Anomalous Reentrant 5/2 Quantum Hall Phase at Moderate Landau-Level-Mixing Strength

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A successful probing of the neutral Majorana mode in recent thermal Hall conductivity measurements opines in favor of the particle-hole symmetric Pfaffian (PH-Pf) topological order, contrasting the theoretical predictions of Pfaffian or anti-Pfaffian phases. Here we report a reentrant anomalous quantized phase that is found to be gapped in the thermodynamic limit, distinct from the conventional Pfaffian, anti-Pfaffian, or PH-Pf phases, at an intermediate strength of Landau level mixing. Our proposed wave function consistent with the PH-Pf shift in spherical geometry rightly captures the topological order of this phase, as its overlap with the exact ground state is very high and it reproduces low-lying entanglement spectra. A unique topological order, irrespective of the flux shifts, found for this phase, possibly corroborates the experimentally found topological order.

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The discovery [1] of the fractional quantum Hall effect in an even denominator filling factor 5/2 way back in 1987 had surprised the entire physics community. The first significant understanding of the possibility of this state had been put forward by Moore and Read (MR) through a unique proposal [2] of the Pfaffian (Pf) wave function, which was later interpreted [3] as a chiral p-wave pairing of the composite fermions [4,5] owing to their effective attractive interactions [6] in the second Landau level. Consequently, a flurry of new experimental techniques (See Ref. [7] for a review) were developed for realizing exotic properties of this state such as quasiparticle charge, non-Abelian braiding statistics of the quasiparticles, and Majorana edge modes. On the other hand, subsequent proposals [8,9] of the anti-Pfaffian (A-Pf) which is topologically distinct from Pf yet degenerate for any two-body interaction, makes the issue intriguing for understanding the true nature of the state. However, the Landau-level mixing (LLM), which is important for the second and higher Landau level quantum Hall states, generates three-body interaction [10–13] that breaks [14] this degeneracy. A topologically distinct phase, namely, the particle-hole symmetric Pfaffian (PH-Pf) as the s-wave pairing of the Dirac composite fermions is also proposed [15], giving rise to yet another competing non-Abelian topological phase in the list for the same state. A PH-Pf-like instability in favor of the Dirac composite fermions with a greater mass arising due to LLM has also been proposed [16] in a gauge theory.

The verdict of the numerical studies [17–21] based on the LLM is possibly in favor of the A-Pf phase over the Pf phase. Both these topological phases and also the PH-Pf phase host the Majorana edge modes whose presence can be probed as half-integral thermal Hall conductance along the edge. These phases are, however, distinguishable because the respective predicted values of thermal Hall conductances [22,23] are 3/2, 7/2, and 5/2 in the unit of $G_0 = \pi^2 k_B^2 T/(3h)$. In contrast to the theoretical expectation [17–22], the recent thermal Hall conductance and shot-noise measurements [23,24] are consistent with the PH-Pf phase. For reconciling this, several probable scenarios have been proposed in the literature such as non-equilibration [22] of the thermal Majorana mode and subsequent proposals of partial equilibration of anti-Pfaffian edge [25–27], formation of puddles of Pf and A-Pf phases [28–30], and stabilization of PH-Pf due to disorder [28,31–33]. However, no general consensus has yet been achieved and thus the 5/2 state remains enigmatic. Moreover, whereas the 5/2 state in GaAs is typically observed [1,10,11,23,24,34–39] in the range of 12–1 Tesla magnetic field which amounts to the LLM parameter $\kappa = 0.7$–2.5, the theoretical studies [17–20] have been performed for $\kappa \lesssim 1$ only. Also, a topological phase transition from a quantum Hall state to an unquantized state in the vicinity of $\kappa \sim 0.7$–1 followed by a hint of a new quantized phase for Pf shift has been found [19] by the numerical calculation of the lowest excitation energies and entanglement entropy. However, the latter phase has not been explored further due to the lack of its understanding in terms of the MR wave function [2] $\Psi_{\rm MR}$ or its particle-hole conjugate [8,9] wave function $\Psi_{\rm MR}^{\rm phc}$.

Here we perform exact diagonalization of the Coulomb Hamiltonian corrected with LLM of strength κ separately at Pf, A-Pf, and PH-Pf flux shifts for few-electron systems (up to $N = 16$) in spherical geometry and determine overlaps of the exact ground states at different κ values if the corresponding ground states are found at total angular momentum $L = 0$. It amazingly shows (generic to all the shifts) clear segregation of two distinct quantum Hall phases formed at low and a moderate range of κ separated by an unquantized (ground state at $L \neq 0$) regime in the vicinity of $\kappa \sim 0.7$. Whereas the overlaps of the ground states for intraphase κ values are nearly unity, it is negligible for interphase κ values. A finite excitation gap for a pair of quasiparticle and quasihole in the thermodynamic limit for the phase at moderate-κ suggests it to be a quantized phase. The entanglement spectra (ES) for this quantized phase shows an entanglement gap with the number of edge state counting as 1-1-2-2 (up to the resolution obtained in finite systems). Further, the ES for all the three flux shifts at this moderate-κ *anomalous* phase ($\mathcal{A}$ phase) has a high degree of resemblance, signifying a unique topological order which is independent of these shifts. We propose a trial wave function $\Psi_{\mathcal{A}}$ for this anomalous $\mathcal{A}$ phase. As its flux shift matches with PH-Pf shift, we determine the overlap of it with the exact ground state at PH-Pf shift and found to be very high. Moreover, the low-lying ES for $\Psi_{\mathcal{A}}$ is consistent with the same for the exact state and hence can be considered as representative for the true topological order of the $\mathcal{A}$ phase. We further analyze the topological properties of $\Psi_{\mathcal{A}}$ including the Majorana mode governing $2.5G_0$ thermal Hall conductance. Therefore, the $\mathcal{A}$ phase can possibly be attributed to the experimentally observed phase [23,24] at moderate κ.

The effective interaction between spin-polarized electrons in the second Landau level with the consideration of LLM available in the literature [10,11] as

$$\hat{H}_{\rm eff}(\kappa) = \sum_{m\,\rm odd}\left[V_m^{(2)} + \kappa\delta V_m^{(2)}\right]\sum_{i<j}\hat{P}_{ij}(m) + \sum_{m\geqslant 3}\kappa V_m^{(3)}\sum_{i<j<k}\hat{P}_{ijk}(m), \quad (1)$$

where $\hat{P}_{ij}(m)$ and $\hat{P}_{ijk}(m)$ are two- and three-body projection operators, respectively, onto pairs or triplets of electrons with relative angular momentum m. Here $V_m^{(2)}$ represents two-body Coulomb pseudopotential in the second Landau level and $\delta V_m^{(2)}$ is its correction due to LLM and $V_m^{(3)}$ is the three-body pseudopotential arising due to LLM whose strength is defined by $\kappa = (e^2/\epsilon\ell_0)/\hbar\omega_c$ that is the ratio between Coulomb energy and cyclotron energy scales. We henceforth (unless mentioned otherwise) perform exact diagonalization of $\hat{H}_{\rm eff}$ with bare Coulomb and LLM-corrected pseudopotentials [40] for GaAs in spherical geometry for the Pf, A-Pf, and PH-Pf shifts, i.e., the respective number of flux quanta $N_\phi = 2N - 3$, $N_\phi = 2N + 1$, and $N_\phi = 2N - 1$ with the shifts from $2N$, for different values of κ. We determine overlaps of the exact ground states when found at $L = 0$ for different values of κ, i.e., $\mathcal{O}_{ij} = \langle\Psi_{\rm gs}(\kappa_i)|\Psi_{\rm gs}(\kappa_j)\rangle$.

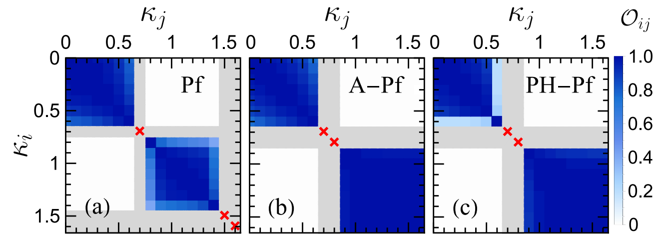

FIG. 1. (a) Overlaps (shown as color map) of the exact ground states (found at $L = 0$) of the Hamiltonian $\hat{H}_{\rm eff}$ in Eq. (1) with LLM-corrected pseudopotentials [40] at Pf flux shift ($N = 14$ and $N_\phi = 25$) for varying κ. Two topologically distinct phases are found as the overlaps of the ground states at different κ values belonging to the same phase are closer to unity and two different phases are closer to zero. These two phases are intermediated by an unquantized regime (shown as crossed marks) of κ where the ground state is not found at $L = 0$. Overlaps have not been calculated for the gray zones, as one of the ground states corresponds to the unquantized regime. (b) Same as (a) but for the A-Pf flux shift ($N = 12$ and $N_\phi = 25$). (c) Same as (a) but for PH-Pf flux shift ($N = 14$ and $N_\phi = 27$).

In Figs. 1(a)–1(c), we show $\mathcal{O}_{ij}$ for Pf, A-Pf, and PH-Pf shifts, respectively. The color mapping clearly shows two distinct topological phases separated by an unquantized phase in the vicinity of an intermediate value $\kappa_c \sim 0.7$ for all the three shifts. The transition between two quantum Hall phases is sharp (see Supplemental Material [41]) even for the finite width of the quantum wells, although the unquantized zone shrinks to zero at larger widths for finite size systems. Whereas $\mathcal{O}_{ij}$ is nearly unity when both κ_i and κ_j belong to the same phase, it is vanishingly small when κ_i and κ_j belong to two different phases. We find (see Supplemental Material [41]) that the $\mathcal{A}$ phase at moderate κ is present even in the absence of all $V_m^{(3)}$ but the presence [40] of all $\delta V_m^{(2)}$. Although a step-by-step addition of $V_3^{(3)}$, up to $V_3^{(8)}$, and finally up to $V_3^{(9)}$ reduces the range of the phase, it gets sharpened and the range reduction is compensated with the increase in N. Although the values of the pseudopotentials are estimated [11,20] perturbatively with perturbation parameter κ, because the $\mathcal{A}$ phase is robust against all these variations of pseudopotentials, we believe that the phase will be restored even for improved pseudopotentials at moderate κ; the only quantitative change will be expected in terms of the change in the range of the phase in κ space.

We further calculate the energy for creating a pair of quasiparticle and quasihole by taking an average [46] of $E_{\rm qh} = E(N, 2N) - E(N, 2N-1)$ and $E_{\rm qp} = E(N, 2N-2) - E(N, 2N-1)$, i.e., $\Delta_c = (E_{\rm qh} + E_{\rm qp})/2$, where $E(N, N_\Phi)$ denotes net ground state energy (after subtraction of the background energy of $N^2/\sqrt{2N_\Phi}$) of the system of N electrons with N_Φ number of flux quanta. In Fig. 2, we show Δ_c at different values of N for $\kappa = 1.1$ and 1.2 belonging to the $\mathcal{A}$ phase. Although each of these Δ_c is negative, their scaling with $1/N$ provides its thermodynamically

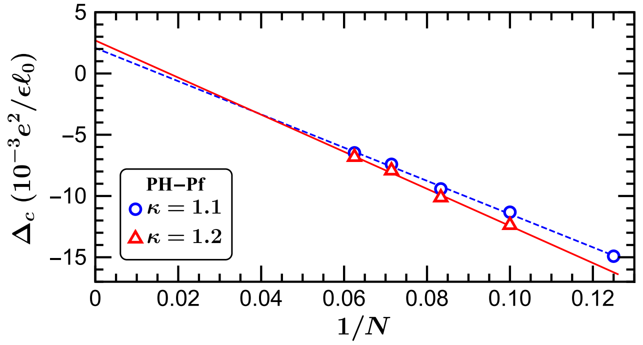

FIG. 2. Scaling of gap for a pair of quasiparticle-quasihole excitation at PH-Pf flux shift in the $\mathcal{A}$ phase for $\kappa = 1.1$. and 1.2 with $1/N$.

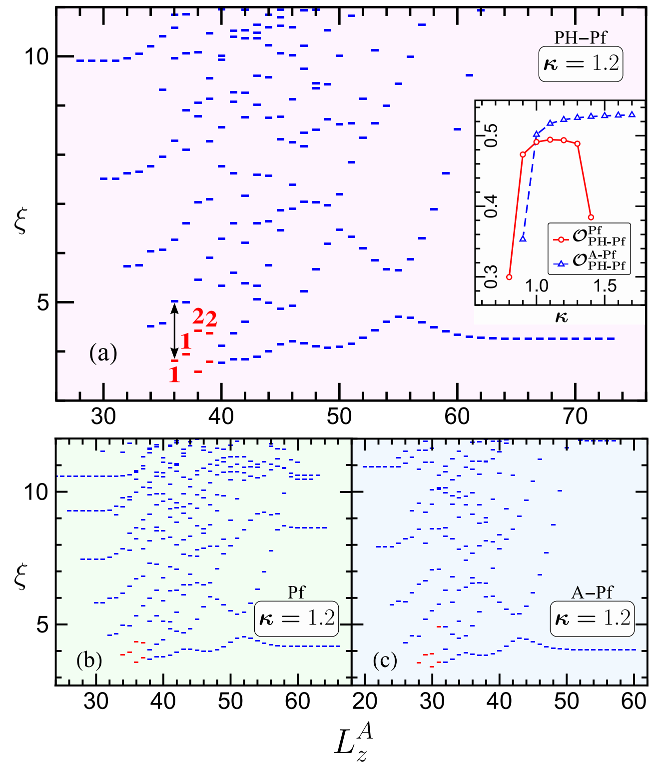

FIG. 3. (a) ES for PH-Pf flux shift with $N_\phi = 27$ at a κ belonging to the $\mathcal{A}$ phase. L_z^A represents the sum of the azimuthal components of angular momenta occupied by the particles in the A part of the partition [41]. An equal number of electrons ($N_A = N_B = 7$) in both partitions are considered for computing the corresponding ES. Here ξ represents entanglement energy in an arbitrary unit. The entanglement gap is shown by a line with an arrow-headed top and bottom, and the counting of edge states is marked. Inset: Overlaps of the ground state of PH-Pf flux ($N_\phi = 27$) at $\kappa = 1.2$ with the ground states at Pf flux ($N_\phi = 25$) and A-Pf flux ($N_\phi = 29$) for different values of κ in the range of the $\mathcal{A}$ phase for $N = 14$. The respective overlaps are $\mathcal{O}^{\rm Pf}_{\rm PH-Pf}$ (circles) and $\mathcal{O}^{\rm A-Pf}_{\rm PH-Pf}$ (triangles). (b) The ES for Pf flux shift with $N_\phi = 25$, $N_A = N_B = 7$, and $\kappa = 1.2$. (c) Same as (b) but for A-Pf flux shift with $N_\phi = 25$, $N_A = N_B = 6$.

extrapolated value $\Delta_c \sim 0.002$–$0.003e^2/(\epsilon\ell_0)$, which amounts to ∼200–300 mK (in the same ballpark of the charged gap found in the experiments) for 5 T magnetic field, where $\epsilon \simeq 13$ is the dielectric constant of the host of the electron gas. The neutral excitations are also gapped (see Supplemental Material [41]). Therefore the $\mathcal{A}$ phase is incompressible and quantized.

We show (Fig. 3) the ES [47–49] for the $\mathcal{A}$ phase for all three flux shifts (see Supplemental Material [41] for other phases). The low-lying ES for all the three shifts are broadly similar [Figs. 3(a)–3(c)] in nature, suggesting its unique topological order. It does not belong to any of the conventional topological sectors, namely, Pfaffian, anti-Pfaffian, or particle-hole symmetric Pfaffian. The Hilbert space for PH-Pf (Pf) flux is a subspace of A-Pf (PH-Pf) flux (see Supplemental Material [41]) for a fixed N. The similar ES occurs because the ground state at PH-Pf flux which is in between Pf and A-Pf fluxes for a fixed N has counter-intuitively (see Supplemental Material [41]) sizable overlap [inset Fig. 3(a)] with the same for the latter two fluxes. The ES is gapped and it displays the counting of edge states as 1-1-2-2-⋯.

We propose a trial ground state wave function for the $\mathcal{A}$ phase of 5/2 state in the spherical geometry as

$$\Psi_{\mathcal{A}}(\{u_i, v_i\}) = \prod_{i<j}^{N}(u_i v_j - u_j v_i) \times \mathcal{S}\left[\prod_{1\le k,l\le N/2}(u_k v_{N/2+l} - u_{N/2+l} v_k)^2\right], \quad (2)$$

where $u_j = \cos(\theta_j/2)e^{i\phi_j/2}$ and $v_j = \sin(\theta_j/2)e^{-i\phi_j/2}$ are the spherical spinors in terms of spherical angles $0 \le \theta_j \le \pi$ and $0 \le \phi_j \le 2\pi$, and $\mathcal{S}$ represents symmetrization in particle indices. This wave function corresponds to $N_\phi = 2N - 1$, i.e., PH-Pf flux shift *a la* the previously proposed PH-Pf wave function [50], although the former does not have particle-hole symmetry. Owing to higher three-body interaction in the $\mathcal{A}$ phase, the system loses the particle-hole symmetry anyway. The wave function $\Psi_{\mathcal{A}}$ in Eq. (2) may be interpreted as two separate condensates of two-flavored composite bosons (electrons attached with one unit of flux quantum) with strong interflavored repulsive correlation. It further signifies that the bosonic wave function (ignoring the ubiquitous Jastrow factor required for Pauli exclusion principle for fermions) will not vanish even if the macroscopic $N/2$ bosons coincide. As per other known wave functions [2,51] with the possibility of coinciding two or more bosons supporting non-Abelian quasiparticles, the wave function $\Psi_{\mathcal{A}}$ in Eq. (2) is likely to support non-Abelian quasiparticles. The wave function in Eq. (2) may also be regarded as a fully antisymmetrized 113 Halperin wave function [52].

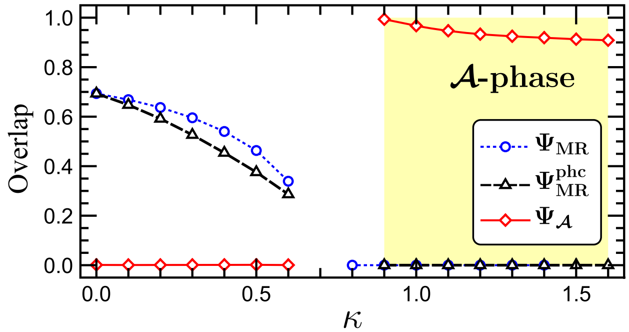

FIG. 4. Overlaps of Ψ_{MR}, Ψ_{MR}^{phc} (particle-hole conjugate of Ψ_{MR}), and $\Psi_{\mathcal{A}}$ in Eq. (2) with the corresponding exact ground states for $(N = 14, N_\phi = 25)$, $(N = 12, N_\phi = 25)$, and $(N = 14, N_\phi = 27)$, respectively, versus κ at zero quantum well width. The unconnected zones refer to the unquantized regimes between two distinct topological quantized phases for each case.

In Fig. 4, we show the overlap of Ψ_{MR}, Ψ_{MR}^{phc}, and $\Psi_{\mathcal{A}}$ with the corresponding exact ground states of $\hat{H}_{eff}$ at their respective flux shifts. While the former two overlaps decrease with the increase of κ and those are exceedingly low at moderate-κ regime, the latter has, in contrast, very *high* overlap [53] in the latter regime which coincides with the experimental regime of $\kappa = 0.8$–1.8. We find, beyond this regime, another transition to an unquantized phase which may have some bearings to the observed nematic phase [54–56]. However, as pointed out in Ref. [19], the perturbatively obtained $\hat{H}_{eff}$ [Eq. (1)] may not be reliable at high κ and thus such a transition may defer to higher κ. Contrary to Ψ_{MR} and Ψ_{MR}^{phc} wave functions, the wave function $\Psi_{\mathcal{A}}$ in Eq. (2) with the exact ground states even at Pf and A-Pf fluxes in the $\mathcal{A}$ phase has a sizable overlap (see Supplemental Material [41]). Therefore, the $\mathcal{A}$ phase appears as independent of the flux shifts and is well characterized by the wave function $\Psi_{\mathcal{A}}$. Further, the low-lying ES corresponding to $\Psi_{\mathcal{A}}$ nicely resembles (see Fig. 5) with the same for the exact ground state. Therefore, $\Psi_{\mathcal{A}}$ seems to represent similar topological order as the exact ground state has for the $5/2$ $\mathcal{A}$ phase.

The topological properties of the $\mathcal{A}$ phase can be extracted by exploiting the two-component structure of $\Psi_{\mathcal{A}}$. The corresponding low-energy effective Lagrangian density [57] is given by

$$\mathcal{L} = -\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\sum_{I,J=1}^{2} K_{IJ} a_\alpha^I \partial_\beta a_\gamma^J - \frac{1}{2\pi}\epsilon^{\alpha\beta\gamma}\sum_{I=1}^{2} t_I A_\alpha \partial_\beta a_\gamma^I \qquad (3)$$

with $K_{11} = K_{22} = 1$, $K_{12} = K_{21} = 3$, and $t_1 = t_2 = 1$. Here a_α^1 and a_α^2 represent two components of Chern-Simons gauge fields, A_α is the external electromagnetic field, and $\epsilon^{\alpha\beta\gamma}$ is the antisymmetric Levi-Cevita tensor. Further introducing quasiparticle vector $l^T = (1, 0)$ and spin vector $s^T = (1/2, 1/2)$ and following Ref. [57], we find topological properties such

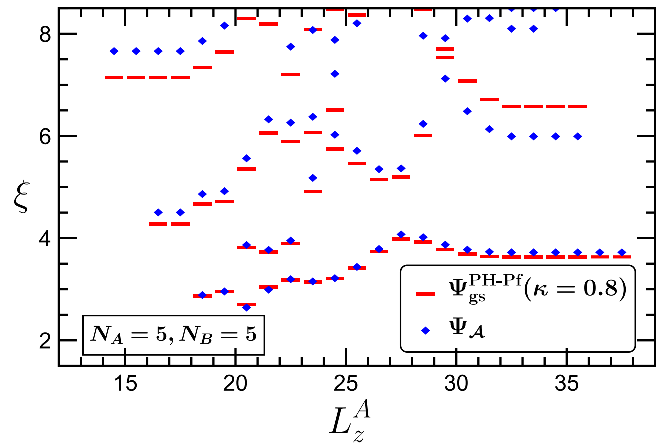

FIG. 5. ES for $N = 10$ and $N_\phi = 19$ for the exact ground state (red dashed) and for $\Psi_{\mathcal{A}}$ (blue diamonds) at $\kappa = 0.8$ belonging to quantized $\mathcal{A}$ phase. Equal number of particles ($N_A = 5$, $N_B = 5$) have been considered for both the partitions for calculating the corresponding ES.

as filling factor $\nu = t^T K^{-1} t = 1/2$, quasiparticle charge $q = e l^T K^{-1} t = e/4$, topological shift $S = (2/\nu) t^T K^{-1} s = 1$ relevant for Hall viscosity, and the ground state degeneracy $D = |\mathrm{Det}(K)|^g = 8^g$, where g is the genus of the geometry of the system. As the two eigenvalues of K are opposite in sign, there will be one downstream charge mode with charge $q = e/4$ and one upstream neutral mode. The wave function $\Psi_{\mathcal{A}}$ has got hidden $\mathbb{Z}_2$ symmetry because the composite bosons are divided into two groups and up to $N/2$ composite bosons can occupy the same position. A similar hidden $\mathbb{Z}_2$ symmetry is present [58] in the reformulated form of the Read-Rezayi [51] wave function for the $5/2$ state. Because of the $\mathbb{Z}_2$ symmetry, one downstream neutral Majorana mode will also be present. Therefore, the net thermal Hall conductance becomes $2.5G_0$ as the fully filled lowest Landau level will provide the contribution of $2G_0$.

In this Letter, we show an *anomalous* topological $5/2$ quantized fractional quantum Hall phase at a moderate strength of the Landau-level-mixing that is in the ballpark of the typical GaAs systems. We have proposed a wave function which turns out to be excellent for describing the ground state of this phase. Because the topological properties provided by this wave function are consistent with the enigmatic $5/2$ state, we attribute the experimentally observed [23] 2.5 unit of thermal Hall conductance (not expected from the theoretical predictions for the conventional phases) in the system with $\kappa \sim 1.1$ to this phase. However, this assertion will be strengthened by the identification of appropriate conformal field operators [2,59] whose correlation can determine a wave function with a topological order that should be adiabatically connected to our proposed wave function and at the same time explains the Majorana mode along with other charge modes for the desired thermal Hall conductance.

Our work opens an avenue for exploring further characteristics of the $\mathcal{A}$ phase that are possibly causes of the recent experimentally observed [23,24] unusual value of thermal Hall conductance at the $5/2$ quantum Hall state. The role of Landau-level mixing in the fractional quantum Hall states in graphene [60,61] and ZnO-based systems [62,63], where it is stronger, should also be an interesting direction of future work; whether or not the predicted $\mathcal{A}$ phase occurs in such systems. Our results suggest a new approach for uncovering phases relevant in the experimental domain for other fractional quantum Hall states in the second Landau level, which are mostly enigmatic.

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[40] LLM-corrected pseudopotentials for disc geometry are taken from Ref. [11]. For zero quantum well width and fully polarized electrons, LLM induced two-body pseudopotentials are $\delta V_1^{(2)} = -0.2143$, $\delta V_3^{(2)} = -0.1039$, $\delta V_5^{(2)} = -0.0353$, $\delta V_7^{(2)} = -0.0115$, and $\delta V_9^{(2)} = -0.0023$ and the three-body pseudopotentials are $V_3^{(3)} = -0.0147$, $V_5^{(3)} = -0.0054$, $V_6^{(3)} = -0.0099$, $V_7^{(3)} = 0.0005$, and $V_8^{(3)} = -0.0009$ in the unit of $e^2/(\epsilon\ell_0)$. Bare Coulomb pseudopotentials are considered for spherical geometry. We have checked that the positive thermodynamic charge gap found for A phase does not alter even if we consider bare pseudopotentials for disk geometry.
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