Semiclassical Husimi Distributions of Schur Vectors in Non-Hermitian Quantum Systems

Joseph Hall, Simon Malzard, and Eva-Maria Graefe
Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom

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We construct a semiclassical phase-space density of Schur vectors in non-Hermitian quantum systems. Each Schur vector is associated to a single Planck cell. The Schur states are organized according to a classical norm landscape on phase space—a classical manifestation of the lifetimes which are characteristic of non-Hermitian systems. To demonstrate the generality of this construction we apply it to a highly nontrivial example: a PT-symmetric kicked rotor in the regimes of mixed and chaotic classical dynamics.

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The correspondence between quantum and classical mechanics has been investigated since the early days of quantum theory. For decades the main focus has been on quantum systems isolated from their environment, described by unitary time evolution. Most realistic systems, however, are subject to losses and dissipation, which can often be modeled by non-Hermitian Hamiltonians and nonunitary time evolution operators [1,2]. The unique dynamical features of such systems have brought them into the spotlight, and there has been much effort to engineer specific non-Hermitian systems for the purpose of control of quantum and other wave dynamics [3,4]. The special class of PT-symmetric non-Hermitian Hamiltonians can lead to purely real eigenvalues and special dynamical properties [5–7]. The question of how insights into quantum-classical correspondence and semiclassical techniques can be extended to nonunitary quantum systems has come into focus in recent years. While considerable progress has been made for simple types of open quantum maps, modeling (partial) escape through openings [8–15], the general question remains challenging.

A common approach is the study of the localization of quantum eigenstates on regions of classical phase space. The Husimi distribution of quantum states, which provides a quasiprobability distribution in phase space, is particularly useful in this context. In closed systems, the Husimi distributions of individual eigenstates localize on classical phase-space structures [16–18]. A unit Planck cell supports a single eigenstate, i.e., to each state we may associate a distinct total area h in phase space, and the sum of the Husimi functions of all eigenstates is the uniform distribution [18,19]. This localization lies at the heart of quantization rules and algorithms for the construction of quasimodes. In systems described by non-normal operators the characteristic lack of orthogonality of the relevant quantum states presents a challenge in forming a similar correspondence. A consequence of the nonorthogonality is that a standard Planck cell partitioning of the phase space is no longer guaranteed, and the sum of the Husimi distributions of all the eigenstates does not uniformly cover the whole phase space [9,20].

In Ref. [21] it was argued for the case of a kicked rotor with escape through an opening in phase space, that considering the Schur vectors (arising from an orthogonalization of the eigenvectors) allows one to count phase-space areas belonging to certain lifetimes consistently, leading to a fractal Weyl law [22]. In Ref. [9] the Husimi distributions of a set of Schur vectors were used for comparison to classical structures for a similar system. The underlying classical dynamics are not altered by the escape, but augmented with an additional lifetime. The support and localization of ordered sets of Schur vectors were heuristically found to correspond to regions of the phase space associated to trajectories with certain classical lifetimes.

In a similar vein in a study of a PT-symmetric kicked top a correspondence has been observed between Husimi distributions of Schur vectors and false color plots of the semiclassical norm [23]. These studies point at a general connection between quantum Schur vectors and the behavior of the classical counterpart of the norm as a function on phase space. The origin of this connection, and a quantitative description of the Husimi distribution of Schur vectors based on the semiclassical norm hitherto remain illusive.

Here we argue that the correspondence of the Husimi distributions to areas of classical phase space belonging to certain values of the semiclassical norm can be traced back to the surprising observation that the eigenvectors of the Hermitian quantum operator $\hat{W}(t) = \hat{U}(t)\hat{U}^{\dagger}(t)$ for large times approach the Schur vectors of the time-evolution operator $\hat{U}$. Guided by this observation we exploit the orthogonality of Schur vectors to invoke the association of the support of individual states to unique Planck cells, which is familiar from Hermitian systems. This allows one to systematically and uniformly cover the entire phase space and recover quantization rules. Here we devise an algorithm that constructs a classical Husimi density conditioned on the counting of Planck cells. This generates a semiclassical approximation of the Husimi density of Schur states based solely on the classical dynamics. We demonstrate the construction for a PT-symmetric generalization of a kicked rotor.

Let us begin by reviewing some basic features of the eigenvectors and eigenvalues of non-Hermitian Hamiltonians. For simplicity we shall introduce these ideas for time-independent systems. They generalize to systems with periodic time dependence.

We consider a non-Hermitian Hamiltonian $\hat{K}$ with eigenstates $|\phi_n\rangle$,

$$\hat{K}|\phi_n\rangle = \epsilon_n|\phi_n\rangle, \tag{1}$$

where the energies $\epsilon_n = \varepsilon_n + i\mu_n$ are complex. The norm of an eigenstate changes in time as

$$\langle\phi_n|\hat{U}^\dagger(t)\hat{U}(t)|\phi_n\rangle = e^{\frac{2\mu_n}{\hbar}t}, \tag{2}$$

where $\hat{U}(t) = e^{-i\hat{K}t/\hbar}$ denotes the nonunitary time-evolution operator. Interpreting the norm as an overall probability, a gain-loss profile is encoded in the imaginary parts of the energy spectrum $\{\mu_n\}$ resulting in an exponential growth ($\mu_n > 0$) or decay ($\mu_n < 0$) of stationary states.

In general, the eigenstates $|\phi_n\rangle$ are not orthogonal, but away from exceptional points they form a complete basis for the relevant Hilbert space. Consider an arbitrary initial state $|\psi(t=0)\rangle$, which is expanded in the eigenbasis as

$$|\psi(t=0)\rangle = \sum_n \psi_n|\phi_n\rangle, \tag{3}$$

where $\psi_n = \langle\chi_n|\psi(t=0)\rangle$, and where $\{\langle\chi_n|\}$ are the left eigenstates of $\hat{K}$ [24], normalized such that $\langle\chi_n|\phi_k\rangle = \delta_{nk}$. The state evolves as

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_n e^{-\frac{i}{\hbar}(\varepsilon_n + i\mu_n)t}\psi_n|\phi_n\rangle, \tag{4}$$

which is similar to the unitary case, with additional relative exponential growth or decay of the coefficients of the different eigenstates. If we order the $|\phi_n\rangle$ such that

$$\mu_1 > \mu_2 > \ldots, \tag{5}$$

an initial state that is a superposition of eigenstates will dynamically decay into the subspace spanned by the $|\phi_n\rangle$ with smaller n. Thus, there is a natural structure of invariant subspaces of the Hilbert space arising from the hierarchy of eigenvectors. To identify classical phase-space areas associated to these quantum subspaces, we need an orthogonal basis. This can be provided by appropriately ordered Schur vectors of the time-evolution operator, as suggested in Ref. [9].

The non-Hermitian Hamiltonian can be expressed as $\hat{K} = \hat{V}\hat{R}\hat{V}^\dagger$ where $\hat{V}$ is a unitary matrix whose columns are the Schur vectors, and $\hat{R}$ is an upper triangular matrix with the eigenvalues $\{\epsilon_n\}$ on the diagonal. The Schur decomposition is not unique, but depends on the order of the eigenvalues along the diagonal. A natural choice is to order them according to growth and decay rates [Eq. (5)] reflecting the hierarchy of the quantum subspaces. It has been observed in two example systems that sums of the Husimi functions of the Schur vectors belonging to the largest imaginary parts are localized on certain regions in classical phase space associated to these quantum subspaces [9,23].

Here we identify these regions as bounded by contour lines of the classical counterpart of the norm as a function of the initial state under propagation with $\hat{U}^\dagger$. This is connected to the surprising finding that in the (generic) case of complex quasienergies with nondegenerate imaginary parts, the Schur vectors of $\hat{K}$, with the ordering [Eq. (5)], emerge as the asymptotic eigenvectors of the operator $\hat{W}(t) = \hat{U}(t)\hat{U}^\dagger(t)$ for large times. In the Supplemental Material [25] we motivate this observation with a concrete example and provide a general proof.

The operator $\hat{W}(t)$ is Hermitian, and thus, we can apply standard semiclassical arguments to associate phase-space areas bounded by contour lines of the classical phase-space function of the operator $\hat{W}$ with the quantum Husimi distributions of its eigenvectors, and by extension the Schur vectors of $\hat{K}$. The expectation value of $\hat{W}$ is the value of the norm of an initial state $|\psi(0)\rangle$ at time t, when we regard $\hat{U}^\dagger(t)$ as the time-evolution operator. As a classical counterpart of this we identify the norm $w(q,p,t)$ of an initial coherent state centered at the respective phase-space point (q,p) after propagation with $\hat{U}^\dagger(t)$ up to time t under a coherent state approximation. Following Ref. [26], this is found by integrating the dynamical equation

$$\dot{w}(t) = 2\Gamma(q,p)w(t) \tag{6}$$

along the phase-space trajectories

$$\dot{q} = -\frac{\partial H}{\partial p} + \frac{\partial \Gamma}{\partial q}, \qquad \dot{p} = \frac{\partial H}{\partial q} + \frac{\partial \Gamma}{\partial p}, \tag{7}$$

where H and Γ denote the expectation values of the Hermitian and anti-Hermitian parts $\hat{H}^\dagger = \hat{H}$ and $\hat{\Gamma}^\dagger = \hat{\Gamma}$ of the Hamiltonian $\hat{K} = \hat{H} + \mathrm{i}\hat{\Gamma}$.

In addition to (potentially) modifying the phase-space trajectories, the gain-loss function Γ introduces a nontrivial dynamics of the variable $w(t)$, the *classical norm*. We represent the qualitative behavior of the classical norm through false-color plots we refer to as *norm landscapes*. The norm landscape is constructed by associating to each point (q, p) in phase space their respective average norm $\langle w(t)\rangle$ at a final time t_f.

Based on the norm landscapes, we use a state counting argument to construct a semiclassical approximation of the Husimi distributions of the quantum Schur vectors. We define the subset $\mathcal{S}_m$ of phase space associated to the mth Schur state as the set of initial conditions that satisfy

$$\mathcal{S}_m = \{(q, p){:}\, \alpha_-^{(m)} \le \langle w_{t_f}\rangle \le \alpha_+^{(m)}\}, \qquad (8)$$

where $\alpha_-^{(m)}, \alpha_+^{(m)} \in \mathbb{R}$, and $\langle w_{t_f}\rangle$ is the average norm at a final time t_f. *A priori* the values $\alpha_-^{(m)}$, $\alpha_+^{(m)}$ are unknown, and are determined self-consistently from a semiclassical state counting argument. To mimic the intrinsic Gaussian nature of the Husimi distribution, we construct the classical density $\mathcal{D}_m(q, p)$ by the convolution of a uniform distribution on the set $\mathcal{S}_m$ with a two-dimensional Gaussian smoothing kernel with standard deviation $\sigma = \sqrt{(\hbar/2)}$ matching the minimum quantum uncertainty. We then condition the classical density by requiring it be normalized with respect to a Planck cell, that is,

$$\int\int \mathcal{D}_m(q, p) dq dp = h. \qquad (9)$$

By simple extension, constructing the classical density for a group of n Schur states, e.g., the n states with largest μ, simply requires the integral [Eq. (9)] to equal nh.

Let us now demonstrate these ideas for a PT-symmetric kicked rotor which features both chaotic and mixed classical dynamics, described by the Hamiltonian

$$\hat{K} = \frac{\hat{p}^2}{2} + \frac{i\gamma}{2\pi}\hat{p} + \frac{k}{4\pi^2}\cos(2\pi\hat{q}) \sum_{n=-\infty}^{\infty} \delta(t-n), \qquad (10)$$

where $\gamma, k, \in \mathbb{R}^+$. We consider the one-period evolution (Floquet) operator $\hat{U}$, where the time evolution is composed of the free evolution followed by a kick. We consider this example, because it yields a nontrivial classical dynamics, which differs from the Hermitian case in both the phase-space evolution and the additional norm. This example is particularly challenging due to nontrivial long-term dynamics brought about by the PT symmetry of the system, in contrast to the behavior of many other non-Hermitian or open systems, that approach limiting configurations on relatively short timescales. To demonstrate that our approach equally applies to open systems without PT symmetry, in the Supplemental Material [25] we apply it to a model of a kicked rotor with partial escape that has been studied in Refs. [9,11,21].

Using a standard quantization on a torus [27], the matrix elements of the Floquet operator $\hat{U}$ for the model [Eq. (10)] in the position representation are

$$U_{ll'} = \frac{1}{N} e^{-\frac{iNk}{2\pi}\cos\left(\frac{2\pi l}{N}\right)} \sum_{m=-N_1}^{N_1} e^{-\frac{i\pi}{N}m^2 + \gamma m + \frac{2\pi i}{N}m(l-l')}, \qquad (11)$$

where $N = 2N_1 + 1$ is the matrix dimension, which plays the role of an effective $\hbar = (1/2\pi N)$. The PT symmetry manifests in the quasienergies ϵ_n being real or appearing in complex conjugate pairs. The growth or decay rate is encoded in the imaginary part of a quasienergy $\mu_n = \mathrm{Im}(\epsilon_n)$. Here we consider a matrix dimension of $N = 1001$. For this value of N even for small values of γ for a broad range of kicking strengths k a large number of quasienergies lie off the real axis in complex conjugate pairs. Spectral instabilities (which typically occur for nontrivial non-normal operators [28]) limit the numerical range of γ for which reliable quantum results can be obtained, and thus we will limit the discussions to small values of $\gamma \le 0.003$. (The classical system can be analyzed for any value of γ.)

The classical map generated by $\hat{U}^\dagger$ is given by

$$p_{n+1} = \mathrm{mod}\left(p_n - \frac{k}{2\pi}\sin(2\pi q_n) + \frac{\gamma}{2\pi} + \frac{1}{2}, 1\right) - \frac{1}{2}$$
$$q_{n+1} = \mathrm{mod}\left(q_n - p_{n+1} + \frac{\gamma}{2\pi}, 1\right). \qquad (12)$$

While arising from a nonunitary evolution the classical map is in fact area preserving, independent of the value of γ, a curiosity that is made possible by the presence of PT symmetry, but that is not necessarily typical for PT-symmetric systems. This area conservation prevents the formation of typical features of dissipative classical systems such as attractors, and chaotic saddles [29,30], yet, as we shall see, the quantum Schur vectors are clearly influenced by the presence of the loss and gain profile. This is reflected in the classical norm map [Eq. (6)]

$$w_{n+1} = e^{2\gamma p_{n+1}} w_n. \qquad (13)$$

In the left plot of the first row of Fig. 1 we show an example of a mixed regular-chaotic dynamics for $k = 1.1$ and $\gamma = 0.001$. For this small value of γ the classical phase-space trajectories differ little from those of the unitary system. The associated norm landscape for $t = 66$ is depicted in the right column of the same figure. We observe a rather distinct partitioning of the phase space into those trajectories whose norm grows, is stable, or decays. These

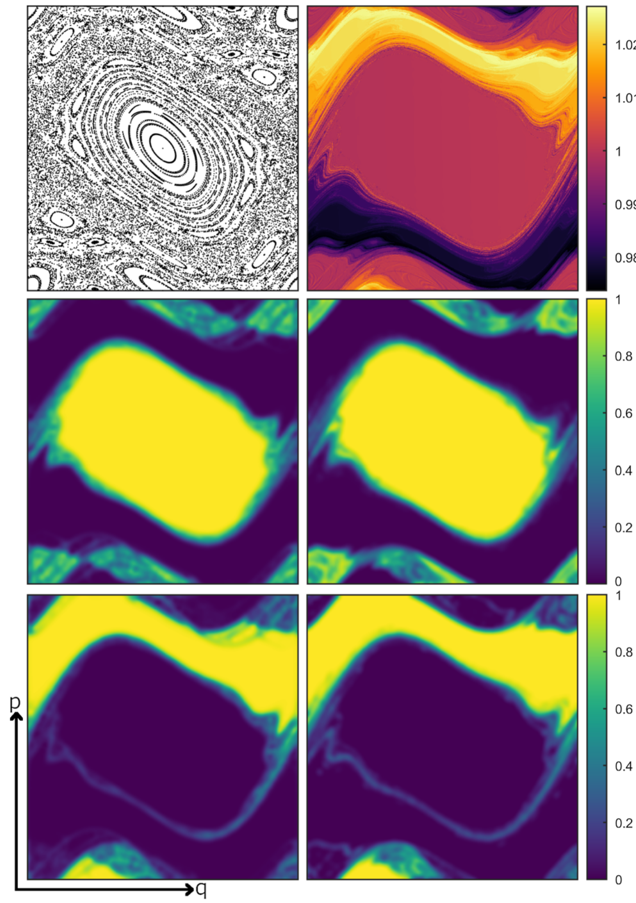

FIG. 1. Quantum-classical correspondence in phase space $(q,p) \in [0,1) \times [-0.5, 0.5)$ for a non-Hermitian kicked rotor for $k = 1.1$ and $\gamma = 0.001$. The top row depicts the Poincaré section (left) and the norm landscape (right) for a final time $t_f = 66$. The middle and bottom rows depicts the Husimi distribution (left) of the stable (middle) and gain (bottom) set and its associated classical density (right) for the same values of k, γ, t_f, and $\Delta_+ = -\Delta_- = 0.76$.

regions can contain trajectories that are regular or chaotic, as can be seen in the regions of gain and loss that contain chains of regular islands. A more captivating example can be seen in the top row of Fig. 2 where the Poincaré section (left) and norm landscape (right) are plotted for $k = 10$, $\gamma = 0.003$. Here the Poincaré section is dominated by a single featureless chaotic sea. In contrast, the norm landscape presents a much richer structure within the region corresponding to the chaotic sea in the Poincaré section.

The emergence of three different regions of norm behavior is a classical manifestation of the PT symmetry. The organization of the Schur vectors around the norm landscape is demonstrated most readily in PT-symmetric systems by considering the Husimi distributions of three sets which we will refer to as gain, stable, and loss sets. The gain (loss) sets consist of those Schur vectors with positive (negative) imaginary parts of the quasienergy, $\mu_n > 0$ ($\mu_n < 0$), and the stable set corresponds to $\mu_n = 0$. The

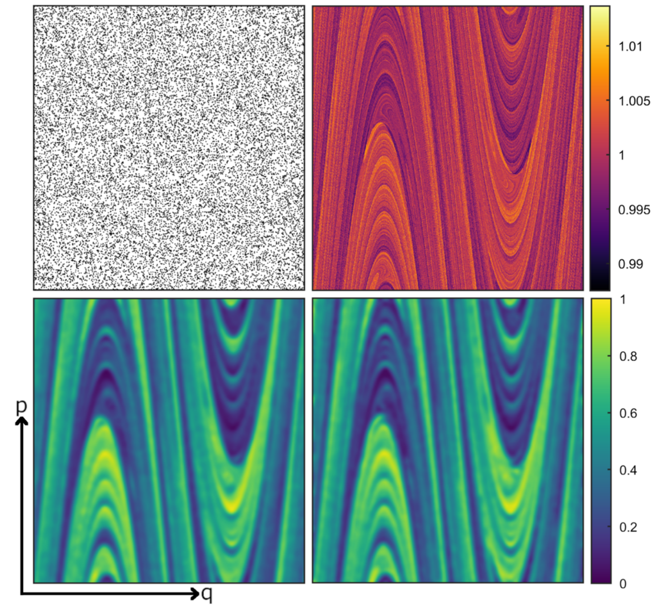

FIG. 2. Quantum-classical correspondence in phase space $(q,p) \in [0,1) \times [-0.5, 0.5)$ for a non-Hermitian kicked rotor with $k = 10$ and $\gamma = 0.003$. From top left to bottom right: classical Poincaré section, norm landscape ($t_f = 14$), Husimi distributions of the gain set, and semiclassical phase-space density $\mathcal{D}_+$ for $t_f = 14$ and $\Delta_+ = 0.048$.

left column of Fig. 1 depicts the Husimi distribution of the gain (middle row) and stable (bottom row) sets of Schur vectors for the case $k = 1.1$ and $\gamma = 0.001$. We observe a clear correspondence between the Husimi distributions of the gain and stable set and the relevant regions of the norm landscape. Correspondingly, we define three classical support sets, $\mathcal{S}_+$ for the support on the gain set, $\mathcal{S}_0$ for the support on the stable set, and $\mathcal{S}_-$ for the support on the loss set, as

$$\begin{aligned} \mathcal{S}_+ &= \{z\colon \langle w_{t_f} \rangle > e^{2\gamma\Delta_+}\} \\ \mathcal{S}_- &= \{z\colon \langle w_{t_f} \rangle < e^{2\gamma\Delta_-}\} \\ \mathcal{S}_0 &= \{z\colon e^{2\gamma\Delta_-} \leq \langle w_{t_f} \rangle \leq e^{2\gamma\Delta_+}\}. \end{aligned} \tag{14}$$

The form $\alpha_\pm = e^{2\gamma\Delta_\pm}$ of the partitioning parameter is chosen to reflect the functional form of the classical norm [Eq. (13)], and we have $\Delta_\pm \in \mathbb{R}^\pm$. Furthermore, the PT symmetry of the system enforces the additional constraint $\Delta_- = -\Delta_+$. Denoting the fraction of gain (loss) states by $f_Q^{(+)}$ ($f_Q^{(-)}$) and the fraction of stable states by $f_Q^{(0)}$, we condition the corresponding classical densities $\mathcal{D}_{\pm,0}(q,p)$ such that

$$f_C^{(+)} = \frac{1}{N} \int\int \mathcal{D}_+(q,p)dqdp = f_Q^{(+)}, \tag{15}$$

and similarly for $f_Q^{(0)}$ and $f_Q^{(-)}$.

The right column of Fig. 1 shows the classical densities associated to the two sets of Schur vectors in the left column, where the final times has been chosen as $t_f = 66$. We observe a striking agreement between the Husimi distributions of the Schur vectors and the classical density for each case. Similarly Fig. 2 shows the Husimi distribution (left) of the gain states and the classical density with $t_f = 14$ (right) for $k = 10$ and $\gamma = 0.001$. Here the Husimi distribution shows a more intricate structure than for $k = 1.1$, which is again accurately reproduced by the classical density.

There typically is a large region of final times yielding similar results. We have verified that the Jensen-Shannon divergence [13,31] between the quantum and classical Husimi densities rapidly decreases with the final time and reaches a plateau region. We have found minima of the Jensen-Shannon divergence at $t_f \approx 66$, $t_f \approx 14$ for $k = 1.1$, and $k = 10$, respectively, which we have chosen for the construction of the classical densities in Figs. 1 and 2. In both cases choosing a t_f in a relatively large range around the minimum value causes little quantitative difference to the results.

In summary, we have introduced a classical density that yields an accurate approximation for the Husimi distributions of quantum Schur states for non-Hermitian systems with complex energies, as demonstrated for a PT-symmetric kicked rotor. This is a highly nontrivial model, with both complex and real quasienergies, with mixed and fully chaotic classical dynamics. The semiclassical Husimi distributions are constructed by associating to each Schur vector an area h in the phase space determined by bands of the classical norm landscape. This is based on the curious observation that the Schur vectors emerge as asymptotic eigenvectors of the operator $\hat{W}(t) = \hat{U}(t)\hat{U}^{\dagger}(t)$, associated to the classical norm. Interestingly, these norm landscapes have been found to be of crucial importance in the classical propagation of Husimi distributions in non-Hermitian systems recently [32]. The deeper understanding of the meaning of the operator $\hat{W}$ and the application of the proposed semiclassical algorithm to a larger class of example systems will be the topics of future investigations. The provision of semiclassical tools for non-Hermitian quantum systems is a crucial task, in particular given the notorious numerical difficulties that interesting non-Hermitian Hamiltonians pose. The construction of semiclassical Schur vectors introduced here paves a new way toward semiclassical quantization rules for eigenvalues.

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