Self-Dual Gravity and Color-Kinematics Duality in AdS_4

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We show that self-dual gravity in Euclidean four-dimensional anti–de Sitter space (AdS_4) can be described by a scalar field with a cubic interaction written in terms of a deformed Poisson bracket, providing a remarkably simple generalization of the Plebanski action for self-dual gravity in flat space. This implies a novel symmetry algebra in self-dual gravity, notably an AdS_4 version of the so-called kinematic algebra. We also obtain the three-point interaction vertex of self-dual gravity in AdS_4 from that of self-dual Yang-Mills by replacing the structure constants of the Lie group with the structure constants of the new kinematic algebra, implying that self-dual gravity in AdS_4 can be derived from self-dual Yang-Mills in this background via a double copy. This provides a concrete starting point for defining the double copy for Einstein gravity in AdS_4 by expanding around the self-dual sector. Moreover, we show that the new kinematic Lie algebra can be lifted to a deformed version of the $w_{1+\infty}$ algebra, which plays a prominent role in celestial holography.

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Introduction.—Self-dual Yang-Mills (SDYM) and gravity (SDG) have provided a very fruitful setting for studying the mathematical structure of perturbative quantum gravity in asymptotically flat background. For example, in the light-cone gauge they can be described by very simple scalar theories [1–9], which make various properties such as color-kinematics duality and the double copy manifest, as shown in [10] and further explored in [11–23]. Color-kinematics duality is a relation between the color structures and kinematic numerators appearing in Feynman diagrams [24], which lies at the heart of the double copy relating gravity to the square of gauge theory, allowing one to reduce complicated calculations in the former to simpler calculations in the latter [25–27].

Another notable feature of SDYM and SDG is their integrability [4–7,28–31], which in the case of SDG may be linked to an infinite-dimensional symmetry known as the $w_{1+\infty}$ algebra. This algebra is closely related to the kinematic algebra in SDG [32] and may play a fundamental role in describing 4D quantum gravity in an asymptotically flat background via a two-dimensional conformal field theory (CFT) living on the sphere at null infinity, known as the celestial CFT [33–49]. Celestial CFT provides a framework to recast soft theorems of scattering amplitudes and their underlying asymptotic symmetries in the language of 2D CFT.

The holographic description of quantum gravity is best understood in anti–de Sitter space (AdS), where the dual description is provided by a CFT on the boundary [50]. Furthermore, the study of boundary correlators in four-dimensional anti–de Sitter space (AdS_4) is relevant for cosmology after Wick rotating to four-dimensional de Sitter space (dS_4) [51–53]. There has recently been a great deal of progress formulating color-kinematics duality and the double copy in (A)dS [54–67] (and there are also the beginnings of a larger program to extend the double copy to curved backgrounds [68–74]), although a systematic understanding is still lacking.

In this Letter, we set out to find a simple description of SDG in AdS_4 in order to gain a deeper understanding of how color-kinematics duality and the double copy work in this background. After generalizing the self-duality equation to a nonzero cosmological constant, we show that the solution for the metric can be elegantly written in terms of a scalar field obeying a simple generalization of the equation of motion found long ago by Plebanski for SDG in flat space [1]. In particular, it describes a scalar field with interactions encoded by a deformed Poisson bracket. From this, we deduce that SDG can be derived from SDYM in this background by replacing the color algebra with a deformed kinematic algebra that reduces to the flat space one as the AdS radius goes to infinity. Even more surprisingly, we find that that this kinematic algebra can be lifted to a deformed version of the $w_{1+\infty}$ algebra, suggesting exciting new connections between AdS/CFT and flat space holography.

This Letter is organized as follows. In *Self-dual Yang-Mills*, we consider SDYM in AdS_4, which obeys the same equation of motion as flat space (the fact that we are working in AdS is only encoded by boundary conditions). In *Self-dual gravity*, we then look at SDG in AdS_4. We introduce an appropriate generalization of the self-duality condition and use it to extract a simple Plebanski-like scalar equation. This exhibits a modified Poisson bracket and double copy structure. In *Color-kinematics duality*, we show that SDG in this background encodes a new kinematic algebra and can be obtained by combining this with the flat space kinematic algebra via an asymmetric double copy. In $w_{1+\infty}$ *algebras*, we then lift the new kinematic algebra to a deformed $w_{1+\infty}$ algebra. We then present our conclusions. There is also Supplemental Material [75] providing more details of the derivation of our SDG solution.

Self-dual Yang-Mills.—We will consider four-dimensional Euclidean AdS_4 with unit radius in the Poincaré patch,

$$ds^2_{\mathrm{AdS}} = \frac{dt^2 + dx^2 + dy^2 + dz^2}{z^2}, \tag{1}$$

where $0 < z < \infty$ is the radial coordinate. In a general background, the self-duality constraint for Yang-Mills theory (YM) reads

$$F_{\mu\nu} = \frac{\sqrt{g}}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}, \tag{2}$$

where $g_{\mu\nu}$ is the background metric and g is its determinant. In conformally flat spaces, such as AdS_4, this just reduces to the self-duality constraint in $\mathbb{R}^4$, since four-dimensional YM is classically scale invariant. Indeed, for the metric in (1), the $\sqrt{g}$ yields a factor of z^{-4}, while the inverse metrics used to raise indices of the field strength give z^4, so these factors just cancel out.

We work in the so-called light-cone coordinates,

$$\begin{aligned} u &= it + z, \qquad v = it - z, \\ w &= x + iy, \qquad \bar{w} = x - iy, \end{aligned} \tag{3}$$

in which the metric is given by

$$ds^2_{\mathrm{AdS}} = \frac{4(dwd\bar{w} - dudv)}{(u - v)^2}, \tag{4}$$

and $\epsilon_{uvw\bar{w}} = -1$. The nontrivial self-duality constraints can then be written as

$$F_{uw} = F_{v\bar{w}} = 0, \qquad F_{uv} = F_{w\bar{w}}. \tag{5}$$

Following [2], we will also impose light-cone gauge $A_u = 0$. We then find that the self-duality constraints are solved by [76]

$$A_w = 0, \qquad A_{\bar{w}} = \partial_u\Phi, \qquad A_v = \partial_w\Phi, \tag{6}$$

where Φ is a scalar field in the adjoint representation that satisfies the following equation of motion:

$$\Box_{\mathbb{R}^4}\Phi + i[\partial_u\Phi, \partial_w\Phi] = 0, \tag{7}$$

where $\Box_{\mathbb{R}^4} = -\partial_u\partial_v + \partial_w\partial_{\bar{w}}$. This can, in turn, be derived from the following Lagrangian by introducing a Lagrange multiplier field $\bar{\Phi}$,

$$\mathcal{L}_{\mathrm{SDYM}} = \mathrm{Tr}\{\bar{\Phi}(\Box_{\mathbb{R}^4}\Phi + i[\partial_u\Phi, \partial_w\Phi])\}. \tag{8}$$

This is the same action that was previously derived for SDYM in flat space [2–5,7–9] since AdS_4 can be conformally mapped to half of $\mathbb{R}^4$. On the other hand, since there is a boundary at $z = 0$, momentum along the z direction will not be conserved, which will become visible when computing boundary correlators in this background. We save a detailed analysis for future work.

With a view to the gravity formulation, we find it useful to split the spacetime coordinates as

$$x^i = (u, w), \qquad y^\alpha = (v, \bar{w}), \tag{9}$$

and introduce the operators

$$\Pi_\alpha = (\Pi_v, \Pi_{\bar{w}}) = (\partial_w, \partial_u), \tag{10}$$

which allows us to write the gauge field in (6) as

$$A_i = 0, \qquad A_\alpha = \Pi_\alpha\Phi. \tag{11}$$

Finally, we define the Poisson bracket [10],

$$\{f, g\} \coloneqq \partial_w f\partial_u g - \partial_u f\partial_w g = \varepsilon^{\alpha\beta}\Pi_\alpha f\Pi_\beta g, \tag{12}$$

and notice that it appears naturally in the scalar equation of motion,

$$\Box_{\mathbb{R}^4}\Phi - \frac{i}{2}[\{\Phi, \Phi\}] = 0, \tag{13}$$

where we introduced the notation

$$[\{f, g\}] = \varepsilon^{\alpha\beta}[\Pi_\alpha f, \Pi_\beta g]. \tag{14}$$

Since the Poisson bracket obeys the Jacobi identity, it is the kinematic analog of a commutator encoding the color algebra. Hence, SDYM manifestly exhibits color-kinematics duality since it possesses both a commutator and a Poisson bracket structure. The double copy involves replacing color structures with kinematic structures, mapping gauge theoretic quantities into gravitational ones. As we will see in the next section, replacing the commutator with another Poisson bracket yields a scalar action for self-dual gravity.

Self-dual gravity.—In this section, we will first review SDG in a flat background, as first derived in [1], and then describe the generalization to AdS_4.

Self-duality in asymptotically flat gravity: In asymptotically flat gravity, the self-duality condition is given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu}^{\eta\lambda}R_{\eta\lambda\rho\sigma}. \tag{15}$$

The above is the appropriate form of the condition in Euclidean signature. One can go to Lorentzian signature by rescaling the right-hand side by a factor of i and the coordinates appropriately. Crucially, the self-duality condition encodes both the equations of motion and the algebraic Bianchi identity for the Riemann tensor, which can be seen by contracting two of the indices in (15) to get

$$R_{\mu\rho} = \frac{1}{2}\epsilon_{\mu}^{\sigma\eta\lambda}R_{\eta\lambda\rho\sigma} = 0. \tag{16}$$

Writing the metric as

$$ds^2 = dw\, d\bar{w} - du\, dv + h_{\mu\nu}dx^\mu dx^\nu, \tag{17}$$

we find that (15), together with the light-cone gauge choice $h_{u\mu} = 0$, leads to

$$h_{i\mu} = 0, \qquad h_{\alpha\beta} = \Pi_\alpha \Pi_\beta \phi, \tag{18}$$

with Π_α as defined in the YM sector (10) and the scalar ϕ satisfying

$$\Box_{\mathbb{R}^4}\phi - \{\{\phi, \phi\}\} = 0, \tag{19}$$

where we introduced the notation

$$\{\{f, g\}\} = \frac{1}{2}\varepsilon^{\alpha\beta}\{\Pi_\alpha f, \Pi_\beta g\}, \tag{20}$$

and $\{,\}$ is the Poisson bracket introduced previously in (12). This then allows us to give elegant double copy rules in the self-dual sector via [10]

$$\Phi \to \phi, \qquad \frac{i}{2}[\{,\}] \to \{\{,\}\}. \tag{21}$$

Self-duality in AdS_4 gravity: We wish to generalize the self-duality condition to AdS_4. To this end, we introduce the tensor

$$T_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}), \tag{22}$$

where Λ is the cosmological constant. We now define our duality relation as

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu}^{\eta\lambda}T_{\eta\lambda\rho\sigma}. \tag{23}$$

Upon contracting with $g^{\nu\sigma}$, we find

$$R_{\mu\rho} - \Lambda g_{\mu\rho} = \frac{1}{2}\sqrt{g}\epsilon_{\mu}^{\sigma\eta\lambda}R_{\eta\lambda\rho\sigma} = 0, \tag{24}$$

where the left-hand side gives the Einstein equation with a cosmological constant in the absence of matter sources,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad R = 4\Lambda, \tag{25}$$

and the right-hand side is again the algebraic Bianchi identity for the Riemann tensor.

As a side comment, we note another way in which $T_{\mu\nu\rho\sigma}$ is the natural generalization of the Riemann tensor appearing in (15) to spaces with a nonzero cosmological constant. To this end, it helps to consider the Weyl tensor in four dimensions,

$$C_{\mu\nu}^{\rho\sigma} = R_{\mu\nu}^{\rho\sigma} - 2R_{[\mu}^{[\rho}g_{\nu]}^{\sigma]} + \frac{1}{3}Rg_{[\mu}^{[\rho}g_{\nu]}^{\sigma]}. \tag{26}$$

In asymptotically flat spaces, upon application of the vacuum equation of motion $R_{\mu\nu} = R = 0$, we get the well-known result that the Weyl tensor becomes equal to the Riemann tensor. However, in the presence of a cosmological constant, the relevant equations are those in (25). Upon plugging these into the Weyl tensor, we recover exactly the form of $T_{\mu\nu\rho\sigma}$ from (22) [77]. This result is also natural from the spinorial formulation of tensors in general relativity, where the so-called Weyl spinors, arising from the Weyl tensor, encode the self-dual and anti-self-dual degrees of freedom, upon applying the equations of motion.

We will now specialize to a background with cosmological constant $\Lambda = -3$, corresponding to AdS_4 background with unit radius.

Solution: In this section, we will show that the solution to the self-duality constraint in AdS_4 is a remarkably simple generalization of the flat space one in (19) when written in terms of a deformed Poisson bracket. Let us begin by introducing the modified Poisson bracket,

$$\{f, g\}_* = \{f, g\} + \frac{2}{u - v}(f\partial_w g - g\partial_w f). \tag{27}$$

Using a deformation of the operators (10),

$$\tilde{\Pi} = (\tilde{\Pi}_v, \tilde{\Pi}_{\bar{w}}) = \left(\partial_w, \partial_u - \frac{4}{u - v}\right), \tag{28}$$

we can write (27) as

$$\{f,g\}_* = \frac{1}{2}\varepsilon^{\alpha\beta}(\Pi_\alpha f\tilde{\Pi}_\beta g - \Pi_\alpha g\tilde{\Pi}_\beta f). \qquad (29)$$

In this form, the Poisson bracket that previously appeared in flat space can be recovered simply by replacing $\tilde{\Pi}$ with its undeformed version Π,

$$\{f,g\} = \{f,g\}_*|_{\tilde{\Pi}\to\Pi}. \qquad (30)$$

We also observe the following relation between the brackets:

$$\{f,g\}_* = (u-v)^4\left\{\frac{f}{(u-v)^2},\frac{g}{(u-v)^2}\right\}. \qquad (31)$$

Let us proceed to solve the self-duality equation in (23). First, we make the following general ansatz:

$$ds^2 = \frac{4(dw\,d\bar{w} - du\,dv + h_{\mu\nu}\,dx^\mu dx^\nu)}{(u-v)^2}, \qquad (32)$$

where $h_{\mu\nu}$ are unfixed functions. Imposing light-cone gauge $h_{u\mu}=0$, we then find the following simple solution:

$$h_{i\mu} = 0, \qquad h_{\alpha\beta} = \Pi_{(\alpha}\tilde{\Pi}_{\beta)}\phi, \qquad (33)$$

where ϕ is a scalar field satisfying the following equation of motion:

$$\frac{1}{u-v}\Box_{\mathbb{R}^4}\left(\frac{\phi}{u-v}\right) - \left\{\left\{\frac{\phi}{u-v},\frac{\phi}{u-v}\right\}\right\}_* = 0, \qquad (34)$$

where the modified double Poisson is defined as follows:

$$\{\{f,g\}\}_* = \frac{1}{2}\varepsilon^{\alpha\beta}\{\Pi_\alpha f,\Pi_\beta g\}_*, \qquad (35)$$

with $\{,\}_*$ defined in (29). Setting $f=g$, this becomes

$$\{\{f,f\}\}_* = \partial_w^2 f\partial_u^2 f - (\partial_u\partial_w f)^2 + \frac{2}{u-v}(\partial_w f\partial_u\partial_w f - \partial_u f\partial_w^2 f). \qquad (36)$$

We provide details of how to solve the self-duality equations in the Supplemental Material [75].

Hence, the equation of motion in (34) provides a natural generalization of the equation of motion for SDG in flat space given in (19). In particular, it exhibits an asymmetric double copy structure,

$$\Phi \to \frac{\phi}{u-v}, \qquad \frac{i}{2}[\{,\}] \to \{\{,\}\}_*, \qquad (37)$$

up to a rescaling of the kinetic term, which will be explored further in the next section. After some algebra, the equation of motion in (34) can also be written as follows:

$$\sqrt{g}(-\Box_{\mathrm{AdS}} + m^2)\phi + 4\left\{\left\{\frac{\phi}{u-v},\frac{\phi}{u-v}\right\}\right\}_* = 0, \qquad (38)$$

where $\Box_{\mathrm{AdS}}\phi = g^{-1/2}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi)$ with $g_{\mu\nu}$ the background AdS_4 metric, and $m^2=-2$ corresponding to a conformally coupled scalar in AdS_4. Recall that a conformally coupled scalar field in AdS_{d+1} has mass $m^2 = -[(d^2-1)/4]$ and can be mapped to a massless scalar field in flat space by a Weyl transformation [78]. The equation of motion in (38) is, in turn, encoded by the following Lagrangian:

$$\mathcal{L}_{\mathrm{SDG}} = \sqrt{g}\bar{\phi}(\Box_{\mathrm{AdS}} - m^2)\phi + 4\bar{\phi}\left\{\left\{\frac{\phi}{u-v},\frac{\phi}{u-v}\right\}\right\}_*, \qquad (39)$$

where $\bar{\phi}$ is a Lagrange multiplier field.

Finally, (34) admits the following solutions, which are related to plane wave solutions by a Weyl rescaling,

$$\phi = (u-v)e^{ikx}, \qquad (40)$$

where $kx \equiv uk_u + vk_v + wk_w + \bar{w}k_{\bar{w}}$ is the flat space inner product and $k_u k_v - k_w k_{\bar{w}} = 0$ (we refer to this as the on shell condition). Note that the momenta are complex since we are working in Euclidean signature. Since there is a boundary at $z=0$, momentum along the z direction will not be conserved and the natural observables are boundary correlators Fourier transformed to momentum space [79–82].

Color-kinematics duality.—It is straightforward to read off Feynman rules from the Lagrangians in (8) and (39). First we expand the scalar fields in the SDYM action as $\Phi = \Phi^a T^a$, where T^a are generators of the gauge group satisfying $\mathrm{Tr}(T^aT^b) = \delta^{ab}$ and $[T^a,T^b] = if^{abc}T^c$. Using on shell plane wave external states for SDYM and external states of the form (40) for SDG, we obtain the following three-point vertices (which would be relevant when computing three-point boundary correlators):

$$V_{\mathrm{SDYM}} = \frac{1}{2}X(k_1,k_2)f^{a_1a_2a_3},$$
$$V_{\mathrm{SDG}} = \frac{1}{2}X(k_1,k_2)\tilde{X}(k_1,k_2), \qquad (41)$$

where

$$X(k_1,k_2) = k_{1u}k_{2w} - k_{1w}k_{2u},$$
$$\tilde{X}(k_1,k_2) = X(k_1,k_2) - \frac{2i}{u-v}(k_1-k_2)_w. \qquad (42)$$

The objects X and $\tilde{X}$ obey Jacobi identities analogous to $f^{a_1a_2a_3}$ and can therefore be thought of as structure constants of kinematic Lie algebras,

$$0 = X(k_1, k_2)X(k_3, k_1 + k_2) + \text{cyclic}$$
$$= \tilde{X}(k_1, k_2)\tilde{X}(k_3, k_1 + k_2) + \text{cyclic}. \quad (43)$$

These relations do not rely on momentum conservation and encode color-kinematics duality. Moreover, we find that the SDG vertex can be obtained from the SDYM one by replacing the color structure constant with the deformed kinematic structure constant,

$$f^{a_1 a_2 a_3} \to \tilde{X}(k_1, k_2), \quad (44)$$

which encodes the double copy. Whereas in flat background there is only one kinematic algebra and the SDG vertex is obtained by simply squaring X [10], in AdS_4 there are two distinct kinematic algebras and SDG arises from an asymmetrical double copy.

The kinematic structure constants naturally arise from Poisson brackets on plane waves,

$$\{e^{ik_1 x}, e^{ik_2 x}\} = X(k_1, k_2)e^{i(k_1+k_2)\cdot x},$$
$$\{e^{ik_1 x}, e^{ik_2 x}\}_* = \tilde{X}(k_1, k_2)e^{i(k_1+k_2)\cdot x}, \quad (45)$$

where kx is defined below (40). Note that when we plug the solutions in (40) into the deformed Poisson bracket in (39), this is indeed equivalent to acting on plane waves since we divide by the conformal factor $(u - v)$. The kinematic Jacobi identity in (43) is a consequence of the following general property of the deformed Poisson bracket:

$$\{f, \{g, h\}_*\}_* + \{g, \{h, f\}_*\}_* + \{h, \{f, g\}_*\}_* = 0, \quad (46)$$

for arbitrary functions f, g, and h. Note that the deformed Poisson bracket satisfies a deformed Leibniz rule,

$$\left\{\frac{fg}{(u-v)^2}, h\right\}_* = \frac{1}{(u-v)^2} f\{g, h\}_* + \frac{1}{(u-v)^2} g\{f, h\}_*, \quad (47)$$

or alternatively,

$$\{fg, h\}_* = f\{g, h\}_* + g\{f, h\}_* - \frac{2fg\partial_w h}{u - v}, \quad (48)$$

although this does not play an important role in our analysis.

$w_{1+\infty}$ algebras.—As shown in [32], the kinematic algebra that appears in SDG can be lifted to a $w_{1+\infty}$ algebra, which plays an important role in the study of scattering amplitudes in the context of celestial CFT [33]. In particular, the $w_{1+\infty}$ algebra contains the extended Bondi-Metzner-Sachs (BMS) algebra underlying soft graviton theorems of scattering amplitudes [45,46]. In this section, we will follow similar steps to those in [32] to show that the deformed kinematic algebra derived in the previous section can be lifted to a deformed $w_{1+\infty}$ algebra.

For an on shell state, the momentum satisfies $k_{\bar{w}}/k_u = k_v/k_w = \rho$, where ρ is some number. It is then possible to expand an on shell plane wave as follows:

$$e^{ikx} = \sum_{a,b=0}^{\infty} \frac{(ik_u)^a (ik_w)^b}{a!b!} \mathfrak{e}_{ab}, \quad (49)$$

where $\mathfrak{e}_{ab} = (u + \rho\bar{w})^a (w + \rho v)^b$. This is naturally interpreted as an expansion in soft momenta. Letting $w_m^p = \frac{1}{2}\mathfrak{e}_{p-1+m,p-1-m}$ and plugging this into the Poisson brackets in (12) and (27) then gives

$$\{w_m^p, w_n^q\} = (n(p-1) - m(q-1))w_{m+n}^{p+q-2},$$
$$\{w_m^p, w_n^q\}_* = \{w_m^p, w_n^q\} + \frac{(m+q-p-n)}{u-v} w_{m+n+1/2}^{p+q-3/2}. \quad (50)$$

We recognize the first line as the $w_{1+\infty}$ algebra [33,83], and the second line appears to be a deformed version of this algebra. In the limit where $z = (u - v) \to \infty$ (which corresponds to the flat space limit), the deformation vanishes. This suggests that self-dual gravity in AdS_4 is integrable.

Constant deformations of the $w_{1+\infty}$ algebra have been classified in [84–87], however, we note that our deformation falls outside of this classification, since it depends on $u - v$.

Conclusion.—We have shown that SDG in AdS_4 can be described by a scalar field whose interactions are encoded by a deformed Poisson bracket, providing a surprisingly simple generalization of the Plebanski action for SDG in flat space. Our action implies a new kinematic algebra dual to the color algebra appearing in SDYM, which is a deformation of the flat space kinematic algebra. Moreover, the new kinematic algebra can be lifted to a deformation of the $w_{1+\infty}$ algebra, implying a new relation between AdS/CFT and flat space holography that extends beyond the flat space limit. Indeed, to our knowledge, $w_{1+\infty}$ symmetry was not previously identified in the context of AdS/CFT. It would be interesting to see how our SDG action compares to previous proposals in [88–96], as well as how it can be realized in twistor space [35,41,97]. In particular, scalar equations were proposed long ago in [96] and more recently in [91], although they appear to be nontrivially related to ours and the deformed Poisson structure is not manifest in those formulations. Moreover, it would be interesting to generalize our approach to other conformally flat backgrounds.

There are a number of other directions for future study. Perhaps the most immediate task is to compute tree-level boundary correlators of SDYM and SDG in AdS_4 and investigate how they encode color-kinematics duality and $w_{1+\infty}$ symmetry. In doing so, we must take into account the fact that momentum along the radial direction is not conserved and that the bulk-to-bulk propagators must satisfy nontrivial boundary conditions as a result of the boundary at $z = 0$. Note that the classical solutions in (40) correspond to bulk-to-boundary propagators and can be mapped to plane waves via a Weyl transformation. One slightly nonstandard aspect of these calculations will be the need to work in the light-cone gauge, since previous treatments usually worked in the axial gauge [79,80]. We can then investigate if the correlators exhibit universal behavior in the soft or collinear limit, analogous to those in flat space and explore how this is encoded by the $w_{1+\infty}$ symmetry. Recent work relating soft theorems to Ward identities in 3D CFT may be of use in this regard [98–102]. In flat space, the scattering amplitudes of SDYM and SDG are one-loop exact rational functions [103,104] that also exhibit color-kinematics duality [105]. It would be very interesting to see if any of these properties extend to loop-level boundary correlators in AdS_4.

As mentioned above, we can obtain SDG in AdS_4 from an asymmetrical double copy by combining the flat space kinematic algebra (which appears in SDYM) with a deformed kinematic algebra. It would be interesting to see what gravitational theory arises from squaring the deformed kinematic algebra, or alternatively, what gauge theory arises from combining the deformed kinematic algebra with the color algebra. Our approach may also provide a framework for defining color-kinematics duality and the double copy in Einstein gravity via an expansion around the self-dual sector. Indeed, the four-point tree-level wave function coefficient for gravitons in dS_4 (which can be obtained by analytic continuation from AdS_4) can be deduced from an ansatz resembling an asymmetric double copy with deformed kinematic numerators [54].

Finally, and perhaps most ambitiously, it would be interesting to identify the CFT dual to SDG in AdS_4. Given that the bulk theory may have an infinite-dimensional symmetry it seems very likely that it is integrable, and it should be possible to prove this by generalizing the arguments in [4–7,12,30,31]. SDG in AdS_4 may therefore provide an exactly solvable toy model of AdS/CFT. Moreover, introducing a Moyal deformation analogous to the one recently implemented for SDG in flat space [42] may describe a chiral higher spin theory in AdS_4 [106]. There are many exciting avenues that we hope to explore in the future.

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