Prethermalization and the Local Robustness of Gapped Systems

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We prove that prethermalization is a generic property of gapped local many-body quantum systems, subjected to small perturbations, in any spatial dimension. More precisely, let H_0 be a Hamiltonian, spatially local in d spatial dimensions, with a gap Δ in the many-body spectrum; let V be a spatially local Hamiltonian consisting of a sum of local terms, each of which is bounded by $\epsilon \ll \Delta$. Then, the approximation that quantum dynamics is restricted to the low-energy subspace of H_0 is accurate, in the correlation functions of local operators, for stretched exponential timescale $\tau \sim \exp[(\Delta/\epsilon)^a]$ for any $a < 1/(2d-1)$. This result does not depend on whether the perturbation closes the gap. It significantly extends previous rigorous results on prethermalization in models where H_0 was frustration-free. We infer the robustness of quantum simulation in low-energy subspaces, the existence of athermal "scarred" correlation functions in gapped systems subject to generic perturbations, the long lifetime of false vacua in symmetry broken systems, and the robustness of quantum information in non-frustration-free gapped phases with topological order.

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Introduction.—Consider an exactly solved many-body quantum Hamiltonian H_0, assumed to be spatially local in d spatial dimensions. Now, consider perturbing the Hamiltonian to $H_0 + V$, where V is made out of a sum of local terms, each of bounded norm ϵ. As long as we take the thermodynamic limit *before* sending $\epsilon \to 0$, general lore states that a perturbation ($\epsilon > 0$) has drastic qualitative effects. For example, the orthogonality catastrophe shows that eigenstates are extraordinarily sensitive to perturbations [1]. A general integrable system generally exhibits a complete rearrangement of the many-body spectrum, transitioning from Poisson ($\epsilon = 0$) to Wigner-Dyson ($\epsilon \neq 0$) energy-level statistics [2,3]. Only in special settings, such as the conjectured many-body localized phase [4–9], might the simple properties of many-body systems remain robust to perturbations.

With that said, it is known that in *gapped* quantum many-body systems, the thermalization timescale (as measured by physical observables, i.e., local correlation functions) may be exponentially long:

$$t_* \sim \exp\left[\left(\frac{\Delta}{\epsilon}\right)^a\right], \tag{1}$$

where Δ is the gap of H_0, and $a > 0$. To understand why, consider the Hubbard model $H \sim \sum_{i\sim j} \epsilon c^\dagger_{\sigma i} c_{\sigma j} + \sum_i \Delta n_{\uparrow i} n_{\downarrow i}$ [10,11]: although two particles on the same site (called a doublon) store enormous energy and "should" thermalize into a sea of mobile excitations by separating, there is no local perturbation that can do this! The doublon has energy Δ, but one no-doublon excitation has energy $\lesssim \epsilon$. One must go to order Δ/ϵ in perturbation theory to find a many-body resonance whereby a doublon can split apart while conserving energy: this implies Eq. (1). Only in the last few years was this intuition put on rigorous ground [12,13].

Existing proofs of prethermalization in the Hubbard model rely fundamentally on peculiar aspects of the problem. The "unperturbed" H_0 consists exclusively of the repulsive potential energy—it is a sum of local operators that (1) act on a single lattice site, (2) mutually commute, and (3) have an "integer spectrum," such that the many-body spectrum of H_0 is of the form $0, \Delta, 2\Delta, \ldots$. The "perturbation" V is the kinetic (hopping) terms. While prethermalization proofs have also been extended to Floquet and other non-Hamiltonian settings [14–19] with various experimental verifications [20–25], assumptions (2) and (3), which lead to exact solvability, among other useful features, essentially remain.

At the same time, one may be surprised on physical grounds by this state of affairs: the intuition for prethermalization does *not* rely on solvability of H_0, nor even a discrete spectrum in the thermodynamic limit. In fact, it should suffice to simply say that if Δ is a many-body spectral gap of H_0, and any local perturbation can add energy at most $\epsilon \ll \Delta$, then one has to go to order Δ/ϵ in perturbation theory to witness a many-body resonance wherein a system, prepared on one side of the gap of H_0, can "decay" into a state on the other side.

Indeed, this argument is consistent with a very different physical scenario: false vacuum decay. Here, we consider a gapped H_0 with degenerate ground states protected by symmetry (in the thermodynamic limit), separated from the rest of the spectrum by gap Δ. An example is an Ising ferromagnet with $\mathbb{Z}_2$ symmetry spontaneously broken in the ground state. If the perturbation V explicitly breaks the symmetry, one of H_0's ground states will generically have "extensive energy" for $H_0 + V$. So V will close the gap, and the false vacuum is one of exponentially many excited states of similar energy. Still, path integral calculations imply the false vacuum is stable for nonperturbatively long times [26]. This is confirmed, as measured by local correlators in specific lattice models [27–31]. If we consider a quench at time $t = 0$, since the rate per spacetime volume of nucleating a bubble of true vacuum scales as $1/t_*$, the probability a local correlator detects the true vacuum is t^{d+1}/t_* in d spatial dimensions, implying thermalization time $\exp[(\Delta/\epsilon)^a]$.

Moreover, we expect gapped topologically ordered phases are robust to perturbations at all times. This could pave the way for topological quantum computing [32,33] and quantum memory [34,35] at zero temperature. However, such stability has been proven only for certain gapped Hamiltonians [36,37].

The gap in H_0 is crucial to all three stories above. In this Letter, we prove that all three phenomena are related to a common result: when *any gapped* H_0 is perturbed to $H_0 + V$, local correlation functions are efficiently approximated by truncating to the low-energy subspace of H_0 for a nonperturbatively long time. Prethermalization, captured by Eq. (1), is independent of the solvability of H_0. This is (1) a substantial generalization of the theory of [13], (2) a proof that false vacuum decay is nonperturbatively slow, and (3) a proof of stability for gapped topological phases over nonperturbatively long times. These diverse applications of our result are summarized in Table I.

Main result.—Let H_0 and V be local many-body Hamiltonians on a d-dimensional lattice Λ, e.g.,

$$V = \sum_{S \subset \Lambda, S\,\text{local}} V_S, \tag{2}$$

where V_S acts nontrivially on the degrees of freedom on sites in the geometrically local S, and trivially elsewhere,

TABLE I. Summary of *rigorous* results on the robustness of gapped systems.

Scenario	Assumption on H_0	$t_* \geq ?$
Prethermalization	Commuting; integer spectrum	$\exp[\Delta/\epsilon]$ [13]
False vacuum decay	Gapped Discrete symmetry breaking (gapped)	$\exp[(\Delta/\epsilon)^a]$ (this Letter)
Stability of topological order	Gapped Frustration-free, local topological order, and gapped	∞ [36]

and $\|V_S\| \leq \epsilon$. H_0 has a similarly local structure, and we require the existence of a "spectral gap" of size Δ, wherein the many-body Hilbert space $\mathcal{H}$ can be decomposed into $\mathcal{H} = \mathcal{H}_< \oplus \mathcal{H}_>$, where $\mathcal{H}_<$ contains eigenvectors of eigenvalue at most E_*, while $\mathcal{H}_>$ contains eigenvectors of eigenvalue at least $E_* + \Delta$; see Ref. [38] for details.

For sufficiently small ϵ/Δ, there is a quasilocal unitary U such that

$$H_0 + V = U(H_* + V_*)U^\dagger, \tag{3}$$

where H_* has no matrix element connecting eigenstates of H_0 whose eigenvalue difference is larger than Δ, while V_* is a sum of local terms of strength

$$\|(V_*)_S\| \lesssim \epsilon \exp\left[-\left(\frac{\Delta}{\epsilon}\right)^a\right], \quad \text{for any } a < \frac{1}{2d-1}. \tag{4}$$

(This a is likely not tight for $d > 1$.) U is generated by finite-time evolution with a quasilocal Hamiltonian protocol $\tilde{H}$ with terms of strength ϵ, where $\tilde{H} \propto -\int_{-\infty}^{\infty} W(t)V(t)dt + \mathrm{O}(\epsilon^2)$ is defined order by order. Here, $W(t)$ is a fast-decaying function, and $V(t) = e^{itH_0}Ve^{-itH_0}$ is dominated by terms with range $\lesssim t$ due to the Lieb-Robinson bound (which proves the locality of operator dynamics in many-body systems [46,47]). These facts imply $\tilde{H}$ is indeed quasilocal.

Numerical demonstration.—To explain the physical implications of these formal statements, it helps to showcase our result with the interacting $d = 1$ spin model

$$H_0 = \sum_{i=1}^{N-1}(Z_i Z_{i+1} + J_x X_i X_{i+1}) + h\sum_{i=1}^{N} X_i, \tag{5a}$$

$$V = \epsilon Z = \epsilon \sum_{i=1}^{N} Z_i, \tag{5b}$$

where $h = 0.9$, $J_x = 0.37$. If $J_x = 0$, H_0 is the transverse-field Ising model with two ferromagnetic ground states, separated from the excited states by a gap $2(1-h) \approx 0.2$ [48]. J_x term is added to break the integrability of H_0, but using exact diagonalization, we find H_0 is still gapped within the ferromagnetic phase; see Fig. 1(a). However, this gap is extremely sensitive to V: the ground state $|\psi_\uparrow\rangle$ of H_0 with $\langle Z_i \rangle > 0$ quickly merges into the excitation spectrum when $\epsilon \sim N^{-1}$. So Eq. (5) models false vacuum decay, generalizing the literature that studies the case $J_x = 0$ [27–31]. For $\epsilon \lesssim \Delta$, we see clear nonthermal dynamical behavior in Fig. 1(b) if the system starts in the true false vacuum $|\psi_\uparrow\rangle$ or even the product state $|\uparrow \cdots \uparrow\rangle$. Prethermalization and slow false vacuum decay are visible in the anomalously large values of $\langle Z_i(t) \rangle$, even at $t > N/\epsilon$. Both preparing the initial state $|\uparrow \cdots \uparrow\rangle$ and measuring $\langle Z_i(t) \rangle$ are achievable in ultracold atom experiments [49].

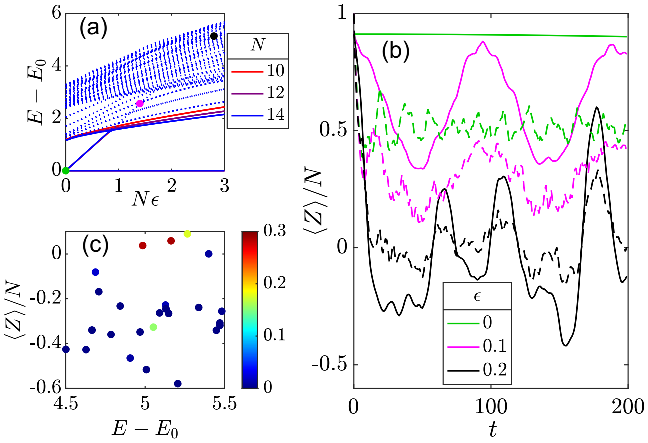

FIG. 1. (a) Spectrum of $H = H_0 + V$ in Eqs. (5a) and (5b) at $N = 14$ (blue lines). The lowest 60 eigenstates are shown; E_0 is the ground state energy. For the lowest 3 eigenstates, data for $N = 10$, 12 are also shown by solid lines of different colors, indicating the gap closes at $\epsilon \sim 1/N$. Solid dots represent $E_\uparrow = \langle\psi_\uparrow|H|\psi_\uparrow\rangle$ at $\epsilon = 0, 0.1, 0.2$; for the latter two values, the false vacuum has been lifted above the gap. (b) Solid lines: $\langle Z\rangle/N$ with initial state $\psi_\uparrow$ for $\epsilon = 0, 0.1, 0.2$. The green line $\epsilon = 0$ has slight dynamics because $\psi_\uparrow$ is superposition of only *almost* degenerate states (with finite system size). Dashed lines: $\langle Z\rangle/N$ starting from $|\uparrow\cdots\uparrow\rangle$ instead. Athermal behavior is observed for times $t > N/\epsilon$, even as $\Delta = 0.2 \sim \epsilon$. (c) Overlap of eigenstates $|E\rangle$ of H with the false vacuum: $|\langle E|\psi_\uparrow\rangle|^2$, as a function of energy around $E_\uparrow$ for $N = 14$, $\epsilon = 0.2$. The color for each eigenstate $|E\rangle$ indicates $|\langle E|\psi_\uparrow\rangle|^2$. $|\psi_\uparrow\rangle$ is supported mainly by three atypical "scar states" with $\langle Z\rangle > 0$.

The nonthermal behavior is also manifest when we analyze the exact eigenstates of $H_0 + V$; see Fig. 1(c). While V strongly prefers $\langle Z_i\rangle < 0$, and most eigenstates near energy $E_\uparrow = \langle\psi_\uparrow|H|\psi_\uparrow\rangle$ (similar for $|\uparrow\cdots\uparrow\rangle$) obey this, there are three atypical eigenstates with $\langle Z\rangle > 0$ on which $\psi_\uparrow$ has large support.

H_0 is neither commuting or frustration-free nor has integer spectrum or topological order. Previous bounds could not prove prethermalization in this model. Our work proves that this numerically demonstrated slow false vacuum decay persists to the thermodynamic limit, even as V closes the gap of H_0.

Scars and the false vacuum.—Let us return to the physical implications of the decomposition [Eq. (3)] in more general models. We focus on the false vacuum decay problem in a ferromagnet for concreteness, but the discussion readily generalizes. First, observe that $UH_*U^\dagger = H'_*$ is block-diagonal in $U\mathcal{H}_< \oplus U\mathcal{H}_>$. Analogously to Fig. 1(a), $U\mathcal{H}_<$ has two eigenstates: ground state $|\Omega'_0\rangle$ with energy E_0, and excited state $|\Omega'_1\rangle$ with energy $E > E_0$. By definition, $|\Omega'_{0,1}\rangle = U^\dagger|\Omega_{0,1}\rangle$, where $|\Omega_{0,1}\rangle$ are (in the thermodynamic limit) degenerate ground state of ferromagnetic H_0. Note that $H'_* = H - UV_*U^\dagger$, H'_* is a local Hamiltonian, only differing from H up to *nonperturbatively small* quasilocal terms ($UV_*U^\dagger$).

Let us now see that the states $|\Omega'_{0,1}\rangle$, when probed by a local (few-body) operator B, behave similarly to the ferromagnetic vacua $|\Omega_{0,1}\rangle$. By the Lieb-Robinson theorem, $\|U^\dagger BU - B\| \sim \epsilon/\Delta$. So if $\langle\Omega_1|B|\Omega_1\rangle \sim 1$, $\langle\Omega_1|B|\Omega_1\rangle = \langle\Omega'_1|U^\dagger BU|\Omega'_1\rangle \approx \langle\Omega'_1|B|\Omega'_1\rangle \sim 1$ as well. Since $|\Omega'_1\rangle$ is an eigenstate of H'_*, $\langle\Omega'_1|B(t)|\Omega'_1\rangle \gtrsim 1 - t/t_*$ stays close to 1 before prethermal time [Eq. (1)]: time dependence in the correlator comes from the nonperturbatively small V_* contained within $B(t) = e^{iHt}Be^{-iHt}$. Thus, the prethermal phenomena will be visible in local correlation functions.

By construction, H'_* generically has exact quantum scars [50–61]: athermal and atypical eigenstates buried in an otherwise chaotic spectrum. In contrast, our theorem does *not* prove that H has exact scars. It is intriguing that prethermalization also has clear fingerprints in the actual eigenstates of $H_0 + V$ in Fig. 1.

In our example, the scar $|\Omega'_1\rangle$ also is the *false vacuum* state. Our framework improves on a subtle shortcoming [62] of the classic [26] path integral calculation of false vacuum decay, by providing a nonperturbative and explicit construction of such a state, including when H_0 is not amenable to a perturbative analysis (a la Feynman diagrams). We also highlight *what it means* to have a long-lived false vacuum: local correlation functions in $|\Omega'_1\rangle$ are close to those of $|\Omega_1\rangle$, even though $|\Omega_1\rangle$ and $|\Omega'_1\rangle$ are effectively orthogonal.

Applications.—An immediate consequence of our result is the generic robustness of quantum simulation of low-energy—often *constrained*—quantum dynamics in the presence of realistic experimental perturbations. For example, one may wish to study exotic quantum dynamics in a Hilbert space where no two adjacent spins in a 1D chain can both be up. Yet in experiment, such a constraint can only be "softly" implemented by penalizing adjacent up spins, e.g., via the Rydberg blockade [51]. Our result proves that for *any* such model with soft constraints, the dynamics is accurately approximated by quantum dynamics in the constrained subspace of physical interest for nonperturbatively long times. Similarly, thermalization (and the timescales after which eigenstate thermalization hypothesis [63–66] can hold) is extraordinarily slow in all perturbations of gapped systems, starting from states in $U\mathcal{H}_<$.

In the Hubbard model, [13] proved the quasiconservation of doublon number, which is a sum of local operators. This proof relies critically on H_0 having an integer spectrum. Since our result does not require this assumption, it is not clear if such a few-body quasiconserved quantity exists, when H_0 contains only a single gap. Still, under the assumption that the low-energy spectrum of H_0 comes from (gapped) quasiparticle excitations, we argue in [38] that our rigorous result suggests the absence of low-energy quasiparticle proliferation [29] before the prethermalization time, starting from any state that has sufficiently low energy ($|\psi_\uparrow\rangle$ or $|\uparrow\cdots\uparrow\rangle$ in the numerical example). Since H_* in Eq. (3) is quasilocal and does not connect eigenstates of H_0 with energy difference larger than Δ, it would not connect between states with differing numbers of low-energy quasiparticles (whose energy is at least Δ).

Most spectral gaps in many-body systems arise in gapped phases of matter, where the ground states are separated by a finite gap Δ from any excited state. In a topological phase, there are exactly degenerate ground states [67], which may serve as a logical qubit. Our prethermalization proof implies such a qubit will remain protected in a low-dimensional subspace for extraordinarily long timescales in the presence of perturbations. This work thus provides an interesting generalization of earlier results [34,36,37,68] that proved the robustness of topological order in frustration-free Hamiltonians. In practice, decoherence of an experimental device may be far more dangerous than any perturbation itself to a qubit [69,70]. We cannot prove the robustness of *accessible* information [35]: logical operators L are often extensive, so even if the rotation U in Eq. (3) is quasilocal, $\|U^\dagger L U - L\| \sim 1$ is possible.

A somewhat similar application of our result arises in SU(2)-symmetric quantum spin models, where states in the Dicke manifold (maximal S^2 subspace) can readily form squeezed states [71] of metrological value [72]. When the Dicke manifold is protected by a spectral gap (as arises in realistic models), our work demonstrates that this protection of squeezed states is robust for exponentially long timescales in the presence of inevitable perturbations. Of course, many practical atomic physics experiments have long-range (power-law) interactions [73], which currently lie beyond the scope of our proof. It will be important in future work to understand whether our conclusions can be extended to this setting.

Proof idea.—We now sketch the proof of our main result (details are in [38]). Although the proof structure mirrors that for Hubbard-like models [13], we need substantial technical improvements because our assumption is much weaker: we only need a single gap in H_0. In what follows, $|n\rangle$ is an eigenstate of H_0 with eigenvalue E_n.

Suppose for the moment that V was so small that $\|V\| \ll \Delta$, and (for convenience) suppose that $\langle m|V|n\rangle = V_{mn} \neq 0$ only if $|m\rangle$ and $|n\rangle$ are on opposite sides of the gap. In this case we would know exactly V does not close the gap, and moreover we could use first order perturbation theory to explicitly rotate the eigenstates

$$|n\rangle_1 = |n\rangle + \sum_{m\neq n} \frac{V_{mn}}{E_n - E_m}|m\rangle. \tag{6}$$

Moreover,

$$\||n\rangle_1 - |n\rangle\| \lesssim \frac{\epsilon}{\Delta}. \tag{7}$$

Higher orders in perturbation theory are tedious but straightforward, and Eq. (7) holds for the exact all-order eigenstates $|n\rangle_{H_0+V}$. Unfortunately, this series is badly behaved in the more realistic setting where each local term in V is bounded by ϵ instead. Now, $\|V\| \sim \epsilon N$ diverges with the number of lattice sites N. Yet this divergence should only be present in many-body states due to the orthogonality catastrophe; local operators should be well-behaved to high order.

The operator counterpart of Eq. (6) is formulated by the Schrieffer-Wolff transformations [74,75], which proceed as follows. First, we project V onto terms acting within [$\mathbb{P}V$] and between [$(1-\mathbb{P})V$] the high and low-energy subspaces of H_0. This can be done by defining

$$\begin{aligned}\mathbb{P}V &= \int_{-\infty}^{\infty} dt\, w(t) e^{iH_0 t} V e^{-iH_0 t} \\ &= \sum_{n,m} \hat{w}(E_n - E_m) V_{nm} |n\rangle\langle m|.\end{aligned} \tag{8}$$

Here, $w(t)$ is a real-valued function with Fourier transform $\hat{w}(\omega)$. The second line of Eq. (8) follows from the Heisenberg evolution $V(t) = \sum_{n,m} V_{nm} e^{i(E_n - E_m)t}|n\rangle\langle m|$. We do not try to calculate $|n\rangle$ or V_{nm}; nevertheless, the formal statement [Eq. (8)] is valuable. If we can find a function where $\hat{w}(\omega) = 0$ if $|\omega| \geq \Delta$, this transformation can project out the off-diagonal terms in V. Such functions are known [76,77], and have asymptotic decay $w(t) \sim e^{-|t|/\ln^2|t|}$ at large t. The Lieb-Robinson theorem [46,47] shows that for any local operator B_x supported on site x, $e^{iH_0 t} B_x e^{-iH_0 t}$ is, up to exponentially small corrections, a sum of operators acting on sites within a distance $d \sim vt$ of x, for finite velocity v. As a result, terms in $\mathbb{P}V$ that act on sites separated by distance r decay faster than $\exp[-r^{1-\delta}]$, for any $\delta > 0$: this is because $w(t)$ decays a little slower than e^{-t}, and $B_x(t)$ has support in a ball of size vt, centered at x.

With the desired projection, we then define

$$D_1 = \mathbb{P}V, \qquad W_1 = (1-\mathbb{P})V, \tag{9}$$

and a first order unitary rotation $U_1 = \mathrm{e}^{A_1}$ where

$$[A_1, H_0] = W_1, \tag{10}$$

to rotate away the off-diagonal W_1. A_1 can be found as i times a quasilocal Hamiltonian in a similar fashion in Eq. (8). Explicit calculation shows that the new Hamiltonian in the rotated frame,

$$U_1^\dagger (H_0 + V) U_1 = H_2 + V_2, \tag{11}$$

is indeed block-diagonal (H_2 piece) for the two gapped subspaces of H_0 up to a $O(\epsilon^2)$ piece V_2. Moreover, although the generator Hamiltonian $-iA_1$ contains terms that decay slowly with its support, we prove V_2 is a sum of local terms that decay as $\exp[-r^{1-\delta}]$ with the support size r. To get this locality bound of V_2, we do require somewhat better Lieb-Robinson bounds, inspired by the equivalence class construction of [78], than the standard ones [46]. Equation (11) with the locality bound completes the first order Schrieffer-Wolff transformation. Similar ideas have previously been developed at finite order of the transformation [79–82]. Here, we not only deal with general models, but iterate this process to very high order, to obtain the nonperturbative bound [Eq. (1)].

At kth order, we are given V_k as the off-diagonal part in the Hamiltonian. We define $D_k = \mathbb{P}V_k$, $W_k = (1-\mathbb{P})V_k$ and $[A_k, H_0] = W_k$. Rotating the Hamiltonian by $U_k = e^{A_k}$ gives the next off-diagonal V_{k+1}. The nontrivial aspect of this iteration is to show that V_k (and $A_k, D_k, \cdots$) is not too nonlocal: after all, our argument for prethermalization relied on $\|U^\dagger B U - B\| \ll \|B\|$, which is only guaranteed when U consists of local rotations. As we use the same projection $\mathbb{P}$ at each step of the process, V_k has increasingly large support for increasing k, and eventually this process becomes uncontrollable: the support of terms in V_k is so large that our error $\|U_k^\dagger B U_k - B\|$ increases with k. In our proof, we can show that

$$\frac{\|V_{k+1}\|_{\mathrm{local}}}{\|V_k\|_{\mathrm{local}}} \lesssim \frac{\epsilon}{\Delta} k^{(2d-1)/(1-2\delta)}. \qquad (12)$$

Here, $\|V\|_{\mathrm{local}}$ roughly denotes the operator norm of terms in V that act nontrivially on one particular site. From Eq. (12), we see that we must stop the Schrieffer-Wolff iterations when

$$k_* = \left(\frac{\Delta}{\epsilon}\right)^a, \quad \text{where } a = \frac{1-2\delta}{2d-1}. \qquad (13)$$

This establishes Eqs. (3) and (4) with $V_* = V_{k_*}$ because $\|V_k\|_{\mathrm{local}}$ decays exponentially before k_* in Eq. (12).

Outlook.—In this Letter, we have proved that the prethermalization of doublons in the Hubbard model is but one manifestation of a universal phenomenon, whereby distinct sectors of a gapped Hamiltonian H_0 remain protected for (stretched) exponentially long times in the presence of local perturbations V. Prethermalization, in all measurable local correlation functions, is generic to any perturbation of a gapped system. We thus immediately provide a rigorous proof that the false vacuum decays nonperturbatively slowly, placing less rigorous field-theoretic calculations [26] on firmer footing.

Our result shows that is always reasonable to simulate quantum dynamics generated by V in constrained models, so long as one studies $H_0 + V$, where H_0's ground state manifold is the constrained subspace of interest, and H_0 has a large spectral gap Δ. Even if $H_0 + V$ is gapless and chaotic, the (locally rotated) ground states of H_0 serve as effective "scar states" that will exhibit athermal dynamics for extraordinarily long times. We anticipate that this observation will have practical implications for the preparation of interesting entangled states on the Dicke manifold in future atomic physics experiments, and for the ease of recovering qubits under imperfect local encoding.

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