators.

Finally, Fig. 2(d) depicts results for the parallel amplitude damping model with

$$K_1 = |0\rangle\langle 0| + \sqrt{p}|1\rangle\langle 1|, \qquad K_2 = \sqrt{1-p}|0\rangle\langle 1|. \quad (16)$$

This model illustrates particulary well how much tighter the novel bounds are, when compared with the previous state-of-the-art ones.

Conclusions and open problems.—With the results presented in this Letter, we dare to say that the theory of single-parameter quantum metrology in the presence of uncorrelated noise is now complete. Universal asymptotically saturable bounds are known as well as efficiently computable bounds in the regime of finite (but potentially large) number of channel uses. This, together with exact algorithms to find optimal protocols for small n, provides a complete landscape of achievable quantum enhancement in realistic quantum metrology. This said, we need to admit that in the case of multiparameter models [58,59], Bayesian models [60,61], and most importantly models involving temporally or spatially correlated noise [33,62–65], the quest for a full understanding of quantum metrological potential is still not complete. Nevertheless, these achievements compare favorably to the ones obtained in the related field of (binary) quantum channel discrimination [66,67]. Interestingly, adaptive strategies are not useful asymptotically for asymmetric hypothesis testing [68–70], while an advantage is possible in the symmetric setting [71,72]. However, easily computable asymptotic bounds, as well as practical strategies to attain them for arbitrary channels are still missing, unlike in quantum metrology. Moreover, the asymptotic analysis of causal superposition strategies for quantum channel discrimination [44,73] is still an open question.

This work was supported by the National Science Center (Poland) Grant No. 2020/37/B/ST2/02134. F. A. acknowledges financial support from MUR under the “PON Ricerca e Innovazione 2014-2020” project EEQU.

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