Shortest Route to Non-Abelian Topological Order on a Quantum Processor

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A highly coveted goal is to realize emergent non-Abelian gauge theories and their anyonic excitations, which encode decoherence-free quantum information. While measurements in quantum devices provide new hope for scalably preparing such long-range entangled states, existing protocols using the experimentally established ingredients of a finite-depth circuit and a single round of measurement produce only Abelian states. Surprisingly, we show there exists a broad family of non-Abelian states—namely those with a Lagrangian subgroup—which can be created using these same minimal ingredients, bypassing the need for new resources such as feed forward. To illustrate that this provides realistic protocols, we show how D_4 non-Abelian topological order can be realized, e.g., on Google's quantum processors using a depth-11 circuit and a single layer of measurements. Our work opens the way toward the realization and manipulation of non-Abelian topological orders, and highlights counterintuitive features of the complexity of non-Abelian phases.

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The quantum statistics of particles is a foundational concept with far-reaching ramifications, and in two spatial dimensions, a remarkably rich set of "anyonic statistics" arises [1,2]. Although not realized by fundamental particles, anyons emerge as effective quasiparticles in two-dimensional condensed matter systems, most notably the fractional Quantum Hall effect [3]. The most exotic extension of quantum statistics occurs with *non-Abelian* anyons [4–7] which always come in degenerate quantum states (Fig. 1). Consequently, while braiding Abelian anyons only lead to a phase factor, braiding "non-Abelions" leads to a matrix action on the degenerate states. This has evoked dreams of a physically fault-tolerant route to perform quantum computing, with quantum gates being executed by the motion of non-Abelian anyons [8]. However, a key obstacle is finding states of matter hosting such non-Abelions, called non-Abelian topological order [9]. The most compelling candidates so far are certain fractional quantum Hall states in the second Landau level ($\nu = 5/2,\ 12/5$) [3,8,10]. However, non-Abelian states are more fragile compared to their Abelian counterparts [8,11] and the extreme conditions required to create quantum Hall states, combining high magnetic fields, pristine samples, and millikelvin temperatures, all call for new approaches to creating such quantum states.

Meanwhile, the significant advances in near-term quantum devices [12] open up the possibility of realizing non-Abelian states, not from cooling into the ground states, but from controlled quantum gates that entangle a product state to resemble ground states with non-Abelian excitations. Indeed, recent theory and experimental work have shown how certain Abelian states can be created in this way, in particular the toric code topological order [13–15]. However, the general strategy adopted in these works is essentially a form of adiabatic state preparation whose depth is required to scale with system size [16], a formidable requirement when one wants to scale system size with limited depth quantum circuits.

Remarkably, a work-around exists which allows us to create certain topological orders in a time independent of system size. For instance, the aforementioned toric code is

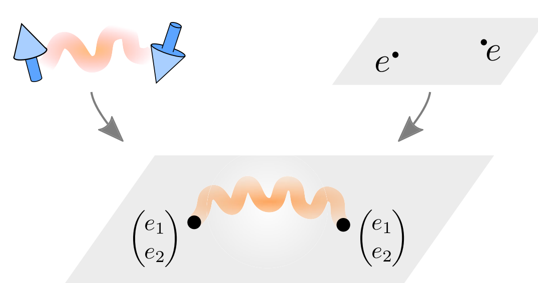

FIG. 1. Artist's impression of a non-Abelion. Non-Abelian anyons as in D_4 topological order bring together two ingredients in a remarkable mix: Bell pairs and gauge charges. Non-Abelions transform under a nontrivial matrix representation of the gauge group, leading to a topological degeneracy. The Bell pair is a robust consequence of forming a gauge neutral singlet. In this work we show how to efficiently prepare D_4 non-Abelian order with a single layer of measurement, whereby non-Abelian entanglement serves as a smoking gun.

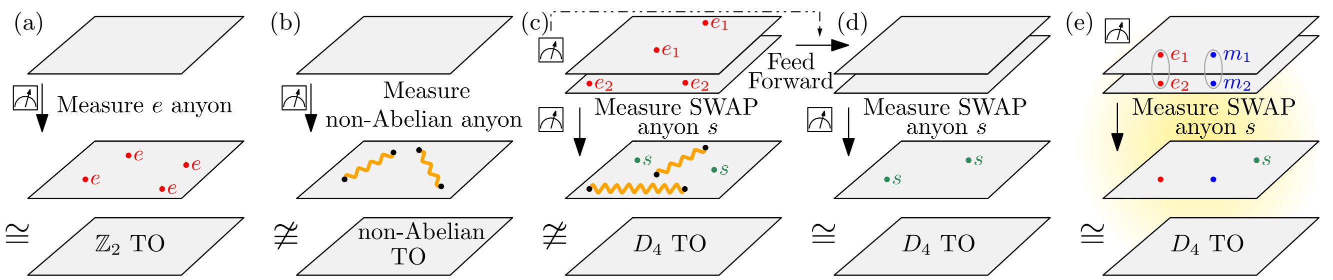

FIG. 2. Topological order from measurement. (a) By measuring the star term A_v (1) of the toric code, $\mathbb{Z}_2$ topological order (TO) is obtained regardless of the measurement outcome. A clean toric code is achieved by pairing up e anyons by a feed forward of single-site Pauli operators. (b) In contrast, measuring non-Abelian gauge charges gives rise to topological degeneracies and non-Abelian entanglement (Fig. 1). Removing these requires a unitary circuit whose depth scales with system size [21]. (c) One route to non-Abelian topological order is to first prepare two copies of the toric code by measuring the e anyons in each layer. We obtain non-Abelian $D_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ topological order if we gauge the swap symmetry of the two layers. However, the Abelian anyon defects of the bilayer then become non-Abelian defects. (d) The aforementioned protocol does work if we use feed forward to obtain two *clean* toric codes before gauging the swap symmetry. (e) In this work, we point out that one can obtain the same phase of matter *without* feed forward, using only a *single* measurement layer. The key is to first prepare the two copies of the toric code by measuring the appropriate charges and fluxes which are *invariant* under the swap symmetry; subsequently gauging the swap symmetry does thus not introduce non-Abelian anyons. In fact, we can measure all three anyons at once. See Fig. 3 or Eq. (2) for an explicit and realistic circuit.

obtained at once by simply measuring its two commuting stabilizers on the links of the square lattice [7,17–20]:

$$A_v = \begin{array}{ccc} & \sigma^z & \\ \sigma^z & & \sigma^z \\ & \sigma^z & \end{array} \quad \text{and} \quad B_p = \begin{array}{ccc} & \sigma^x & \\ \sigma^x & & \sigma^x \\ & \sigma^x & \end{array}. \tag{1}$$

Stronger yet, starting from a product state $|\psi\rangle = |+\rangle^{\otimes N}$, one needs to measure only A_v [see Fig. 2(a)]. The random measurement outcomes for A_v do not affect the $\mathbb{Z}_2$ topological order: the resulting "e anyons" ($A_v = -1$) are static Abelian charges which simply redefine our notion of vacuum state. If one, moreover, wants to prepare the "clean" case ($A_v = B_p = 1$), we note that these e anyons come in pairs and can be removed by a single feed-forward layer of σ^x-string operators [7].

The above approach generalizes to various other Abelian topological orders [22]. However, the richer non-Abelian topological order does not admit such a simple stabilizer description, but at best only a commuting projector Hamiltonian [7,23]. Indeed, due to the intrinsic degeneracies associated with non-Abelions, the excited states do not resemble the ground state—in fact, they are not the ground state of any local gapped Hamiltonian. Hence, if one naively measures the terms in their parent Hamiltonian, one typically produces non-Abelian charges [Fig. 2(b)], which cannot even be paired up by any finite-depth unitary string operator [21]. Intuitively, this is linked to the "Bell pair" mentioned in Fig. 1.

This raises the question: is non-Abelian topological order out of the reach of a simple measurement protocol? Partial results are known where measurement helps: it has recently been shown that certain non-Abelian topological orders can be obtained in finite time by *several* layers of measurement, interspersed with feed forward [24–27]. In light of these sophisticated protocols, and the aforementioned issue, it seems nigh impossible to obtain non-Abelian topological order from a *single* layer of measurements. This is of more than mere conceptual interest: feed forward remains a very costly ingredient, with many quantum simulators and computing platforms not yet allowing for it. A protocol which avoids it, as for the toric code above, is thus of conceptual and practical significance.

Here, we show that a class of non-Abelian topological order can be created by a single layer of measurements, thereby thus not requiring feed forward. Surprisingly, this shows that there exists a class of non-Abelian states which are no more complex to prepare than their Abelian counterparts, but nevertheless display richer behavior.

As a conceptually simple route toward non-Abelian order, let us imagine starting with *two* copies of the toric code. These can be prepared by measuring the star term A_v (1) on each layer, producing e_1 and e_2 anyons on the two layers [Fig. 2(c)]. Such a bilayer has a natural "swap" symmetry interchanging the two copies. If this *global physical* symmetry were turned into a *local gauge* symmetry, we would achieve non-Abelian topological order. Indeed, the e_1 and e_2 anyons then transform as a doublet under the gauge group, which can be identified with $D_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ [28,29]. To obtain this gauge symmetry, we can proceed as in the toric code case, i.e., by simply measuring the gauge charge operator (or more precisely, its Gauss law operator); soon we make this more explicit. This has two effects: first, this produces a speckle of Abelian anyons associated with the swap gauge symmetry; this is as harmless as in the toric code case. A more serious issue is that the Abelian anyons of the toric code now turn into non-Abelian anyon defects [Fig. 2(c)].

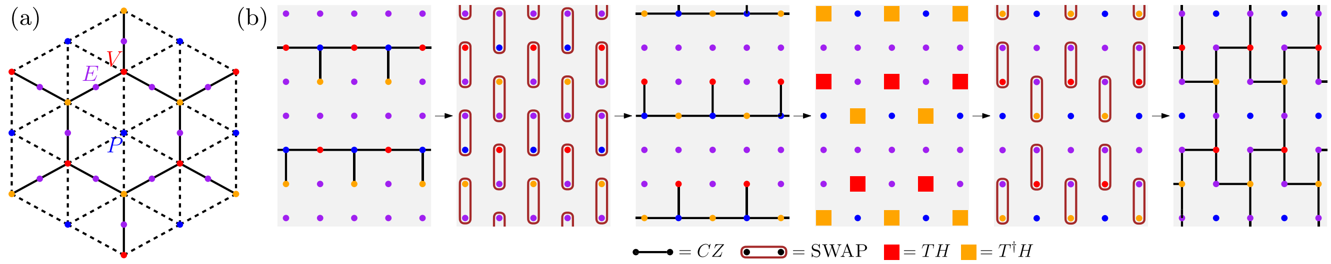

FIG. 3. Preparing non-Abelian D_4 topological order with a single layer of measurement: from theory to Sycamore chip. (a) The intuitive approach sketched in Fig. 2(e) is formalized in Eq. (2) for qubits on edges (E), vertices (V), and plaquettes (P) of the honeycomb lattice. Initializing all qubits in $|+\rangle$, the circuit consists of three steps: (1) CZ gates connecting plaquettes to vertices (dashed lines), which form the dice lattice. (2) TH on the red vertices, and $T^\dagger H$ on the orange vertices (to wit, $T \propto e^{-i(\pi/8)Z}$). (3) CZ gates connecting vertices to edges (solid lines), which forms the heavy-hex lattice. Finally, we measure all vertices and plaquettes in the X basis, producing D_4 topological order for any measurement outcome, using only nine layers of nonoverlapping two-body gates. (b) Implementation on device with square lattice connectivity, e.g., Google's Bristlecone and Sycamore chips. The first three panels prepare the dice lattice cluster state, the fourth panel performs the single-site basis rotation, and the last two panels apply the cluster state entangler for the heavy-hex lattice. Finally, measuring all but the purple-color qubits produces D_4 non-Abelian topological order. This protocol is independent of system size, requiring only 11 layers of nonoverlapping two-body gates.

So far, the above example thus hits on the same stumbling block: in the quest to produce non-Abelian order via measurement, we produce defects which destroy the desired phase of matter. One possible solution is to clean up the e-anyon defects before gauging the swap symmetry [Fig. 2(d)]; this gives a multistep measurement protocol with feed forward [24–27] which—while interesting—we here seek to avoid. We surmise that this stumbling block cannot be avoided if one measures only charges. However, we show the issue can be resolved by using the larger freedom of measuring charges *or* fluxes [to wit, the fluxes of the toric code are also called m anyons, as detected by B_p in Eq. (1)]. Indeed, rather than producing the toric code bilayer by measuring e_1 and e_2, we can also produce it by measuring a different set of Abelian anyons: the composites e_1e_2 and m_1m_2 [Fig. 2(e)]. Crucially, these are a singlet under the swap symmetry. Hence, now proceeding as before, measuring the "swap anyons" produces *only* Abelian defects. We have thereby produced D_4 topological order in finite time, without feed forward. Observe that this approach works even if we measure the anyons $\{e_1e_2, m_1m_2, s\}$ all at once.

Let us now turn the above conceptual discussion into a concrete protocol for preparing D_4 topological order for qubits living on the edges (E) of the honeycomb lattice. To effectively measure the type of many-body operators discussed above, we will use two-body entangling gates and perform single-site measurements on ancilla qubits on the vertices (V) and plaquettes (P) of the honeycomb lattice. We claim that the topological order is obtained by the following sequence [Fig. 3(a)]:

$$|D_4\rangle_E = \langle x|_{PV} \prod_{\langle v,e\rangle} CZ_{ve} \prod_v e^{\pm(\pi i/8)Z_v} H_v \prod_{\langle p,v\rangle} CZ_{pv} |+\rangle_{PEV}, \tag{2}$$

where X, Y, Z denote Pauli matrices, H is the Hadamard gate, CZ is the controlled-Z gate, and $x = \pm 1$ denotes the arbitrary outcome upon measuring all the ancillas in the X basis.

We can break the above procedure down into three steps. First, performing CZ_{pv} prepares the dice lattice cluster state, whereby measuring the plaquettes results in the color code. This is unitarily equivalent to two copies of the toric code [42], playing the role of the bilayer in Fig. 2(e). The single-site gates on the vertices rotate the color code into a basis where the swap symmetry is realized by $\prod_v X_v$. Last, we gauge this symmetry by measuring its associated Gauss law operator on each vertex, $X_v \prod_{e \supset v} Z_e$, which is achieved by a single-site measurement preceded by the CZ_{ve} unitary.

Importantly, any measurements in Eq. (2) can be delayed to the last step. A similar formula appeared in Ref. [24], with the crucial difference that the single-site rotation was different. As a consequence, the latter requires feed forward, corresponding to the scenario in Fig. 2(d).

Certain quantum processors have restricted connectivity, and might thus not be able to directly apply the gates in Fig. 3(a). In such cases it is still possible to create the D_4 state by using SWAP gates to attain the desired connectivity. To illustrate this, we propose an implementation for Google's quantum processor, which has the connectivity of a square lattice as shown in Fig. 3(b). We find that the non-Abelian state can be prepared with a two-body depth of 11 layers, independent of total chip size. (This becomes 13 layers once we decompose the SWAP layers into Google's native CZ gates; see Supplemental Material [29] for further discussion.)

While we have discussed the minimal case of D_4 topological order in great detail, we note that the idea of our efficient protocol extends to other topological orders which admit a so-called *Lagrangian subgroup* [43,44]. This is defined to be a subgroup of Abelian anyons with trivial self- and mutual statistics such that every other anyon in the theory braids nontrivially with it. In the case of D_4, this corresponds to the group generated by $\{e_1e_2, m_1m_2, s\}$ as encountered in Fig. 2(e). Phrased in the language of quantum doubles [7], e_1e_2 and s correspond to the sign representations of D_4, while m_1m_2 corresponds to the conjugacy class of the center of D_4 [28,29]. It is known that if one condenses the anyons in the Lagrangian subgroup, one obtains a trivial state. By playing this argument in reverse, one can argue that measuring the Gauss law operators associated with these anyons, one obtains its non-Abelian topological order with only a single layer of measurement [29]. Other examples which can in principle be obtained in this way are, say, the quaternion Q_8 quantum double [29], or even the doubled Ising topological order [23,45] (by measuring the Gauss law for ϵ and $\bar{\epsilon}$, though this requires physical fermions). It would be interesting to work out explicit protocols amenable to quantum processors, as we did for D_4 above.

In conclusion, we have established the shortest route to non-Abelian topological order. Indeed, while the preparation time for a purely unitary protocol must scale with system size, we found that the minimal nonunitary element of a single measurement layer could efficiently prepare certain non-Abelian orders. This furthermore avoids the need for feed forward which is intrinsic to multimeasurement approaches [24–27]. For the illustrative case of D_4, we found that roughly ten unitary layers (prior to single-site measurements) were already sufficient, even for realistic qubit connectivity as on the Google chips. Naturally, it would be worthwhile to work out concrete protocols for other existing architectures. On the conceptual side, the existence of a single-shot protocol for certain non-Abelian states motivates us to identify the minimal number of measurement layers (alongside finite-depth unitaries) for obtaining various types of quantum states. We will examine this measurement-induced hierarchy of quantum states in a forthcoming work [46].

Last, if a non-Abelian state is realized, how do we tell? One interesting probe is the aforementioned non-Abelian entanglement (Fig. 1), which we can now turn into an advantage. Indeed, the successful preparation of non-Abelian order can be confirmed by noting that if we insert non-Abelian excitations, the entanglement entropy is changed according to its quantum dimension [47]. For instance, for our particular D_4 protocol [Eq. (2)], this is achieved by acting with Pauli-Z operators on the vertices at any point prior to the single-site rotations. That such a deceptively simple tweak can have such a drastic consequence underlines the exotic nature of non-Abelian states, and points the way to the first realization and detection in a quantum simulator.

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