

## Kinematic Lie Algebras from Twistor Spaces

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We analyze theories with color-kinematics duality from an algebraic perspective and find that any such theory has an underlying  $BV^\bullet$ -algebra, extending the ideas of Reiterer [A homotopy BV algebra for Yang–Mills and color–kinematics, [arXiv:1912.03110](https://arxiv.org/abs/1912.03110)]. Conversely, we show that any theory with a  $BV^\bullet$ -algebra features a kinematic Lie algebra that controls interaction vertices, both on shell and off shell. We explain that the archetypal example of a theory with a  $BV^\bullet$ -algebra is Chern-Simons theory, for which the resulting kinematic Lie algebra is isomorphic to the Schouten-Nijenhuis algebra on multivector fields. The  $BV^\bullet$ -algebra implies the known color-kinematics duality of Chern-Simons theory. Similarly, we show that holomorphic and Cauchy-Riemann Chern-Simons theories come with  $BV^\bullet$ -algebras and that, on the appropriate twistor spaces, these theories organize and identify kinematic Lie algebras for self-dual and full Yang-Mills theories, as well as the currents of any field theory with a twistorial description. We show that this result extends to the loop level under certain assumptions.

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*Introduction and summary.*—Color-kinematics (CK) duality [1,2] (see reviews in Refs. [3–7]) is a surprising property of certain field theories, that allows for their scattering amplitudes to be split into a kinematic component or numerators and gauge Lie algebra or color numerators, such that the kinematic numerators mirror the algebraic properties of the color numerators. This was first observed for the tree-level amplitudes of Yang-Mills theory, but many other theories also exhibit CK duality. CK duality is the cornerstone of the double copy prescription [1,2], which constructs gravity scattering amplitudes from a simple combination of pairs of corresponding kinematical numerators for Yang-Mills theory, suggesting deep, illuminating connections between the known fundamental theories of nature and providing cutting-edge predictions in gravitational-wave astronomy; see, e.g., Refs. [8–10].

Concretely, a theory is CK-dual if its  $n$ -point amplitudes  $\mathcal{A}_n$  can be written as

$$\mathcal{A}_n \sim \sum_{\gamma \in \Gamma_n} \frac{\mathbf{c}_\gamma \mathbf{n}_\gamma}{d_\gamma}, \quad (1)$$

where  $\Gamma_n$  denotes the set of cubic Feynman graphs with  $n$  external lines,  $d_\gamma$  are the product of  $(1/p_\ell^2)$  over all propagator lines  $\ell$ , with  $p_\ell$  their momenta,  $\mathbf{c}_\gamma$  are the color numerators, i.e., products of gauge Lie algebra structure constants, as prescribed by the diagram  $\gamma$ , and  $\mathbf{n}_\gamma$  are the kinematic numerators, built from momenta and polarization tensors. Furthermore,  $\mathbf{c}_\gamma$  and  $\mathbf{n}_\gamma$  obey the same antisymmetry under the interchange of edges in  $\gamma$ , and  $\mathbf{c}_{\gamma_1} + \mathbf{c}_{\gamma_2} + \mathbf{c}_{\gamma_3} = 0$  implies  $\mathbf{n}_{\gamma_1} + \mathbf{n}_{\gamma_2} + \mathbf{n}_{\gamma_3} = 0$ . CK duality thus suggests the existence of a *kinematic Lie algebra* (KLA) from which the  $\mathbf{n}_\gamma$  are constructed.

The geometric and algebraic underpinnings of this KLA remain a central question [11–26]. In the case of self-dual Yang-Mills (SDYM) theory, the KLA consists of area-preserving diffeomorphisms on  $\mathbb{C}^2$  [27]; see Refs. [28–32]. A cubic Lagrangian for the nonlinear sigma model with Feynman rules obeying ( $\leq 1$ )-loop CK duality [33] has been employed to identify the KLA of the maximally-helicity-violating (MHV) sector. Beyond the MHV sector, tensor currents and fusion rules elucidate the KLA of Yang-Mills (YM) theory [34–36]. A closed form expression for tree-level CK-dual numerators was obtained from a covariant CK duality [37] that identified an underlying kinematic Lorentz

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algebra. Moreover, it has been shown that the tree-level currents of super Yang-Mills theory come with a KLA [38].

Instead of working at the level of amplitudes, we consider CK duality and the KLA directly at the level of actions [39–42]: given tree-level CK duality, one can always render the action CK-dual using infinitely many auxiliary fields, but at the cost of unitarity, which is broken by Jacobians arising from field redefinitions [40–42]. It therefore remains to identify an organizational principle for the resulting tower of auxiliaries and avoid nonlocal field redefinitions altogether. Following Ref. [11], we find this organizational principle in the form of  $BV^\square$ -algebras, which ensure the existence of a kinematic Lie algebra. (A more detailed algebraic characterization of  $BV^\square$ -algebras, which also closes the gap to CK duality, is in preparation [26]).

The prime example of a theory with  $BV^\square$ -algebra is Chern-Simons (CS) theory, and many field theories can be equivalently formulated as CS-type theories on twistor spaces. Using this picture, we are able to reproduce and generalize, e.g., the results of Refs. [26,27,43]. We show that the currents of such field theories come with a KLA which we can readily identify. In many cases, this KLA extends to the amplitudes, and for a special class, it implies conventional CK duality. For theories for which the anomalies discussed in Ref. [44] are absent, our arguments extend to the loop level.

Our results significantly improve the understanding of KLAs and comprise concrete and new examples. They highlight the power of the action perspective on CK duality as an organizational principle, and our improved algebraic understanding has the power to streamline the computation of the kinematic numerators important in the double copy construction of gravity scattering amplitudes, cf. Ref. [26].

*Chern-Simons theory.*—We start with the illustrative example of ordinary non-Abelian CS theory, which demonstrates all essential features. For our purposes, it is convenient to work with Batalin-Vilkovisky (BV) quantization [45,46], which introduces unphysical fields called antifields in addition to the physical fields and ghosts, and use differential form notation to hide Lorentz indices. The BV action of CS theory reads as

$$S_{\text{CS}} = \int \text{tr} \left( \frac{1}{2} A \wedge dA + \frac{1}{3!} A \wedge [A, A] + A^+ \wedge (dc + [A, c]) + \frac{1}{2} c^+ \wedge [c, c] \right), \quad (2)$$

where  $A$  is a gauge potential one-form;  $c$  is the ghost field, a Grassmann-odd scalar function;  $A^+$  and  $c^+$  are the corresponding antifields (an odd two-form and an even three-form); and all fields take values in a color Lie algebra  $\mathfrak{g}$ . This action is the Maurer-Cartan action for the differential graded (dg) Lie algebra of differential forms with values in  $\mathfrak{g}$ ,  $(\Omega^* \otimes \mathfrak{g}, d)$ , whose Lie bracket is the wedge product composed with the Lie bracket of  $\mathfrak{g}$ , whose differential is

the exterior derivative  $d$ , and whose grading is such that  $\Omega^p$  carries ghost number  $1 - p$ ; see, e.g., Ref. [47]. After color stripping, we are left with the dg commutative algebra  $(\Omega^p, d)$  of ordinary differential forms under wedge product and exterior derivative.

It is well known that CS scattering amplitudes are trivial. Instead, Ref. [43] considers correlators of harmonic differential forms. To compute these, note that

$$dd^\dagger + d^\dagger d = -\square, \quad (3)$$

where the codifferential  $d^\dagger$  is defined as  $d^\dagger \alpha = -(-1)^p \star d \star \alpha$  for a  $p$ -form  $\alpha$  using the Hodge operator with respect to the Minkowski metric, and  $\square$  the  $d^\dagger$  Alembertian. The propagator is now given by  $[(-d^\dagger)/\square]$ . Using Eq. (3), we may decompose the identity operator on differential forms as

$$1 = d \frac{-d^\dagger}{\square} + \frac{-d^\dagger}{\square} d + \Pi_{\text{Harm}}, \quad (4)$$

where  $\Pi_{\text{Harm}}$  projects onto the harmonic forms. The operator  $-d^\dagger$  allows us to introduce a “derived bracket” on  $\Omega^*$ ,

$$(-1)^p [\alpha, \beta] = -d^\dagger(\alpha \wedge \beta) + d^\dagger \alpha \wedge \beta + (-1)^p \alpha \wedge d^\dagger \beta \quad (5)$$

for all  $\alpha \in \Omega^p$  and  $\beta \in \Omega^*$ . Since  $-d^\dagger$  is a second-order differential operator,  $[-, -]$  defines a so-called Gerstenhaber bracket on  $\Omega^*$ , which is a degree-shifted Lie algebra. Furthermore, the bracket  $[-, -]$  maps pairs of physical fields to physical fields and encodes their interactions. Thus, this defines the KLA ( $[1]$  indicates the degree shift to form an ordinary Lie algebra.),

$$\mathfrak{K} = (\Omega^*[1], [-, -]), \quad (6)$$

for correlators of harmonic forms, which therefore can be brought into the form Eq. (1). One can show [48,49] that this KLA is isomorphic to the Schouten-Nijenhuis algebra of totally antisymmetric tensor fields, the natural Gerstenhaber algebra on three-dimensional Minkowski space. Truncating  $\mathfrak{K}$  to degree 0 yields the KLA  $\mathfrak{K}_0$  commonly discussed in the literature, which here is the spacetime diffeomorphism algebra.

The above straightforwardly extends to holomorphic CS theory on  $\mathbb{C}^3$ , with the real  $p$ -forms and the KLA replaced by the complex  $(0, p)$ -forms and the evident holomorphic version of the Schouten-Nijenhuis algebra, respectively.

More generally, the structure  $(\Omega^*, d, \wedge, -d^\dagger)$  is an instance of what is known as a  $BV^\square$ -algebra [11,26] with  $\blacksquare = \square$ , which we explore below.

*Color-kinematics duality algebraically.*—Consider a field theory whose tree amplitudes [Eq. (1)] arise from the Feynman diagram expansion of a BV action. The corresponding vector space of fields  $\mathfrak{Z}$  is graded by the ghost number, and so we may write  $\mathfrak{Z} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{Z}_p$ , where the elements of  $\mathfrak{Z}_p$  carry ghost number  $1 - p$ . The free part

of the action is captured by a kinematic operator, which is a linear map  $\mathfrak{d}$  on  $\mathfrak{Q}$  with  $\mathfrak{d}^2 = 0$  that decreases ghost number by 1 (mapping antifields to fields and fields to zero). All interaction terms are cubic which are captured by a product on  $\mathfrak{Q}$  that conserves ghost number. Gauge invariance implies that this  $\mathfrak{Q}$  forms a dg Lie algebra [41,47,50], which generalizes our previous  $\Omega^p \otimes \mathfrak{g}$ .

The Feynman expansion [Eq. (1)] implies that a propagator  $\mathfrak{h}$  exists that inverts  $\mathfrak{d}$  on propagating (off shell) fields. Thus, the identity operator on fields decomposes as

$$1 = \mathfrak{d}\mathfrak{h} + \mathfrak{h}\mathfrak{d} + \Pi_{\text{on shell}}, \quad (7)$$

where  $\Pi_{\text{on shell}}$  is the projector on shell fields, generalizing Eq. (4). It is always possible to choose  $\mathfrak{h}$  such that  $\mathfrak{h}^2 = 0$  [51]. Splitting  $\mathfrak{h} = (\mathfrak{b}/\blacksquare)$  into the denominator  $\blacksquare$  and numerator  $\mathfrak{b}$  leads to  $\mathfrak{d}\mathfrak{b} + \mathfrak{b}\mathfrak{d} = \blacksquare$ .

Color stripping now amounts to factorizing  $\mathfrak{Q} = \mathfrak{g} \otimes \mathfrak{B}$  into the color Lie algebra  $\mathfrak{g}$  and a dg commutative algebra  $(\mathfrak{B}, \mathfrak{d}, \mathfrak{m})$  with differential  $\mathfrak{d}$  and product  $\mathfrak{m}$  [41]. Denoting the color-stripped propagator also by  $(\mathfrak{b}/\blacksquare)$ , we have

$$\mathfrak{d}\mathfrak{b} + \mathfrak{b}\mathfrak{d} = \blacksquare \quad (8)$$

which generalizes Eq. (3), and  $\blacksquare$  is a second-order differential operator with respect to  $\mathfrak{m}$ . It is algebraically natural to assume that  $\mathfrak{b}$  is also a second-order differential operator that squares to zero [52] (and this is also true in physically relevant situations [25]). In this case, the derived bracket defined by

$$(-1)^{|x|}[x, y] = \mathfrak{b}\mathfrak{m}(x, y) - \mathfrak{m}(\mathfrak{b}x, y) - (-1)^{|x|}\mathfrak{m}(x, \mathfrak{b}y) \quad (9)$$

for all  $x, y \in \mathfrak{B}$  is a Gerstenhaber bracket on  $\mathfrak{B}$  of which Eq. (5) is a special instance. As in the CS case, the KLA (with all BV fields) is then simply

$$\mathfrak{K} = (\mathfrak{B}[1], [-, -]). \quad (10)$$

Truncating  $\mathfrak{K}$  to degree 0 yields the usual KLA  $\mathfrak{K}_0$ .

Mathematically,  $(\mathfrak{B}, \mathfrak{d}, \mathfrak{m}, \mathfrak{b})$  is a BV algebra [52] with Gerstenhaber bracket given by Eq. (9), and  $\blacksquare$  promotes this BV algebra to a  $\text{BV}^\blacksquare$ -algebra [11,26].

Conversely, in a theory with a  $\text{BV}^\blacksquare$ -algebra, the cubic vertices in Feynman diagrams are governed by a KLA and a ‘‘color’’ Lie algebra. When  $\blacksquare = \square$  coincides with the d’Alembertian, this implies CK duality (up to potential anomalies). Otherwise, complications may arise such that it is not possible to write the amplitudes in the form of Eq. (1). Then, we merely speak of a *theory with a KLA*.

Apart from the CS theories already discussed, these ideas extend to field theories whose linearized equations of motions are encoded in a differential with a natural codifferential. This is the case for all theories whose solutions can be described in terms of flat connections on twistor spaces. In the following, we discuss two examples in detail. We stress that even in the absence of an action principle, we still have a

$\text{BV}^\blacksquare$ -algebra and, thus, a KLA for the numerators of the corresponding tree-level currents.

*Self-dual Yang-Mills theory.*—The twistor space  $Z$  of maximally supersymmetric (MS) SDYM theory is the superspace  $\mathbb{R}^{4|8} \times \mathbb{C}P^1$  with  $\mathbb{C}P^1$  the Riemann sphere; see, e.g., Refs. [53,54] for reviews. Holomorphic CS theory on  $Z$  is semiclassically equivalent (see Refs. [55,56] and [54] for details) to MSSYM theory given by the Siegel action [57] on  $\mathbb{R}^4$ ,

$$S_{\text{Siegel}} = \int \text{tr} \left( G^- \wedge F - \frac{1}{2} \phi \square \phi + \chi \nabla \chi + \frac{1}{2} \phi [\chi, \chi] \right), \quad (11)$$

where  $F$  is the gluon field strength two-form, the Lagrange multiplier  $G^-$  is an anti-self-dual two-form,  $\phi$  denotes the six scalar fields, and  $\chi$  denotes the four gluinos; all fields transform under the adjoint representation of the gauge group. Adopting coordinates  $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}})$  on  $\mathbb{R}^{4|8}$ , with  $x^{\alpha\dot{\alpha}}$  a commuting four-vector and  $\eta_i^{\dot{\alpha}}$  anticommuting spinors, and homogeneous coordinates  $\lambda_{\dot{\alpha}}$  on  $\mathbb{C}P^1$  ( $\alpha, \dot{\alpha} = 1, 2; i = 1, \dots, 4$ ), the antiholomorphic vector fields on  $Z$  are spanned by

$$(\hat{E}_\alpha, \hat{E}^i, \hat{E}_0) = \left( \lambda^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \lambda^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^{\dot{\alpha}}}, |\lambda|^2 \lambda_{\dot{\beta}} \frac{\partial}{\partial \bar{\lambda}_{\dot{\beta}}} \right), \quad (12)$$

where  $|\lambda|^2 = \lambda_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}$  (with  $\bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}$ ). The holomorphic vector fields  $(E_\alpha, E^i, E_0)$  are the corresponding conjugates. We denote the antiholomorphic differential one-forms dual to Eq. (12) by  $(\hat{e}^\alpha, \hat{e}_i, \hat{e}^0)$ . We then have the BV action

$$S_{\text{hCS}} = \int \Omega \wedge \text{tr} \left( \frac{1}{2} A \wedge \bar{\partial}_{\text{red}} A + \frac{1}{3!} A \wedge [A, A] + A^+ \wedge (\bar{\partial}_{\text{red}} c + [A, c]) + \frac{1}{2} c^+ \wedge [c, c] \right), \quad (13)$$

where  $\Omega$  is the holomorphic volume form on twistor space [58] and  $\bar{\partial}_{\text{red}} = \hat{e}^\alpha \hat{E}_\alpha + \hat{e}^0 \hat{E}_0$  is the Dolbeault differential restricted to legs along commuting coordinates. The fields are  $\mathfrak{g}$ -valued antiholomorphic differential forms on  $Z$  that are holomorphic with respect to the anticommuting coordinates  $\eta_i = \eta_i^{\dot{\alpha}} \lambda_{\dot{\alpha}}$  and have no legs along antiholomorphic anticommuting coordinates.

The space of color-stripped fields forms a dg commutative algebra, with the product given by the wedge product and differential  $\bar{\partial}_{\text{red}}$ . It further forms a  $\text{BV}^\blacksquare$ -algebra  $\mathfrak{B}_{\text{SDYM}}$  together with the differential operator

$$\mathfrak{b} = -\frac{4}{|\lambda|^2} \varepsilon^{\alpha\beta} \iota_{E_\alpha} \iota_{\hat{E}_\beta} \partial_{\text{red}} + 2\varepsilon^{\alpha\beta} \iota_{\hat{E}_\alpha} \iota_{\hat{E}_\beta} \hat{e}^0 \wedge, \quad (14)$$

where  $\iota_X$  denotes the contraction of a differential form with a vector field  $X$ . A quick calculation shows that  $\bar{\partial}_{\text{red}} \mathfrak{b} + \mathfrak{b} \bar{\partial}_{\text{red}} = \square_{\mathbb{R}^4}$ .

Recall that the actions in Eqs. (13) and (11) are equivalent, i.e., they share the same tree-level amplitudes. We can compute these by embedding external states on  $\mathbb{R}^4$ , given by harmonic gauge potentials, into  $A \in \Omega_{\text{red}}^{0|0,1|0}$ , respecting the gauge condition  $\mathfrak{b}A = 0$ . We then use the trivial Feynman rules derived from Eq. (13) together with the propagator  $\mathfrak{h} = (\mathfrak{b}/\square_{\mathbb{R}^4})$ . This Feynman diagram expansion manifests the KLA contained in  $\mathfrak{B}_{\text{SDYM}}$  with  $\blacksquare = \square_{\mathbb{R}^4}$ . Hence, MSSDYM theory possesses CK duality, and the twistor action produces a CK-duality-manifesting spacetime action for MSSDYM theory after Kaluza-Klein (KK)-expanding along  $\mathbb{C}P^1$ . Integrating out the KK tower of auxiliary fields reproduces the Siegel action.

The full KLA  $\mathfrak{K}$  is isomorphic to the Schouten-Nijenhuis-type Lie algebra  $\widehat{\mathfrak{K}}$  of bosonic holomorphic totally antisymmetric tensor fields on  $Z$  with Lie bracket  $[U, V]_{\text{red}} = (U^\alpha E_\alpha V^\beta - V^\alpha E_\alpha U^\beta)E_\beta$ .

This construction generalizes to dimensionally reduced SDYM theory and theories with any amount of supersymmetry, following Refs. [59,60] or, e.g., Ref. [61].

To match the literature, note that the lowest order in  $\eta_i$  of the gauge potential  $A$  describes the gluon. Since to this order,  $t_{\hat{E}_0} A$  can be gauged away [ [54], §5.2], only holomorphic multivector fields spanned by the  $E_\alpha$  in the Schouten-Nijenhuis-type Lie algebra  $\widehat{\mathfrak{K}}$  contribute to  $\widehat{\mathfrak{K}}_0$ . On spacetime, these parametrize translations along self-dual planes spanned by  $[(\partial/\partial u), (\partial/\partial w)] = [(\partial/\partial x^{11}), (\partial/\partial x^{21})]$ , reproducing the KLA identified in Ref. [27].

Beyond trees, (MS)SDYM theory possesses finite one-loop amplitudes. Unlike SDYM theory, the twistorial action for MSSDYM theory captures the correct one-loop amplitudes [44]. Our arguments therefore also demonstrate CK duality for MSSDYM theory one-loop amplitude integrands.

*Maximally-supersymmetric Yang-Mills theory.*—Similar arguments also hold for full maximally supersymmetric Yang-Mills (MSYM) theory. (Recall that  $\mathcal{N} = 3$  supersymmetric Yang-Mills (SYM) theory is perturbatively equivalent to  $\mathcal{N} = 4$  SYM theory.) However, the KLA will be such that CK duality is not immediate.

In this case, the twistor space is a CR manifold (a generalization of the notion of a complex manifold, cf. Ref. [62]), namely the CR ambitwistor space  $L = \mathbb{R}^{4|24} \times \mathbb{C}P^1 \times \mathbb{C}P^1$ . Holomorphic CS theory on  $L$  is semiclassically equivalent to four-dimensional MSYM theory [63,64]. We use Cartesian coordinates in spinor notation  $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}, \theta^{i\alpha})$  on  $\mathbb{R}^{4|24}$  and homogeneous coordinates  $(\lambda_{\dot{\alpha}}, \mu_\alpha)$  on  $\mathbb{C}P^1 \times \mathbb{C}P^1$ . The antiholomorphic vector fields

$$(\hat{E}_F, \hat{E}_L, \hat{E}_R) = \left( \mu^\alpha \lambda^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, |\lambda|^2 \lambda_{\dot{\beta}} \frac{\partial}{\partial \lambda_{\dot{\beta}}}, |\mu|^2 \mu_\beta \frac{\partial}{\partial \mu_\beta} \right) \quad (15)$$

form a basis of the space of antiholomorphic vector fields along commuting directions; we also define the conjugate

holomorphic vector fields  $(E_F, E_L, E_R)$  and dual antiholomorphic one-forms  $(\hat{e}^F, \hat{e}^L, \hat{e}^R)$ .

The relevant action here is again of the form of Eq. (13), with  $Z$  replaced by  $L$ ,  $\Omega^{3|4,0|0}$  replaced by the holomorphic measure identified in Refs. [63,64], and the dg commutative algebra  $(\Omega_{\text{red}}^{0|0,\bullet|0}, \bar{\partial}_{\text{red}})$  replaced by the restricted bosonic CR differential forms  $\Omega_{\text{CR}}^{0|0,\bullet|0}$  that depend holomorphically on  $(\eta_i, \theta^i) = (\eta_i^{\dot{\alpha}} \lambda_{\dot{\alpha}}, \theta^{i\alpha} \mu_\alpha)$  for  $i = 1, 2, 3$  and with no antiholomorphic fermionic directions, which are endowed with the differential  $\bar{\partial}_{\text{CR}} = \hat{e}^F \hat{E}_F + \hat{e}^L \hat{E}_L + \hat{e}^R \hat{E}_R$ . The second-order differential operator

$$\mathfrak{b} = -\frac{8}{|\lambda|^2 |\mu|^2} t_{E_F} t_{\hat{E}_F} \bar{\partial}_{\text{CR}} \quad (16)$$

leads to

$$\blacksquare = \bar{\partial}_{\text{CR}} \mathfrak{b} + \mathfrak{b} \bar{\partial}_{\text{CR}} = \square_{\mathbb{R}^4} + 8 \frac{\mu^{(\alpha} \hat{\mu}^{\beta)} \lambda^{(\dot{\alpha}} \hat{\lambda}^{\dot{\beta})}}{|\lambda|^2 |\mu|^2} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \frac{\partial}{\partial x^{\beta\dot{\beta}}}. \quad (17)$$

Thus, we obtain the  $\text{BV}^\blacksquare$ -algebra  $\mathfrak{B}_{\text{SYM}} = (\Omega_{\text{CR}}^{0|0,\bullet|0}, \bar{\partial}_{\text{CR}}, \wedge, \mathfrak{b})$  containing a KLA of evident Cauchy-Riemann automorphisms of  $L$ .

While  $L$  is not compatible with Wick rotation,  $\mathfrak{B}_{\text{SYM}}$  and the contained KLA are: KK expand the theory along  $\mathbb{C}P^1 \times \mathbb{C}P^1$ , obtaining a cubic field theory with an infinite tower of KK fields on  $\mathbb{R}^4$ , on which  $(\bar{\partial}_{\text{CR}}, \mathfrak{b}, \blacksquare)$  act as “ $(\infty \times \infty)$ -dimensional matrices of differential operators”.

The complexified KK fields all carry complex Spin(4) representations. (Unlike ordinary KK expansions, compact directions here carry nontrivial Lorentz representations: Wick rotation destroys the geometric interpretation.), and imposing reality conditions suitable for Minkowski space Wick rotates both the fields and the operators  $(\bar{\partial}_{\text{CR}}, \mathfrak{b}, \blacksquare)$ . (We note that both KK expansion and Wick rotation preserve semiclassical equivalence between Cauchy-Riemann CS theory and MSYM theory. Perturbation theory with propagator  $(\mathfrak{b}/\blacksquare)$  and gauge  $\mathfrak{b}A = 0$  for the field  $A$  reproduces MSYM amplitudes on four-dimensional Minkowski space: semiclassical equivalence fixes the interaction vertices; the propagator  $(\mathfrak{b}/\blacksquare)$  is the inverse of the kinematic operator almost everywhere (i.e., modulo measure-zero sets) in momentum space).

In view of CK duality, our above propagator involving the inverse of  $\blacksquare$ ,

$$\frac{1}{\blacksquare} = \frac{\eta^{MN} - K_{\mu\nu}^{MN} \frac{k^\mu k^\nu}{k^2} + K_{\mu\nu}^{MP} K_P^N \frac{k^\mu k^\nu k^\sigma}{k^4} - \dots}{k^2}, \quad (18)$$

where  $M, N, \dots$  label KK modes, has a striking similarity to the YM propagator  $\mathfrak{h}_\xi^{\mu\nu} = (1/k^2)\eta^{\mu\nu} + (1-\xi)(k^\mu k^\nu/k^4)$  in  $R_\xi$  gauge in that both lead to unphysical singularities (e.g.,  $(k^\mu k^\nu/k^4)$  for  $\mathfrak{h}_\xi^{\mu\nu}$ ) in individual Feynman diagrams.

These singularities signal the propagation of unphysical longitudinal modes. With all external states physical, their contributions have to cancel, and for  $R_\xi$  gauge this follows from Ward identities. It is natural to expect that the same occurs for (potentially Wick-rotated) Cauchy-Riemann CS theory, and there is a KK tower of generalized Ward identities that allows one to replace the propagator  $(\mathbf{b}/\blacksquare)$  with  $(\mathbf{b}/\square)$ . If true, then the BV $^\square$ -algebra guarantees CK-dual numerators: the numerators computed with the propagator  $(\mathbf{b}/\square)$  are automatically CK-dual. We study this in upcoming work.

The extension to the loop level depends on the assumption that the relevant (ambi)twistor theories correctly describe the loop amplitudes. The semiclassical equivalences between spacetime field theories and twistorial CS-type theories only extends to the loop level if certain twistor space anomalies vanish [44]. Provided that Cauchy-Riemann CS theory on  $L$  is anomaly free with no further problems reducing the path-integral measure from twistor space fields to spacetime fields, then Cauchy-Riemann CS theory on  $L$  also captures loop amplitudes, and we obtain a loop-level KLA.

Let us sketch an argument that this is true for the Cauchy-Riemann CS anomaly. A codimension  $k$  Levi-flat Cauchy-Riemann manifold  $M$  foliates into holomorphic leaves  $M_t$ . If the space of leaves  $T$  is a  $k$ -dimensional manifold, one easily checks that  $L$  satisfies these conditions, then the Cauchy-Riemann CS partition function is  $\log Z(M) = \int_{t \in T} \omega \log Z(M_t)$ , where  $\omega$  is a volume form (defining the path-integral measure) on  $T$ , and where  $Z(M_t)$  is the partition function of holomorphic CS theory (with the same field content) on  $M_t$ : the full theory on  $M$  is anomaly free if the corresponding holomorphic theory on  $M_t$  is anomaly free.

Thus, it suffices to study holomorphic CS theory on  $M_t$ , or even (as anomalies are integrals of local objects) on a small patch in  $M_t$ , where global issues (e.g., nonzero-degree bundles) disappear. In the weak-coupling limit, anomaly contributions from bosons and fermions are equal and opposite [44], since their linearized actions in the presence of an external gauge field coincide up to statistics: supersymmetry ensures anomaly cancellation in holomorphic CS theory and Cauchy-Riemann CS theory. Without supersymmetry, this argument fails.

Indeed, for nonsupersymmetric twistorial holomorphic CS theory (semiclassically equivalent to SDYM theory) there is an anomaly [44] that, via the preceding argument, implies an anomaly for nonsupersymmetric twistorial Cauchy-Riemann CS theory (semiclassically equivalent to YM theory). Even if the KLA algebra *does* imply tree-level CK duality, it will be anomalous. This is consistent with, and elucidates, the conclusion that CK duality can be realized as an anomalous symmetry of a semiclassically equivalent YM-BV action [42], as well as the proof (by exhaustion) that there are no CK-dual four-point two-loop numerators for

bosonic YM theory assuming that they can be derived from local Feynman rules [65].

*Data management.*—No additional research data beyond the data presented and cited in this work are needed to validate the research findings in this work.

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