Prediction-Retrodiction Measurements for Teleportation and Conditional State Transfer

Sergey A. Fedorov and Emil Zeuthen*
Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen Ø, Denmark

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Regular measurements allow predicting the future and retrodicting the past of quantum systems. Time-nonlocal measurements can leave the future and the past uncertain, yet establish a relation between them. We show that continuous time-nonlocal measurements can be used to transfer a quantum state via teleportation or direct transmission. Considering two oscillators probed by traveling fields, we analytically identify strategies for performing the state transfer perfectly across a wide range of linear oscillator-field interactions beyond the pure beam-splitter and two-mode-squeezing types.

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Classically, specifying the parameters of a closed system at one time allows inferring their values in the past and the future. No information can be added by making new measurements at later times or revealing the outcomes of measurements made before. In quantum mechanics, successive observations of a system completely specified at one time can nevertheless add new information [1]. Such situations were first considered in the context of relativistic quantum theory [1,2], where it was argued that the notion of a quantum state has to be extended in order to logically describe systems between measurements, and, to this end, multitime states were introduced [3,4]. Later, these ideas were extended to open systems and became instrumental in deriving the general statistical theory of past observations [5–8]. Recently, these theories were applied to optical homodyne records for improving the signal-to-noise ratio in sensing [9,10] and verifying quantum trajectories [11].

To date, the analysis of prediction and retrodiction, whether involving multitime [3] or past quantum states [7], concentrated on the statistics of outcomes during the measurement interval $[0, T]$. Here, we consider their preparative aspects, i.e., effects on monitored systems that measurements leave beyond the measurement interval. As illustrated in Figs. 1(a) and 1(b), prediction prepares usual forward-evolving quantum states $|\psi\rangle_{t=T}$, for which the future is determined and the past is unknown, whereas retrodiction prepares backward-evolving states $\langle\phi|_{t=0}$, for which the past is known but the future is not [3]. When states of both types are prepared by a single sequence of measurements, they form a product two-time state $|\psi\rangle_{t=T}\langle\phi|_{t=0}$ [4]. Yet another possibility, shown in Fig. 1(c), is the preparation of superpositions of initial and final states—superposition two-time states $\int |\psi_\alpha\rangle_{t=T}\langle\phi_\alpha|_{t=0} d\alpha$ [4]. We will refer to measurements that accomplish this as *pretrodiction*.

The concept of pretrodiction measurements enables new analytical insights into the problem of conditional state transfer between quantum systems. We demonstrate this for localized harmonic oscillators continuously probed by traveling electromagnetic fields, deriving results that go beyond and complement those of previous analyses based on temporal modes [12–16], Kalman filtering [17,18], and path integrals [19]. We consider general linear interactions between the oscillators and fields, thereby encompassing optical cavities, optomechanical devices, macroscopic spin ensembles, and microwave resonances in superconducting circuits. In the first configuration, shown in Fig. 1(d), one field interacts sequentially with two oscillators over the time $t \in [0, T]$ and is measured by a homodyne detector.

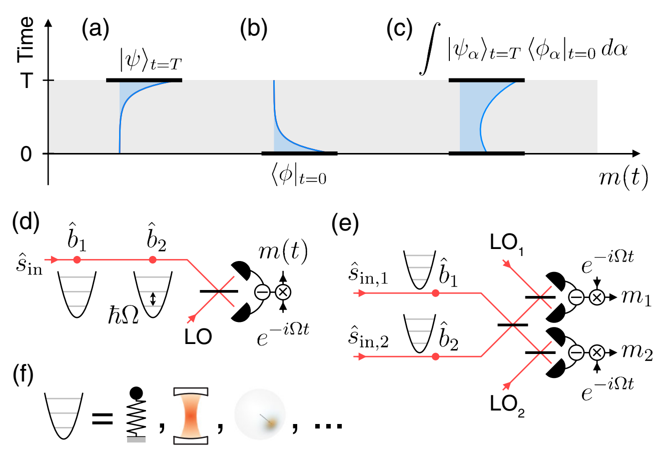

FIG. 1. (a)–(c) Preparations based on measurement records $m(t)$ of (a) a forward-evolving quantum state by prediction measurement, (b) a backward-evolving state by retrodiction measurement, and (c) a two-time state by pretrodiction measurement. (d),(e) The interaction configurations considered in this work: (d) sequential and (e) parallel. Traveling electromagnetic fields (red lines) interact with localized oscillators 1 and 2 (red dots) via Hamiltonians (5), and result in homodyne measurement currents $m(t)$ and $m_{1,2}(t)$. LO, local oscillator. All beam splitters are 50:50. (f) The localized oscillators in the schemes can be realized by mechanical resonators, optical cavities, collective spins, etc.

In the second configuration, shown in Fig. 1(e), two independent fields interact with two oscillators in parallel, are combined on a 50:50 beam splitter, and then measured by two homodyne detectors. The measurements are much slower than the oscillation period in both cases, and the homodyne local oscillators have the same carrier frequency as the driving field, meaning that the signals on the photodetectors are concentrated around the common resonance frequency Ω.

The state transfer from oscillator 2 to 1 can in both setups be accomplished via teleportation, i.e., in spite of no quantum field traveling from 2 to 1. In the sequential scheme in Fig. 1(d), it is also possible to transfer a state from 1 to 2 via direct transmission, which can be conditional on the homodyne measurements. We find that all these types of transfer can be realized perfectly for a wide range of field-oscillator interactions, even when the input traveling fields have nonzero thermal population. The latter fact generalizes a known result for electromagnetic cavities [20–24].

Teleportation via time-nonlocal measurements.—The teleportation of a quantum state [25–27] is traditionally seen as a stepwise protocol of first creating entanglement between Alice and Bob (corresponding to the interaction between the traveling field and oscillator 1 in our schemes), then measuring the states of Alice and Charlie in the EPR basis (the interaction of the field with oscillator 2 followed by homodyne detection in our schemes), and finally applying feedback to the system of Bob. Vaidman was the first to notice the relation of teleportation to time-nonlocal measurements [26]. We formulate the task of teleportation of an oscillator state from Charlie to Bob using a procedure that spans the time from 0 to T as the measurement of the time-nonlocal observables,

$$\bar{x} = x_B(T) - x_C(0), \tag{1a}$$

$$\bar{p} = p_B(T) - p_C(0), \tag{1b}$$

and then performing feedback on Bob's oscillator at $t = T$. Alice (the traveling field) is merely the meter in this process. A measurement of $\bar{x}$ and $\bar{p}$ does not condition a well-defined usual quantum state belonging to $\mathcal{H} = \mathcal{H}_{C,t=0} \otimes \mathcal{H}_{B,t=T}$, the direct product of spaces of ket vectors of Charlie at $t = 0$ and Bob at $t = T$, because the operators corresponding to $\bar{x}$ and $\bar{p}$ on $\mathcal{H}$ do not commute. Instead, it conditions a two-time state Ψ belonging to $\mathcal{H}' = \mathcal{H}^{\dagger}_{C,t=0} \otimes \mathcal{H}_{B,t=T}$ [3,4], where $\mathcal{H}^{\dagger}_{C,t=0}$ is the space of Charlie's bra vectors (our two-time states are conjugate compared to the original definition [1,3]). The operators corresponding to $\bar{x}$ and $\bar{p}$ on $\mathcal{H}'$ do commute.

To find an explicit expression for Ψ, we calculate the product of two time-nonlocal projectors Π enforcing Eqs. (1),

$$\Pi_{x_B(T)-x_C(0)=\bar{x}}\Pi_{p_B(T)-p_C(0)=\bar{p}} \propto \Psi\Psi^{\dagger}, \tag{2}$$

where

$$\Pi_{x_B(T)-x_C(0)=\bar{x}} = \int (|x+\bar{x}\rangle\langle x+\bar{x}|)_{B,T}(|x\rangle\langle x|)_{C,0}dx, \tag{3}$$

and the expression for $\Pi_{p_B(T)-p_C(0)=\bar{p}}$ is analogous. This calculation yields the two-time state,

$$\Psi = \int e^{i\bar{p}x}|x+\bar{x}\rangle_{B,t=T}\langle x|_{C,t=0}dx, \tag{4}$$

which acts as a quantum channel mapping the initial state of Charlie on the final state of Bob (see Ref. [28], Sec. A). Between two harmonic oscillators, it can be created using only linear interactions, homodyne measurements, and input fields in Gaussian states, while it is able to transfer arbitrary (including non-Gaussian) input states.

Continuous measurements.—The time-nonlocal measurements of $\bar{x}$ and $\bar{p}$ required to perform teleportation (and, more generally, conditional state transfer) can be implemented via continuous measurements. From this point on, to make the description more symmetric, we label the oscillators 1 and 2 instead of Bob and Charlie, and consider measurements of the sums rather than differences of their x and p quadratures, since their common relative sign is only a matter of convention. We neglect the detection losses and the intrinsic decoherence of the oscillators during the interaction.

The oscillators are described by annihilation operators, $\hat{b}_1$ and $\hat{b}_2$, and have identical frequencies Ω. Each of them linearly interacts with the field, described by annihilation operator $\hat{s}$, via the Hamiltonian

$$\hat{H}_{\text{int}} = \mu(\hat{s}^{\dagger}\hat{b} + \hat{b}^{\dagger}\hat{s}) + \nu(\hat{s}^{\dagger}\hat{b}^{\dagger} + \hat{s}\,\hat{b}), \tag{5}$$

where μ and ν are real, non-negative, and, in general, time dependent. The field satisfies the commutator $[\hat{s}(t), \hat{s}^{\dagger}(t')] = \delta(t-t')$ and the input-output relation $\hat{s}_{\text{out}} = \hat{s}_{\text{in}} - i(\mu\hat{b} + \nu\hat{b}^{\dagger})$; its input state is vacuum or thermal. The interaction is additionally characterized by the measurement rate Γ, the type $\zeta \in [-1, 1]$, interpolating between beam-splitter ($\zeta = 1$), position-measurement ($\zeta = 0$), and entanglement ($\zeta = -1$) interactions, and the optical damping rate γ,

$$\Gamma = \frac{(\mu+\nu)^2}{2}, \qquad \zeta = \frac{\mu-\nu}{\mu+\nu}, \qquad \gamma = 2\zeta\Gamma. \tag{6}$$

The Hamiltonian (5) describes a range of physical systems including optomechanical cavities [31], gravitational wave detectors, and atomic ensembles [32]. In cavity optomechanics, $\zeta = 0$ corresponds to the probe laser tuned to the cavity resonance, and $\zeta = 1$ ($\zeta = -1$) to the laser red (blue) detuned from the resonance by one mechanical frequency in the sideband-resolved regime. In all cases mentioned, the measurement rate Γ and the damping γ are parametrically controlled by the input optical power. We therefore limit our consideration to time-dependent $\Gamma(t)$ [and $\gamma(t)$] and time-independent ζ. The time dependence of the measurement rates is a crucial part of our analysis and can be seen as a means of matching the temporal field modes that interact with the two oscillators.

Measurements on the oscillators are made via homodyne detection of the output fields. In the sequential configuration [Fig. 1(d)], the demodulated photocurrent constitutes the complex measurement record $m(t) = \hat{p}_{\mathrm{out}}(t)e^{i\Omega t}$, given by

$$m(t) = \hat{p}_{\mathrm{in}}(t)e^{i\Omega t} - \sqrt{\Gamma_1(t)}\hat{b}_{I,1}(t) - \sqrt{\Gamma_2(t)}\hat{b}_{I,2}(t), \quad (7)$$

where $\hat{p}_{\mathrm{in(out)}}(t) = [-i\hat{s}_{\mathrm{in(out)}}(t) + i\hat{s}^{\dagger}_{\mathrm{in(out)}}(t)]/\sqrt{2}$ are the phase quadratures of the input and the output fields, and $\hat{b}_{I,j} = \hat{b}_j e^{i\Omega t}$, $j \in \{1,2\}$, are the slowly varying annihilation operators in the interaction picture. The continuous record $m(t)$ contains information that can be irrelevant to a given task, and the relevant measurement outcome $\mathcal{M}$ is obtained after applying a filter $f(t)$ as $\mathcal{M} = \int_0^T f(t)m(t)dt$. Finding the appropriate $f(t)$ is part of the task. In the parallel configuration [Fig. 1(e)], measurement records similar to Eq. (7) are obtained and processed analogously (see Ref. [28], Sec. E).

The value $\mathcal{M}$ can correspond to the outcome of a time-nonlocal measurement. To see this, we need to express one of the $\hat{b}_{I,j}(t)$ in Eq. (7) via its initial and the other via its final condition by integrating their evolution equations derived from the Hamiltonian (5). Although we are free to choose either boundary condition when expressing the evolution of each oscillator, for given $\Gamma_{1,2}(t)$ and $\zeta_{1,2}$ at most one choice will allow perfect time-nonlocal measurements (i.e., with the contribution from the field degrees of freedom vanishing as the measurement strength increases).

Optimal state transfer by pretrodiction.—We can determine to what extent a certain time-nonlocal observable can be measured with the help of variational analysis. Considering the teleportation of a state from oscillator 2 to 1, we express $\mathcal{M}$ in terms of $\hat{b}_{I,1}(T)$ and $\hat{b}_{I,2}(0)$,

$$\mathcal{M} = \hat{\varepsilon} - M_1\hat{b}_{I,1}(T) - M_2\hat{b}_{I,2}(0), \quad (8)$$

where $M_{1,2}$ are the transfer coefficients, and $\hat{\varepsilon}$ is an operator that absorbs all degrees of freedom of the input field, whose variance determines the measurement error. Both oscillators can have arbitrary initial states (not chosen from a restricted family), and therefore we impose the constraint $M_1 = M_2 = 1$. Under this constraint, $\mathcal{M}$ is related to the pretrodiction observables equivalent to those in Eqs. (1) via $\mathcal{M} = \hat{\varepsilon} - (\bar{x} + i\bar{p})\sqrt{2}$, where the relevant positions and momenta are the rotating-frame quadratures of each oscillator:

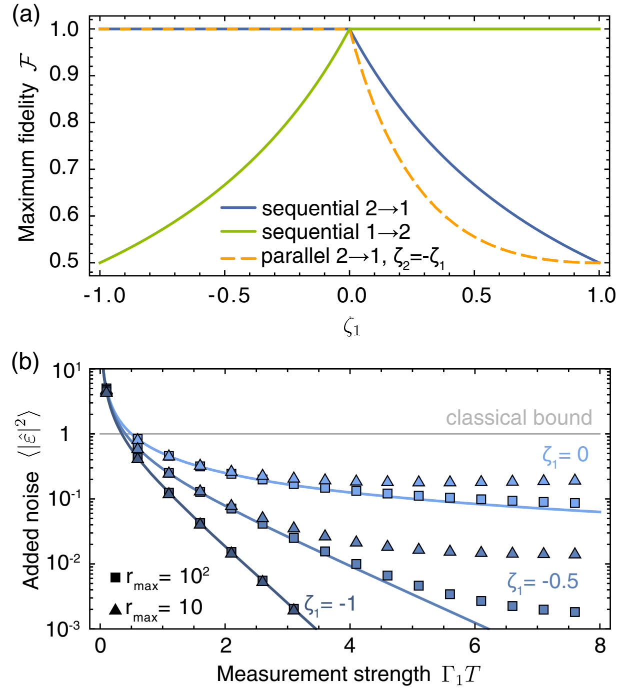

FIG. 2. (a) The state-transfer fidelities for teleportation [Eqs. (11) and (12)] and direct transmission [Eq. (15)] for infinite measurement strength $\Gamma_1 T \to \infty$. (b) The minimum error for teleportation in the sequential configuration [Eq. (11)] as a function of the measurement strength assuming a vacuum input field. The solid lines represent the idealized case with divergent Γ_2, while the values indicated by triangles and squares are obtained with Γ_2 truncated at a finite value $r_{\max} = \Gamma_{2,\max}/\Gamma_1 = 10$ and 10^2, respectively (see Ref. [28], Sec. I).

$$\hat{x} = (\hat{b}_I + \hat{b}_I^{\dagger})/\sqrt{2}, \qquad \hat{p} = (\hat{b}_I - \hat{b}_I^{\dagger})/(i\sqrt{2}). \quad (9)$$

The measurement error for either of $\bar{x}$ and $\bar{p}$ is given by $\langle|\hat{\varepsilon}|^2\rangle \equiv \langle(\hat{\varepsilon}\hat{\varepsilon}^{\dagger} + \hat{\varepsilon}^{\dagger}\hat{\varepsilon})/2\rangle$, where the averaging is over the input state of the field. The state-transfer fidelity [32] is found from this error as $\mathcal{F} \equiv 1/(1 + \langle|\hat{\varepsilon}|^2\rangle)$. To find the maximum fidelity, we analytically perform the nonlinear, constrained variational minimization,

$$\langle|\hat{\varepsilon}|^2\rangle \to \min, \qquad M_1 = M_2 = 1, \quad (10)$$

over $f(t)$ and $\Gamma_2(t)$ (see Ref. [28], Sec. C). The measurement rate of only one of the oscillators needs to change in time: we choose Γ_2 to be time dependent and Γ_1 to be constant.

In the sequential configuration [Fig. 1(d)] with vacuum input fields, the minimum error for teleportation is

$$\langle|\hat{\varepsilon}|^2\rangle_{\mathrm{seq},2\to1} = \frac{\zeta_1}{1 - \exp(-\gamma_1 T)}. \quad (11)$$

The error depends on the interaction type of the first oscillator ζ_1, but not on that of the second ζ_2. Whenever $\zeta_1 < 0$, i.e., the interaction of first oscillator is entanglement dominated, the teleportation fidelity approaches one as the measurement strength is increased, $\Gamma_1 T \to \infty$. In the parallel configuration [Fig. 1(e)], the additional condition $\zeta_2 = -\zeta_1$ must be satisfied for the fidelity to approach one as $\Gamma_1 T \to \infty$, in which case the minimum error is

$$\langle|\hat{\varepsilon}|^2\rangle_{\mathrm{par},2\to 1} = \frac{2\zeta_1/(1+\zeta_1^2)}{1-\exp(-\gamma_1 T)}. \tag{12}$$

The optimum filter function $f(t)$ is in both cases

$$f(t) = \sqrt{\Gamma_1}\frac{(\zeta_1+\zeta_2)e^{\gamma_1 t/2} + (\zeta_1-\zeta_2)e^{-\gamma_1 t/2}}{2\sinh(\gamma_1 T/2)}, \tag{13}$$

and the required time-dependent Γ_2 is

$$\frac{\Gamma_2(t)}{\Gamma_1} = \frac{\sinh^2(\gamma_1 t/2)}{\sinh^2(\gamma_1 T/2) - \sinh^2(\gamma_1 t/2)}. \tag{14}$$

While $\Gamma_2(t)$ diverges as $t \to T$, it can be truncated at some finite value. The effect of this is a fidelity reduction that depends on the truncation point and can be made negligible [see Fig. 2(b) and Ref. [28], Sec. I].

The conditional direct transmission of the state from oscillator 1 to 2 in the sequential configuration [Fig. 1(d)] is treated similarly to the teleportation; it does not have a counterpart in the parallel configuration. The minimum error,

$$\langle|\hat{\varepsilon}|^2\rangle_{\mathrm{seq},1\to 2} = \frac{\zeta_1}{\exp(\gamma_1 T) - 1}, \tag{15}$$

is independent of ζ_2 and realized by the filter function $f(t)$ and the measurement rate $\Gamma_2(t)$ that are obtained from Eqs. (13) and (14) by the replacement $t \to t - T$ (see Ref. [28], Sec. D).

When the input field is in a thermal state with a nonzero population n_{in}, as in the case of microwave transmission lines [23,24], the errors in Eqs. (11), (12), and (15) are multiplied by $(2n_{\mathrm{in}}+1)$. For concreteness, we will keep assuming $n_{\mathrm{in}} = 0$, as is typical of optical fields, when presenting the results.

The performance of our state-transfer protocols is summarized in Fig. 2. Figure 2(a) shows the teleportation and direct transfer fidelities for $\Gamma_1 T \to \infty$. The parameter regions where unit fidelity can be reached for teleportation, $\zeta_1 \le 0$, and direct transfer, $\zeta_1 \ge 0$, intersect at $\zeta_1 = 0$. However, this does not permit perfect *exchange* of the initial states between the oscillators, because different directions of the state transfer require different time dependencies of the measurement rate. For example, if an arbitrary initial state of oscillator 2 is perfectly teleported to oscillator 1 by measuring $\hat{b}_{I,1}(T) + \hat{b}_{I,2}(0)$, the error for the measurement of $\hat{b}_{I,1}(0) + \hat{b}_{I,2}(T)$ is infinite. Figure 2(b) shows the error for finite $\Gamma_1 T$ for teleportation in the sequential configuration. In the regime dominated by optical antidamping of oscillator 1, $\zeta_1 \gtrsim -1$, the teleportation error decreases exponentially $\langle|\hat{\varepsilon}|^2\rangle \approx e^{-\Gamma_1 T}$, whereas in the position-measurement regime, $\zeta_1 \approx 0$, the error decreases inversely proportional to the measurement strength $\langle|\hat{\varepsilon}|^2\rangle \approx 1/(2\Gamma_1 T)$.

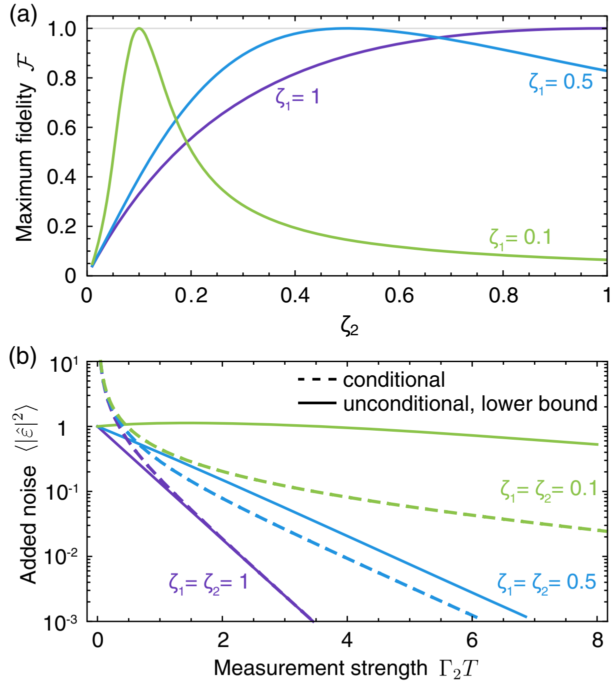

FIG. 3. (a) The state-transfer fidelities of unconditional direct transmission for different $\zeta_{1,2}$. (b) The minimum added noise during conditional and unconditional direct state transfer, respectively, as function of the measurement strength for $\zeta_1 = \zeta_2$.

How conditional is the state transfer?—The variances of the outcomes $\bar{x}$ and $\bar{p}$ of perfect pretrodiction measurements with $\langle|\hat{\varepsilon}|^2\rangle \to 0$ determine the state-transfer error if these outcomes were to be discarded. In this case, the *unconditional* state-transfer fidelity [32] is given by $\mathcal{F}_{\mathrm{uc}} = 1/(1+\langle|\mathcal{M}|^2\rangle)$. This fidelity depends on the interaction setting and the input state of the traveling field. In the teleportation settings, $\langle|\mathcal{M}|^2\rangle \to \infty$ and $\mathcal{F}_{\mathrm{uc}} = 0$, consistent with the causality principle. In the direct transfer setting [$1 \to 2$, Fig. 1(d)], $\mathcal{F}_{\mathrm{uc}}$ is generally nonzero and even reaches one, meaning that the unconditional state transfer can be perfect. To see this, in Fig. 3(a) we compare the unconditional state-transfer fidelities in the sequential configuration for different interaction types $\zeta_{1,2}$, infinite measurement strength, and optimum rates $\Gamma_{1,2}$ minimizing the pretrodiction measurement error (this choice generally does not optimize the unconditional performance). We let the measurement rate of the second oscillator Γ_2 be constant, and allow $\Gamma_1(t)$ to change in time (see Ref. [28], Sec. F); in this case the measurement strength is given by $\Gamma_2 T$. We find that $\mathcal{F}_{\mathrm{uc}}$ can be above zero when $\zeta_{1,2} > 0$, and that $\mathcal{F}_{\mathrm{uc}} = 1$ when $\zeta_1 = \zeta_2 > 0$. In the specific case $\zeta_1 = \zeta_2 = 1$, which describes, e.g., catching a state leaking from one electromagnetic cavity with another, our expression for the time-dependent $\Gamma_1(t)$ converges as $\Gamma_2 T \to \infty$ to the solution obtained in Ref. [20] using a different fidelity measure (see Ref. [28], Sec. G).

Even though a perfect state transfer may be unconditional as $\Gamma_2 T \to \infty$, at finite measurement strengths discarding the outcomes $\bar{x}$ and $\bar{p}$ introduces extra error. This is illustrated in Fig. 3(b), where we compare the conditional and unconditional state-transfer errors for $\zeta_1 = \zeta_2 > 0$, the case in which both go to zero as $\Gamma_2 T \to \infty$. The figure shows that, as expected, the unconditional errors are always higher than the conditional errors at large measurement strengths. At small measurement strengths, our method of comparison breaks down. While the unconditional errors cannot be evaluated for arbitrary initial states, the fact that the initial-state contribution to the error decays over time allows us to lower bound it by assuming that the initial states of both oscillators are vacuum. Such prior knowledge is absent in the evaluation of the conditional error, whence it can exceed the unconditional value at small measurement strengths (see Ref. [28], Sec. F).

Conclusions and outlook.—We introduced prediction-retrodiction measurements as a new primitive in quantum measurement theory and showed its application to the problem of state transfer between localized oscillators. A similar analysis may yield new insights into discrete-variable protocols. Elements of such an approach can be found in Ref. [33], where retrodiction was applied in the Bell-measurement step of teleportation between two qubits coupled to the same cavity. To extend this to a pretrodiction analysis would require incorporating the prediction component constituted by the entanglement step.

In the settings considered, the state transfer is only possible one way, from oscillator 1 to 2, or 2 to 1. However, Vaidman has proposed a two-way teleportation scheme based on "crossed" time-nonlocal measurements that accomplishes exchange of initial states [26]. The pretrodiction framework may help to find a realization of this scheme via continuous measurements.

The result that perfect conditional and unconditional state transfer between oscillators can be realized with a wide range of linear couplings between oscillators and fields, i.e., not only with pure beam-splitter or two-mode-squeezing interaction, can find applications in emergent optical and microwave quantum information processing. In particular, the resilience of our protocols to thermal noise in the input field is a useful trait for microwave quantum links [23,24].

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*zeuthen@nbi.ku.dk

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